The stochastic mass conserved Allen-Cahn equation with nonlinear diffusion

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We study the initial boundary value problem for the stochastic nonlocal Allen-Cahn equation :

$$(P) \quad \begin{cases} \partial_t u = div(A(\nabla u)) + f(u) - \dfrac{1}{|D|}\displaystyle\int_D f(u) + \dot{W}(x,t), & x \in D, \quad t \geq 0 \\ A(\nabla u).n = 0, & \text{on} \quad \partial D \times \mathbb{R}^+ \\ u(x,0) = u_0(x), & x \in D \end{cases}$$

where:

- D is an open bounded set of $\mathbb{R}^n$ with a smooth boundary ∂D;
- $f(s) = \displaystyle\sum_{j=0}^{2p-1} b_j s^j$ with $b_{2p-1} < 0, p \geq 2$;
- The operator A is Lipschitz continuous from $\mathbb{R}^n$ to $\mathbb{R}^n$, namely there exists a positive constant C such that

  $$|A(a) - A(b)| \leq C|a - b|$$

  for all $a, b \in \mathbb{R}^n$, and A is coercive, namely there exists a positive constant C_0 such that

  $$(A(a) - A(b))(a - b) \geq C_0(a - b)^2, \tag{1}$$

  for all $a, b \in \mathbb{R}^n$. Moreover we suppose that $A(0) = 0$.

  We remark that if A is the identity matrix, the nonlinear diffusion operator $-div(A(\nabla u))$ reduces to the linear operator $-\Delta u$.

- The function $W = W(x,t)$ is a Q-Brownian motion. More precisely, let Q be a nonnegative definite symmetric operator on $L^2(D)$ with $\mathrm{Tr}\, Q < +\infty$, $\{e_k\}_{k\geq 1}$ be an orthonormal basis in $L^2(D)$ diagonalizing Q, and $\{\lambda_k\}_{k\geq 1}$ be the corresponding eigenvalues, so that

  $$Qe_k = \lambda_k e_k, \qquad \text{for all } k \geq 1.$$

  Since Q is of trace-class,

  $$\mathrm{Tr}\, Q = \sum_{k=1}^{\infty} \langle Qe_k, e_k \rangle_{L^2(D)} = \sum_{k=1}^{\infty} \lambda_k \leq \Lambda_0. \tag{2}$$

  for some positive constant Λ_0. Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space equipped with a filtration $(\mathcal{F}_t)$ and $\{\beta_k(t)\}_{k\geq 1}$ be a sequence of independent $(\mathcal{F}_t)$-Brownian motions defined on $(\Omega, \mathcal{F}, \mathrm{P})$; the process W defined by

  $$W(x,t) = \sum_{k=1}^{\infty} \beta_k(t) Q^{\frac{1}{2}} e_k(x) = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\beta_k(t)e_k(x) \tag{3}$$

  is a Q-Wiener process in $L^2(D)$. We recall that a Brownian motion $\beta(t)$ is called an $(\mathcal{F}_t)$ Brownian motion if it is $(\mathcal{F}_t)$-adapted and the increment $\beta(t) - \beta(s)$ is independent of $\mathcal{F}_s$ for every $0 \leq s < t$.

The deterministic problem in the case of linear diffusion was first introduced by Rubinstein and Sternberg [4] as a model for the phase separation in a binary mixture, and the well-posedness and the stabilization of the solution for large times were proved by [2].

A singular limit of the stochastic Problem (P) with linear diffusion has been studied by Antonopoulou, Bates, Blömker and Karali [1] to model the motion of a droplet. In this talk, we prove the existence and uniqueness of the weak solution of Problem (P); this had remained as an open problem even in the linear diffusion case studied by [1].

The first step is to perform the change of unknown function $v(t) = u(t) - W_A(t)$ where W_A satisfies the problem

$$(P_1) \quad \begin{cases} \dfrac{\partial w}{\partial t} = div(A(\nabla w)) + \dot{W}(x,t), & x \in D, \quad t \geq 0, \\ A(\nabla w).n = 0, & \text{on} \quad \partial D \times \mathbb{R}^+ \\ w(x,0) = 0, & x \in D \end{cases}$$

We apply a Galerkin method, and search for a priori estimates which lead us to bound uniformly the approximate solution in $L^\infty(0,T;L^2(\Omega \times D))$, $L^2(\Omega \times (0,T);H^1(D))$ and $L^{2p}(\Omega \times (0,T) \times D)$. We deduce that the approximate solution v_m weakly converges along a subsequence to a limit $\bar{v}$ as $m \to \infty$. The main problem is then to identify the limit of the term $f(v_m + W)$ as $m \to \infty$, which we do by means of the monotonicity method [3]. We also prove the uniqueness of the weak solution which in turn implies the convergence of the whole sequence $\{v_m\}$.

Our next purpose is to extend this study to the case of a multiplicative noise.

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Références

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