Methods and Applications of Tropical Algebra: a Guided Tour

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SMAI MAIRCI
Orange Labs, March 19, 2010
Issy Les Moulineaux

Max-plus or tropical algebra

In an exotic country, children are taught that:

$$\text{“}a + b\text{”} = \max(a, b) \qquad \text{“}a \times b\text{”} = a + b$$

So

- “$2 + 3$” $=$

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- $“\sqrt{-1}” =$

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- $\text{“}2^3\text{''} = \text{“}2 \times 2 \times 2\text{''} = 6$
- $\text{“}\sqrt{-1}\text{''} = -0.5$

The sister algebra: min-plus

$$\text{“}a+b\text{”} = \min(a,b) \qquad \text{“}a\times b\text{”} = a+b$$

- $\text{“}2+3\text{”} = 2$
- $\text{“}2\times 3\text{”} = 5$

These algebras were invented by various schools in the world (from Birmingham to St-Petersburg, passing through Paris).

CNET, 87: first meeting on “algèbres exotiques”

SÉMINAIRE SUR LES ALGÈBRES EXOTIQUES ET LES SYSTÈMES A ÉVÉNEMENTS DISCRETS

3 et 4 juin 1987

Organisé par :

le Centre National de la Recherche Scientifique
le Centre National d’Etudes des Télécommunications
l’Institut National de la Recherche en Informatique et Automatique au :

Centre National d’Etudes des Télécommunications
38-40, rue du Général Leclerc
92131 Issy-les-Moulineaux

Pour la modélisation des processus continus, on dispose aujourd’hui de théories ayant atteint une certaine maturité. Il n’en va pas de même pour ce qu’il est désormais convenu d’appeler « systèmes à événements discrets » et que l’on rencontre dans l’étude des ateliers flexibles, des réseaux d’ordinateurs ou de télécommunications, des circuits VLSI spécialisés en traitement du signal, pour ne citer que quelques exemples. Diverses approches et théories de ces systèmes s’appuyant sur des outils mathématiques variés ont néanmoins émergé.

Ce séminaire à caractère didactique, organisé dans le cadre de l’ATP-CNRS « Méthodologie de l’Automatique et de l’Analyse des Systèmes », avec le concours du CNET et de l’INRIA, a pour objectifs d’une part d’initier les participants à certaines de ces théories et aux outils correspondants, et d’autre part de constituer un lieu de rencontre et de confrontation de ces approches.

Conférenciers invités (liste provisoire) : P. Caspi, IMAG Grenoble ; P. Chretienne, Univ. de Paris VI ; R. A. Cuninghame Green, Univ. de Birmingham, UK ; G. Cohen, Ecole des Mines Fontainebleau ; N. Halbwachs, IMAG Grenoble ; M. Minoux, STEI Issy-les-Moulineaux ; P. Moller, IIASA Vienne, AUT ; G. J. Olsder, Univ. Delft, Pays-Bas ; J. P. Quadrat, INRIA Rocquencourt ; Ch. Reutenauer, Univ. Paris VI ; M. Viot, CNRS et Ecole Polytechnique Palaiseau.

Comité d’organisation : P. Chemouil, CNET Issy-les-Moulineaux ; G. Cohen, Ecole des Mines Fontainebleau ; J. P. Quadrat, INRIA Rocquencourt ; M. Viot, CNRS et Ecole Polytechnique Palaiseau.

Toutes les personnes intéressées sont invitées à contacter le plus vite possible :

Monsieur G. Cohen
CAI-ENSMP
35, rue Saint-Honoré
77305 Fontainebleau Cedex
Tél. (1) 64.22.48.21

The term “tropical” is in the honor of Imre Simon, 1943 - 2009

who lived in Sao Paulo (south tropic).

Motivation 1: max-plus linear discrete event systems

$$x_i(k) = \max_{j\in[d]} A_{ij} + x_j(k-1) \ ,$$

$x_i(k)$: departure time of the kth train on line i, completion time of job k, etc

$$\text{“}x(k) = Ax(k-1)\text{”}$$

Baccelli, Cohen, Quadrat, Olsder's book (92)

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$$\text{“}x(k) = Ax(k-1)\text{”}$$

$x(0) = v, Av = \lambda v \implies x(k) = \lambda^k v$

so $x_i(k) = k\lambda + v_i$.

λ: cycle time; v: stationnary schedule.

Baccelli, Cohen, Quadrat, Olsder's book (92)

Motivation 2: Perron-Frobenius theory

$y = f(x) := Mx,\ M_{ij} \geq 0,\ f : \mathbb{R}_+^n \to \mathbb{R}_+^n$

typically population dynamics an networks (trafic is nonnegative)

The log of the population is significant, $X = \log x$, $Y = \log y$

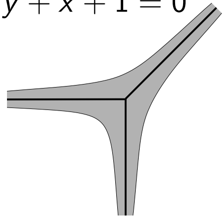
$$Y_i = \log\Big(\sum_{1 \leq j \leq n} M_{ij} e^{X_j}\Big)$$

log-glasses send us to the tropical world...

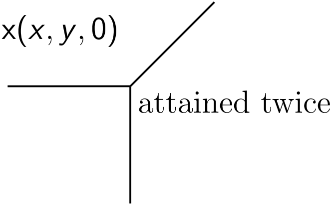
Gelfand, Kapranov, and Zelevinsky defined the amoeba of an algebraic variety $V \subset (\mathbb{C}^*)^n$ to be the “log-log plot”

$$A(V) := \{(\log|z_1|, \dots, \log|z_n|) \mid (z_1, \dots, z_n) \in V\} \ .$$

$y + x + 1 = 0$



$\max(x, y, 0)$



If a sum of numbers is zero, then, two of them much have the same magnitude

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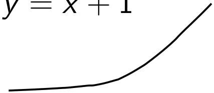
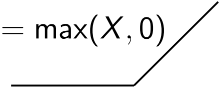
$y = x + 1$



$X = \log(e^X + 1)$

$Y = \max(X, 0)$



See Passare & Rullgard, Duke Math. 04 for more information on amoebas

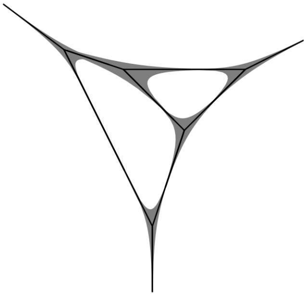
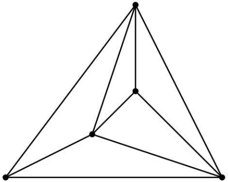
Maslov's "dequantization"

Define:

$$a \oplus_h b := h \log(e^{a/h} + e^{b/h}) \ .$$

Then, $(\mathbb{R} \cup \{-\infty\}, \oplus_h, +) \simeq (\mathbb{R}_+, +, \times)$, but

$$\lim_{h \to 0^+} a \oplus_h b = \max(a, b)$$

Amoeba of

$$0 = 1 + z_1^5 + 80 z_1^2 z_2 + 40 z_1^3 z_2^2 + z_1^3 z_2^4$$

reproduced from Passare, Rüllgard

Max-plus algebra can help you to unify . . .

the analogy between Perron-Frobenius theory and max-plus linear discrete event systems is explained by a common setting

Non-linear Perron-Frobenius theory

deals with generalizations of nonnegative matrices, i.e., nonlinear maps satisfying

(M): $x \leq y \iff f(x) \leq f(y)$

plus additional properties, like

(H): $f(\lambda x) = \lambda f(x), \forall \lambda \geq 0$;

(SH): $f(\lambda x) \leq \lambda f(x), \forall \lambda \geq 1$;
(related to nonexpansiveness in Hilbert's or Thompson's metric)

or f convex, or f concave, or ... (ad lib)

- Alexandroff and Hopf (1935) gave a fixed point proof of the Perron-Frobenius theorem.
- Kreïn and Rutman (48) infinite dim
- Birkhoff (57) introduced Hilbert's projective metric techniques
- Solow and Samuelson (53), Morishima (64): economic growth
- Krasnoselskii (64) infinite dim
- two AMS Memoirs of Nussbaum (88, 89) advanced results.
- results by Hirsch and Smith on monotone dynamical systems (Smith's book).
- articles by Krause
- Dellacherie: non-linear potential theory
- many applications

Application: reputation systems

or self-referential aspects of web ranking

the rank r_i of page i is the frequency of visit of this page by a random walker on the web graph.

W adjacency matrix of the web, $W_{ij} = 1$ if there is a link from page i to page j, and $W_{ij} = 0$ otherwise.

$$r = rP, \quad r \geq 0, \quad \sum_i r_i = 1, \qquad P_{ij} = \frac{W_{ij}}{\sum_k W_{ik}}$$

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$$r = r(1-\gamma)P + \gamma f$$

with probability γ, the user resets his exploration, and moves to page i with proba. f_i.

The pagerank influences the behaviour of web users

...which ultimately determines the pagerank.

A simple model Akian, SG, Ninove, Posta'06, arxiv:0712.0469

If the current pagerank is r, the user moves from page i to page j with probability:

$$P_T(r)_{ij} = \frac{W_{ij} e^{r_j/T}}{\sum_k W_{ik} e^{r_k/T}}$$

Temperature $T =$ measures the insensitivity of the surfer to reputation; $T = \infty$: standard pagerank.

The pagerank of tomorrow is the stationnary distribution of the walk of today

$$r_T^{k+1} = r_T^{k+1} P_T(r^k)$$

Let u_T denote the map $r_T^k \to r_T^{k+1}$.

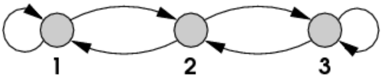
$$u_T(x) = h_T(x)/(\sum_k h_T(x)_k)$$

$$h_T(x)_l = \big(\sum_k W_{ik} e^{x_k/T}\big)\big(\sum_{R\to l} \prod_{(i,j)\in R} W_{ij} e^{x_j/T}\big) \enspace ,$$

the latter sum being taken over river networks R with sea l (Tutte's matrix tree theorem)
So h is order preserving.

Theorem. Akian, SG, Ninove, Posta'06, arxiv:0712.0469 For $T \geq n$, the T-pagerank is unique. For T small, always several T-pageranks.

Proof idea. For $T \geq n$, the map h becomes nonexpansive in Thompson's metric, local $\implies$ global uniqueness Nussbaum (AMS memoir, 1988)



If $T \approx 0$ and $r_T^0 = \left(\frac{1+\varepsilon}{3}\ \frac{1}{3}\ \frac{1-\varepsilon}{3}\right)$ then $\lim_{t\to\infty} r_T^k \approx (1\ 0\ 0)$.

If one believes that the pagerank measures quality, then the pagerank might become meaningless.

Similar idea applied by de Kerchove and Van Dooren (SIAM News 08) to reputation systems. Rater i gives mark x_{ij} to object j.
The reputation of object j is $r_j = \sum_i x_{ij} f_i / \sum_k f_k$
The weight of rater i is decreased if he diverges from the average rate:
$f_i(r) = d - \sum_j (x_{ij} - r_j)^2$, d fixed.
$d \to \infty$: uniform weights of raters.
Applied to "spam" elimination.

Current work Fercoq, Akian, Bouhtou, and SG (NL PF and optimization of web ranking)

Nonlinear PF with log-glasses = mean payoff games

If f is (M)+(H),

$$g := \log \circ f \circ \exp$$

satisfies the axioms of dynamic programming (M)+ (AH),

$$(\mathrm{AH}): \qquad g(\lambda + x) = \lambda + g(x), \forall \lambda \in \mathbb{R}$$

and so:

$$(\mathrm{N}): \qquad \|g(x) - g(y)\|_\infty \leq \|x - y\|_\infty$$

Crandall & Tartar, Kohlberg, Neyman, Sorin

Every M+N self-map of $\mathbb{R}^n$ can be written as a dynamic programming operator, zero-sum two player:

$$g : \mathbb{R}^n \to \mathbb{R}^n \qquad g_i(x) = \inf_{a \in A(i)} \sup_{b \in B(i,a)} (r_i^{ab} + P_i^{ab} x)$$

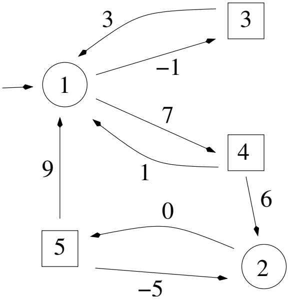
$P_i^{ab} := (P_{ij}^{ab})$, proba. of moving $i \to j$, $\sum_j P_{ij}^{ab} \leq 1$.
r_i^{ab}: payment of Player I to player II.
$(g^k(0))_i$ value of the game in horizon k with starting point i

Kolokoltsov 92; transition probabilities can even be deterministic: Rubinov and Singer 01, Gunawardena and Sparrow (Gunawardena TCS03).

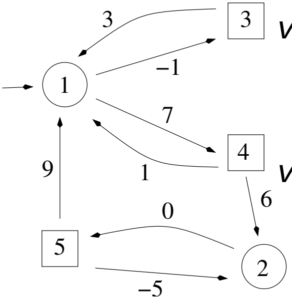
A combinatorial game with mean payoff

$G = (V, E)$ bipartite graph. r_{ij} price of the arc $(i, j) \in E$. "Max" and "Min" move a token. Min always pays to Max the amount of the arc.

Circle: Max plays. Square: Min plays.



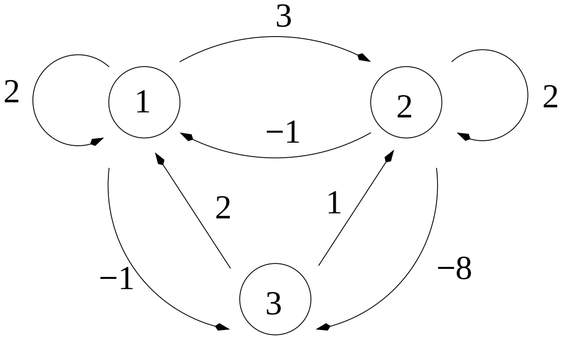
Circle: Max plays. Square: Min plays. v_i^k value in horizon k starting from node i



$$v_1^{k+1} = \max(2 + v_1^k, \min(8 + v_1^k, 13 + v_2^k))$$

$$v_2^{k+1} = \min(9 + v_1^k, -5 + v_2^k)$$

Max and Min flip a coin to decide who makes the move. Min always pay.



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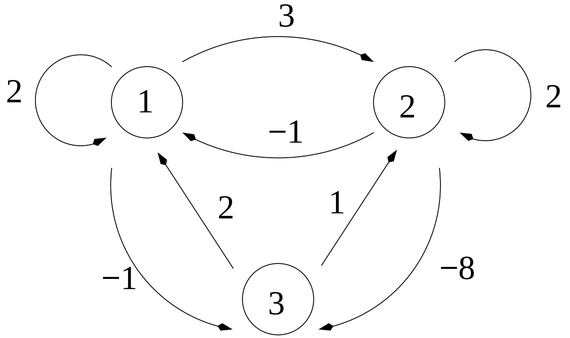

$$v_i^{k+1} = \frac{1}{2}(\max_{j:\, i\to j}(c_{ij} + v_j^k) + \min_{j:\, i\to j}(c_{ij} + v_j^k)) \quad .$$

- The limit of $v^k/k = g^k(v^0)/k$ (mean payoff) exists (finite state and actions), Kohlberg 1980.

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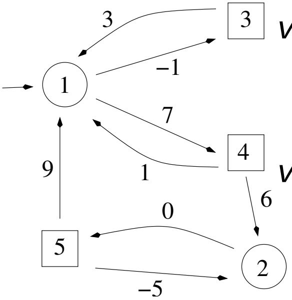
- attained by optimal stationnary feedback strategies

Policy iteration algorithm to compute the mean payoff

Howard 60: 1 player; Hoffman and Karp 66: two player, ergodic;
Cochet, SG, Gunawardena 99: two player degenerate (deterministic);
Vöge, Jurdjinzky 00: parity games...

Choose a strategy: replace each min by one term.



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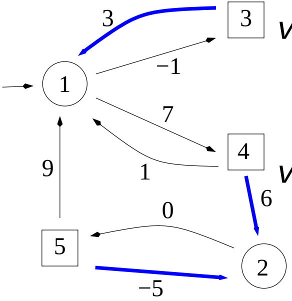


$$v_1^{k+1} = \max(2 + v_1^k, 13 + v_2^k)$$

$$v_2^{k+1} = -5 + v_2^k$$

max-plus map, polynomial time, mean payoff $(2, -5)$.

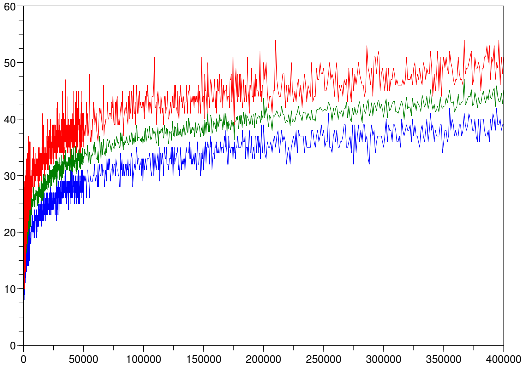
To improve the strategy, we use certificates of optimality:

In Cochet, SG, Gunawardena 99 and Cochet, Gunawardena 98, extended to Cochet, SG 06: stochastic games, Kohlberg's invariant half-lines:

$\exists u \in \mathbb{R}^n$, $\eta \in \mathbb{R}^n$, $g(u + t\eta) = u + (t+1)\eta$, for large t.
Then, $\lim_k g^k(v^0)/k = \eta$.

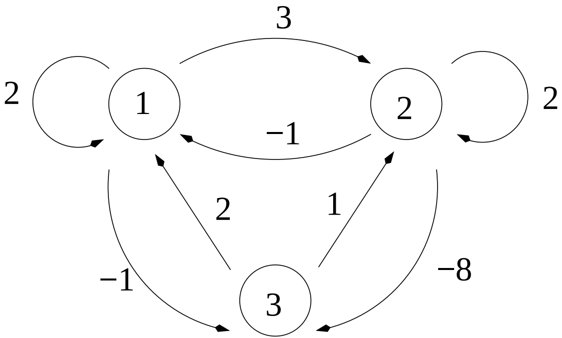
Convergence proof relies on a result of non-linear potential theory.
Different approach: Puri 95, solve the discounted game with discount factor close enough to 1.

Works well in practice



Friedman 09 showed that standard policy iteration rules for parity (and so meanpayoff) games may lead to an exponential number of iterations.

The complexity of mean payoff games is an open problem (in NP $\cap$ coNP, see Gurvich, Karzanov, Khachyan 86, Condon 92, Zwick and Paterson 96).



$$\begin{bmatrix}5\\0\\4\end{bmatrix}\quad \begin{aligned} v_1 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 3+v_2, -1+v_3) + \min(2+v_1, \overline{3+v_2}, \overline{-1+v_3})\\ v_2 &= \tfrac{1}{2}(\max(\overline{-1+v_1}, 2+v_2, -8+v_3) + \min(-1+v_1, 2+v_2, \overline{-8+v_3})\\ v_3 &= \tfrac{1}{2}(\max(\overline{2+v_1}, 1+v_2) + \min(2+v_1, \overline{1+v_2})\end{aligned}$$

this game is fair

The same games appear in program verification

. . .

Static analysis of programs by abstract interpretation

Cousot 77: finding invariants of a program reduces to computing the smallest fixed point of a monotone self-map of a complete lattice L

To each breakpoint i of the program, is associated a set $x^i \in L$ which is an overapproximation of the set of reachable values of the variables, at this breakpoint.

The best x is the smallest solution of $x = f(x)$

```
void main() {
  int x=0;           // 1
  while (x<100) {    // 2
    x=x+1;           // 3
  }                  // 4
}
```

Let $x_2^+ := \max x_2$. We arrive at

$$x_2^+ = \min(99, \max(0, x_2^+ + 1)) \quad .$$

The smallest x_2^+ is 99.

Lattice of templates

S. Sankaranarayanan and H. Sipma and Z. Manna (VMCAI'05)

Polyhedra with limited degrees of freedom

A subset of $\mathbb{R}^n$ is coded by the discrete support function

$$\sigma_X : \mathcal{P} \to \mathbb{R}^n, \qquad \sigma_X(p) := \sup_{x \in X} p \cdot x$$

$\mathcal{P}$ finite, fixed. Eg, $\mathcal{P} = \{e_i - e_j \mid i \neq j\}$: expresses only $x_i - x_j \leq C_{ij}$ (Miné's zones).

Theorem (SG, Goubault, Taly, Zennou, ESOP'07)

When the arithmetics of the program is affine, *abstract interpretation over a lattice of templates reduces to finding the smallest fixed point of a map* $f : (\mathbb{R} \cup \{+\infty\})^n \to (\mathbb{R} \cup \{+\infty\})^n$

$$f_i(x) = \inf_{a \in A(i)} \sup_{b \in B(i,a)} (r_i^{ab} + M_i^{ab} x)$$

with $M_i^{ab} := (M_{ij}^{ab})$, $M_{ij}^{ab} \geq 0$, *but possibly* $\sum_j M_{ij}^{ab} > 1$ *(negative discount rate!)*

```
void main() {
 i = 1; j = 10;
 while (i <= j){ //1
  i = i + 2;
  j = j - 1; }
}
```

$$i \le +\infty$$
$$i \ge 1$$
$$j \le 10$$
$$j \ge -\infty$$
$$i \le j$$
$$i + 2j \le 21$$
$$i + 2j \ge 21$$

$(i, j) \in [(1, 10), (7, 7)]$ (exact result).

- solved by policy iteration for games
- often more accurate than value iteration with accelerations of convergence -widening/narrowing- used classically in the static analysis community
- computing the smallest fixed point: works by Assale, SG, Goubault (MTNS 08), Seidl and Gawlitza, ...

The templates are special max-plus bases / finite elements

Fleming, McEneaney 00-; Akian, Lakhoua, SG 04-

Approximate the value function $\mathbb{R}^d \to \mathbb{R}$ by a "linear comb." of "basis" functions with coeffs. $\lambda_i(t) \in \mathbb{R}$:

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The w_i are given finite elements, to be chosen depending on the regularity of $v(\cdot)$.

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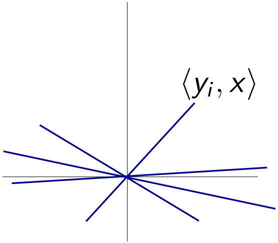
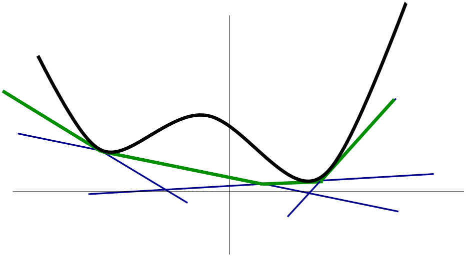
The w_i are given finite elements, to be chosen depending on the regularity of $v(\cdot)$. The sets defined by the templates w_i are precisely the sublevel sets: $\{x \in \mathbb{R}^d \mid v(x) \leq 0\}$.

Best max-plus approximation

$$P(f) := \max\{g \leq f \mid g \text{ "linear comb." of } w_i\}$$

linear forms $w_i : x \mapsto \langle y_i, x \rangle$



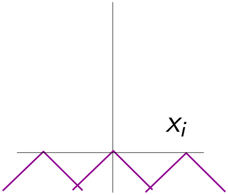


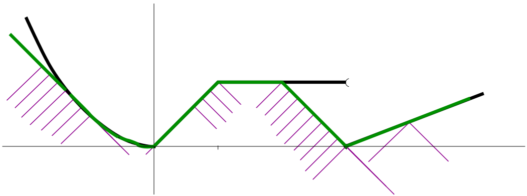
adapted if v is convex

Best max-plus approximation

$$P(f) := \max\{g \leq f \mid g \text{ "linear comb." of } w_i\}$$

cone like functions $w_i : x \mapsto -C\|x - x_i\|$





adapted if v is C-Lip

McEneaney's curse of dimensionality reduction

We want to solve the Hamilton-Jacobi equation

$$\frac{\partial v}{\partial t} = H(x, \frac{\partial v}{\partial x}) \qquad v(0,\cdot) = \phi$$

with $H(x,p)$ convex in p, $x \in \mathbb{R}^d$ (1 player, deterministic game).
Suppose

$$H = \sup_{i \in [r]} H_i, \;\; H_i = -(\frac{1}{2} x^* D_i x + x^* A_i^* p + \frac{1}{2} p^* \Sigma_i p)$$

(=LQ with switching).
Write v as a max-plus linear combination of base functions (computed via Riccati).

Inessential terms are trimmed dynamically using Shor relaxation (SDP) $\longrightarrow$ solution of a typical instance in dim 6 on a Mac in 30'

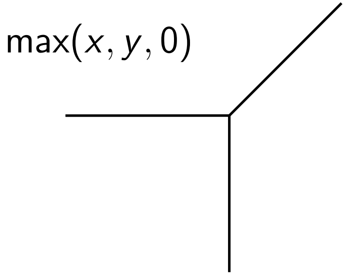
McEneaney, Desphande, SG; ACC 08

Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

$$\text{“}ax + by + c$$

is attained at least twice.



Some elementary tropical geometry

A tropical line in the plane is the set of (x, y) such that the max in

$$\max(a + x, b + y, c)$$

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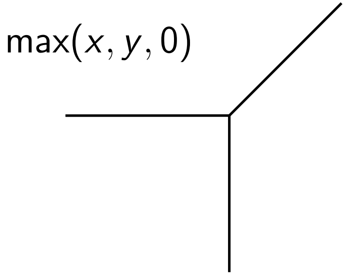
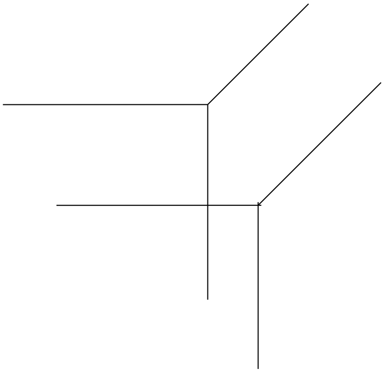
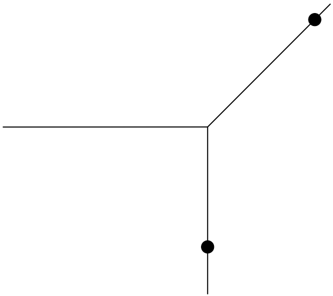


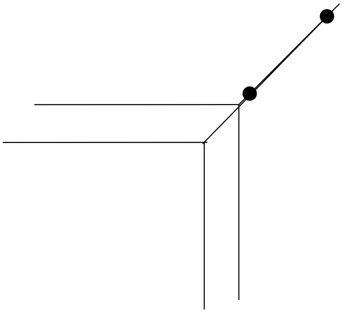
Two generic tropical lines meet at a unique point



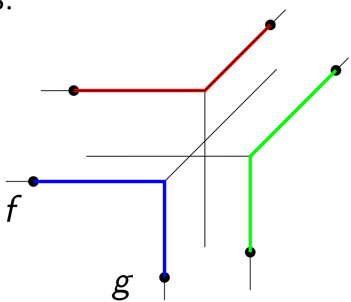
By two generic points passes a unique tropical line



non generic case

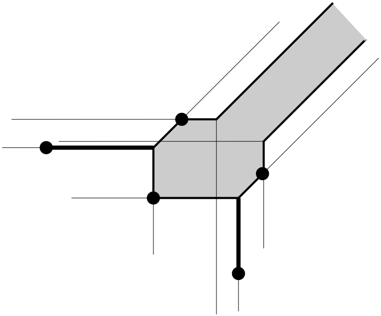


Tropical segments:



$[f,g] := \{\sup(\lambda + f, \mu + g)\ \lambda, \mu \in \mathbb{R}\cup\{-\infty\},\ \max(\lambda,\mu) = 0\}.$

Tropical convex set: $f, g \in C \implies [f, g] \in C$



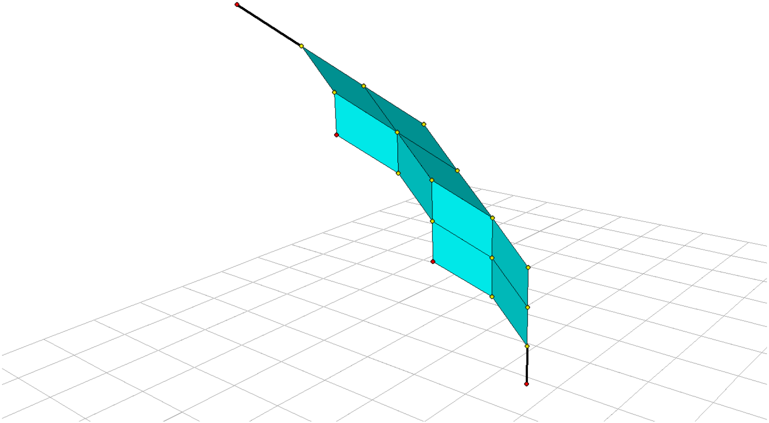
A tropical half-space is a set of the form

$$\{x \in \mathbb{R}^n_{\max} \mid \max_j a_j + x_j \leq \max_j b_j + x_j\}$$

A tropical polyhedral cone are defined as the intersection of finitely many tropical half-spaces, or equivalently, the convex hull of finitely many rays.

- Equivalence in SG 92, STACS'97,
- Tropical Minkowski theorem (extreme rays) in SG, Katz RELMICS 06, LAA07; Butkovič, Sergeev, Schneider LAA07
- more on the external representation in SG and Katz, arXiv:0908.1586).
- Sturmfels, Develin, Joswig, Yu...: tropical geometry point of view.

A max-plus "tetrahedron"?



Equivalence between tropical linear programming and mean payoff games

Checking whether a tropical polyhedron, defined as the intersection of p half-spaces

$$\max_{j\in[d]} A_{ij} + x_j \leq \max_{j\in[d]} B_{ij} + x_j$$

is reduced to the $-\infty$ vector is equivalent to deciding whether an associated game has at least one winning position.

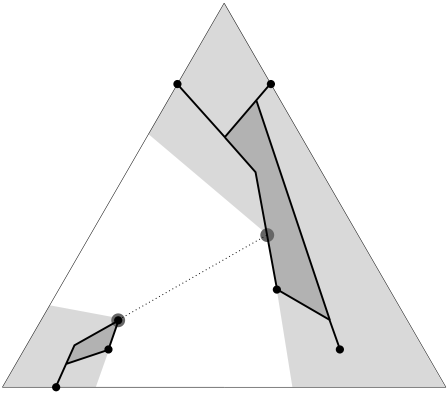
Akian, SG, Guterman, arXiv:0912.2462]

Tropical convex geometry works

- Separation
- projection
- minimisation of distance
- Choquet theory (generation by extreme points)
- discrete convexity: Radon, Helly, Caratheodory, Minkowski, colorful Caratheodory, Tverberg

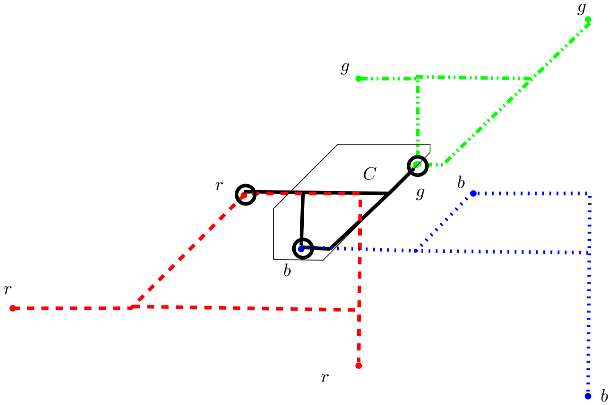
carry over !

Separation of two convex sets



SG & Sergeev arXiv:0706.3347 J. Math. Sci. (07)

Tropical version of Barany's Colorful Caratheodory Theorem



SG, Meunier, 09, arXiv:0804.1361 Disc. Comp. Geom, 2010

Tropical polyhedra can be efficiently handled

Allamigeon, SG, Goubault, arXiv:0904.3436, STACS'2010

They have often fewer (and cannot have more) extreme points as usual polyhedra

Allamigeon, SG, Katz, arXiv:0906.3492, to appear in JCTA

Tropical convexity has been applied. . .

- Discrete event systems Cohen, SG, Quadrat; more recently Katz
- Horoboundaries of metric spaces Walsh
- Static analysis (disjunctive invariants) Allamigeon, SG, Goubault SAS 08
- relations with tree metrics Develin, Sturmfels

Conclusion

Maxplus algebra

- is useful in applications (optimal control, discrete event systems, games)
- has proved to be a gold mine of counter examples and inspiration, for several other fields of mathematics (combinatorics, asymptotic analysis, geometry)
- is quite fun.

Some references available on http://minimal.inria.fr/gaubert/

Advertisement

A work with Bouhtou (Orange Labs) and Sagnol (CMAP/INRIA)

optimization of the measurement of internet trafic (placement of netflow) = experiment design + submodular functions + semidefinite and second order cone programming = scalable method (tropical algebra lies in the SDP cone like the "lièvre dans le paté").

We will be happy to discuss about this with any interested colleagues

FT Opentransit backbone (100 nodes, 267 links, 5591 OD pairs), deployment on 10 nodes, greedy (red) versus Second Order cone programming (circle)