Exponential approximation and enriched FEM for rods and Timoshenko beams

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This study proposes a new enriched finite element method to handle vibrations of rods (traction-compression) and beams (bending) with varying cross-sections. In time-harmonic domain, closed-form analytical solutions for Timoshenko beams exist only for exponential cross-sections [1], and we therefore focus on finite elements enriched with such solutions. More precisely, for a given beam or rod with varying section, we use the enrichment corresponding to an optimal approximation by an exponential-by-parts beam or rod. The best approximation is established using an energetic criterion. Thereafter, the proposed method is presented for a rod whose section is approximated by a unique exponential profile, and extensions are then briefly discussed.

Approximation by an exponential rod. Working with non-dimensional quantities, the longitudinal displacement $u(x)$ in a rod made of an homogeneous material obeys the wave equation $(\mathcal{A}u')' + \mathcal{A}\omega^2 u = f$, where $\mathcal{A}(x)$ is the cross-section, ω is the considered frequency and f represents distributed forces.
Given a rod with a given cross-section $\mathcal{A}$, we define its optimal exponential approximation as the rod of section $\mathcal{A}_\delta(x) = \mathcal{A}_0 e^{2\delta x}$ (where $\mathcal{A}_0 > 0$ and $\delta \in \mathbb{R}$) which has the closest total energy (kinetic plus elastic) under any kind of boundary excitations (i.e. for $f = 0$).
This criterion can be written in terms of (i) the solutions u corresponding to $\mathcal{A}$, and (ii) the solutions u_δ corresponding to the sought $\mathcal{A}_\delta$, which are of the form:

$$u_\delta(x) = e^{-\delta x}(c_+ e^{\mathrm{i}\tilde{k}x} + c_- e^{-\mathrm{i}\tilde{k}x}), \qquad \text{with } \tilde{k} = \sqrt{\omega^2 - \delta^2}. \tag{1}$$

Preliminary results are obtained for linear and quadratic rods, for which there also exist analytical solutions [3]. The optimality condition can therefore be expressed explicitly, resulting in semi-analytical determination of optimal $(\mathcal{A}_0, \delta)$. For an arbitrary cross-section $\mathcal{A}$, an analysis of the error $u - u_\delta$ must be conduced.

Enriched finite elements. To address the case $f \neq 0$, and foreseeing an extension to the time domain, a finite element discretization is studied. The variable $\tilde{u}$ defined by $\tilde{u}(x) = u(x)e^{\delta x}$ is used instead of u, where δ is determined as described above. Following e.g. [2], the considered finite element basis (e.g. $\mathbb{P}_1$ basis) is then enriched by the basis $\{e^{\pm\mathrm{i}\tilde{k}x}\}$ of solutions corresponding to the exponential approximation. For linear and quadratic rods, using a regular mesh of element length h, this enrichment brings significant improvements of the FE solution, reaching $O(h^4)$ convergence rate for enriched $\mathbb{P}_1$ basis (instead of the well-known $O(h^2)$ rate for $\mathbb{P}_1$ elements).

Extensions. To address rods with roughly varying cross-section, such rod is divided into subdomains, and the optimal approximation procedure and corresponding enrichment are then applied to each subdomain. Finally, the whole method is transposed to the Timoshenko model for beams.

Références

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