Parareal-Schwarz Waveform relaxation method for the heat equation

Duc Quang BUI , Caroline JAPHET and Pascal OMNES

03/2017

Parareal method was first presented in [3] as a numerical method to solve time-evolution problems in parallel. This method uses two propagators : the coarse - generated from a coarse time mesh or a low-order method, fast and inaccurate - and the fine - generated from a finer time mesh or a higher order method, slow but accurate. Instead of running the fine propagator on the whole time interval, we divide the latter into small time intervals, where we can run the fine propagator to obtain the desired solution, with the help of the coarse propagator and through parareal steps. Furthermore, each local subproblem can be solved by an iterative method, while solutions can be seen as the limits. Therefore, instead of doing the local iterative method until convergence, one may perform only a few iterations of it, during parareal iterations (see [4]). Propagators then become much cheaper but sharply lose their accuracy, and we hope that the convergence will be achieved across parareal iterations.

Here, we propose to couple both parareal and classical Schwarz Waveform Relaxation (SWR) methods, the latter being an iterative as well as a space-time parallel method. This coupled method, called Parareal-Schwarz Waveform Relaxation method, was first proprosed in [2] (with classical SWR) and in[1] (with Optimized SWR) and allows to solve the original problem in parallel, both for space and time. To our knowledge, the convergence through parareal iterations of such an approach, with incomplete OSWR iterations, has not been analyzed.

In this contribution, we present the analysis of the method for 1-dimensional heat equation. As the main cost of parareal lies in the fine propagator, we use a simple coarse propagator deduced from Backward Euler method, and with only few SWR iterations in the fine propagator. Convergence for this case is at least linear. We also give some numerical results for 1D and 2D heat equation, which shows that the convergence is much faster in practice.

[1] M. J. Gander, Y-L. Jiang and R-J. Li, *Parareal Schwarz Waveform Relaxation Methods*, Springer Berlin Heidelberg, Domain Decomposition Methods in Science and Engineering XX, Volume 91 of the series Lecture Notes in Computational Science and Engineering (2013), pp. 451-458.

[2] R. Guétat, *Parallel-in-time methods : Application to domain decomposition methods*,PhD thesis, UPMC & EPT, 2011.

[3] J.-L. Lions, Y. Maday, and G. Turinici, *A "parareal" in time discretization of PDE's*, C. R. Acad. Sci. Paris Ser. I Math., 332 (2001), pp. 661-668

[4] Y. Maday and G. Turinici, *The parareal in time iterative solver : A further direction to parallel implementation*, Springer Berlin, Proceedings of the 15th International Domain Decomposition Conference, Lect. Notes Comput. Sci. Eng. 40 (2003), pp. 441-448.