On an inverse Cauchy problem arising in tokamaks

Yannick Fischer

INRIA Sophia-Antipolis projet APICS

joint work with

L. Baratchart*, J. Leblond*

*INRIA (Sophia-Antipolis)

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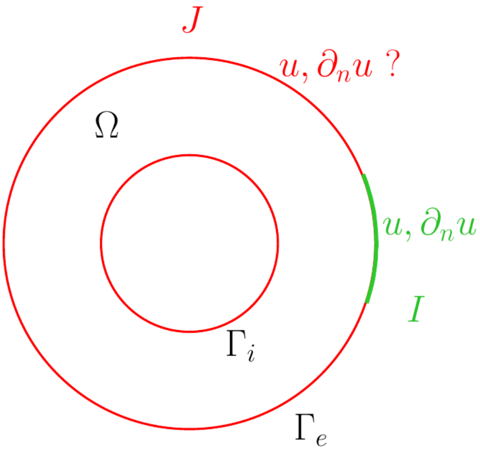
Cauchy problem

$\Omega \subset \mathbb{R}^2$: annular domain with smooth boundary $\partial\Omega = \Gamma_i \cup \Gamma_e$

σ : smooth function (Lipschitz) with $0 < c \leq \sigma \leq C$

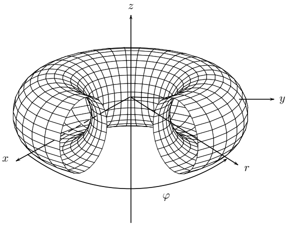


$\nabla \cdot (\sigma \nabla u) = 0$ a.e in Ω
with u and $\partial_n u$
prescribed on $I \subseteq \partial\Omega$

Can we recover u and
$\partial_n u$ on $J = \partial\Omega \setminus I$?

Application to tokamak

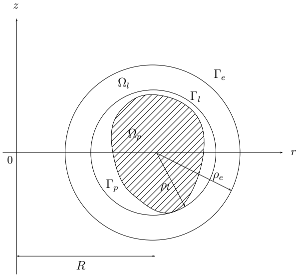
Physical motivation : application to Tokamak (Tore Supra)



Axisymmetric configuration (3D-problem)

$\Longrightarrow$

Study of the equilibrium in poloidal section (2D-problem)



$$\text{Maxwell equation in the vacuum } \Omega_l : \nabla.(\frac{1}{r}\nabla u) = 0$$

where $u(r, z)$ is the magnetic poloidal flux and $\sigma = \frac{1}{r}$ is regular in Ω_l. How to recover u and $\partial_n u$ on Γ_l from (finite) measurements on Γ_e ?

The conjugate Beltrami equation

Idea : Astala and Päivärinta (2006)

From real equation

$$\nabla \cdot (\sigma \nabla u) = 0 \quad \text{a.e in } \Omega \qquad (CD) \qquad \sigma \in W^{1,\infty}_{\mathbb{R}}(\Omega)$$

to complex equation (but $\mathbb{R}$-linear)

$$\overline{\partial} f = \nu \overline{\partial f} \quad \text{a.e in } \Omega \qquad (CB) \qquad \nu \in W^{1,\infty}_{\mathbb{R}}(\Omega)$$

Proposition

$$f = u + iv \in W^{1,2}(\Omega) \quad \text{satisfies (CB) with} \quad \nu = \tfrac{1-\sigma}{1+\sigma}$$

$$\Longrightarrow \nabla.(\sigma \nabla u) = \nabla.(\sigma^{-1} \nabla v) = 0 \quad \text{a.e in } \Omega \quad \text{and}$$

$$\begin{cases} \partial_x v &= -\sigma \partial_y u \\ \partial_y v &= \sigma \partial_x u \end{cases} \quad \text{a.e in } \Omega \quad (\text{or } \partial_t v = \sigma \partial_n u \quad \text{a.e on } \partial\Omega)$$

The conjugate Beltrami equation

Advantages :

- Symmetric roles played by u and v

$$\begin{array}{c}\text{Dirichlet} + \text{Neumann conditions for } u \\ \Downarrow \\ \text{Dirichlet condition for } u + \text{Dirichlet condition for } v = \int_{\partial\Omega} \sigma \partial_n u \\ \Downarrow \\ \text{ONLY Dirichlet conditions for } f\end{array}$$

- Allow regularization of the Cauchy problem for data in $L^2(\partial\Omega)$

Generalized Hardy classes

Definition

$H^2_\nu(\Omega)$ = Lebesgue measurable fonctions f on Ω such that

$$\|f\|_{H^2_\nu(\Omega)} := ess \sup_{\varrho<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta \right)^{1/p} < +\infty \qquad (1)$$

and solving (CB) in the sense of distributions in Ω

- $H^2_\nu(\Omega)$ is a Hilbert space. When $\nu = 0$ and $\Omega = \mathbb{D}$, recover the classical $H^2(\mathbb{D})$ space of holomorphic functions in $\mathbb{D}$ satisfying (1)
- $\|.\|_{H^2_\nu(\Omega)} \sim \|.\|_{L^2(\partial\Omega)}$
- $tr\ H^2_\nu(\Omega)$ is closed subspace of $L^2(\partial\Omega)$

Density result

Theorem

Let $I \subset \partial\Omega$ be a mesurable subset such that $|I|, |J| > 0$

$$tr\, H^2_\nu(\Omega)_{|_I} \text{ is dense in } L^2(I)$$

As $tr\, H^2_\nu(\Omega)$ is a closed subset of $L^2(\partial\Omega)$, if $(f_k)_{k\geq 1} \in H^2_\nu(\Omega)$ is such that

$$\|tr\, f_k - f\|_{L^2(I)} \underset{k}{\longrightarrow} 0,$$

there are only two possibilities :

$$f = (tr\, F)_{|_I} \text{ with F } \in H^2_\nu(\Omega) \quad \text{or} \quad \|tr\, f_k\|_{L^2(J)} \to +\infty$$

This leads to a bounded extremal problem.

Bounded extremal problems...

- If (u, v) are compatible data on $I \longrightarrow$ unique solution by extrapolation
- If (u, v) are **NOT** compatible data on I

  Idea : constrain solutions on J.

Definition

For $M > 0$ and $\varphi \in L^2_{\mathbb{R}}(J)$

$$\mathcal{B} = \left\{ f \in tr\, H^2_\nu(\Omega);\ \|f - \varphi\|_{L^2(J)} \leq M \right\}_{|_I} \subset L^2(I)$$

extrapolation problem $\longleftrightarrow$ well-posed L^2 approximation problem

...Bounded extremal problems

Then the approximation problem admits a unique solution

Theorem

Fix $M > 0$

$$\forall f \in L^2(I),\ \exists!\, g_0 \in \mathcal{B}\ /\ \|f - g_0\|_{L^2(I)} = \min_{g \in \mathcal{B}} \|f - g\|_{L^2(I)}$$

Moreover, if $f \notin \mathcal{B}$*, then* $\|g_0 - \varphi\|_{L^2(J)} = M$

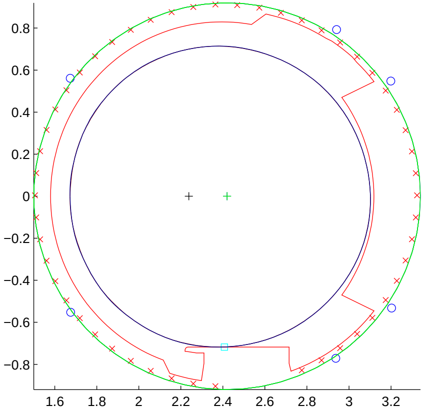
The solution g_0 is given by

$$g_0 = g_0(\lambda) = (I + \lambda \mathcal{P}_\nu \chi_J)^{-1} \mathcal{P}_\nu (\chi_I f \vee (1+\lambda)\varphi)$$

with $\lambda \in (-1, \infty)$.

Algorithm

plasma boundary = outermost closed magnetic surface in the limiter



1) $u = \sum_{i=1}^{N} \alpha_i b_i$ from u and $\partial_n u$ on Γ_{ext}

2) $u_0 = \max u$ on the limiter and $\Gamma^1_{int} = \{(x,y); u(x,y) = u_0\}$

$$\partial\Omega^1 = \Gamma_{ext} \cup \Gamma^1_{int}$$

3) BEP in Ω^1

$\Rightarrow g_0 = \min \|f - g\|$ with $\|Re\ g_0 - u_0\| = M$

4) $u_1 = \max Re\ g_0$ on the limiter and $\Gamma^2_{int} = \{(x,y); Re\ g_0 = u_1\}$

etc...

Simulations

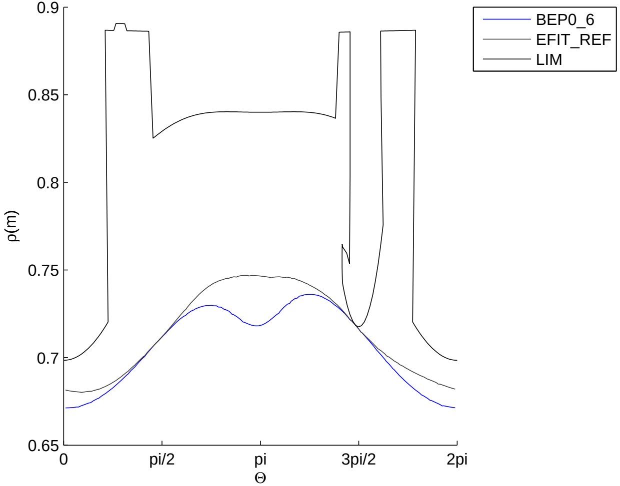
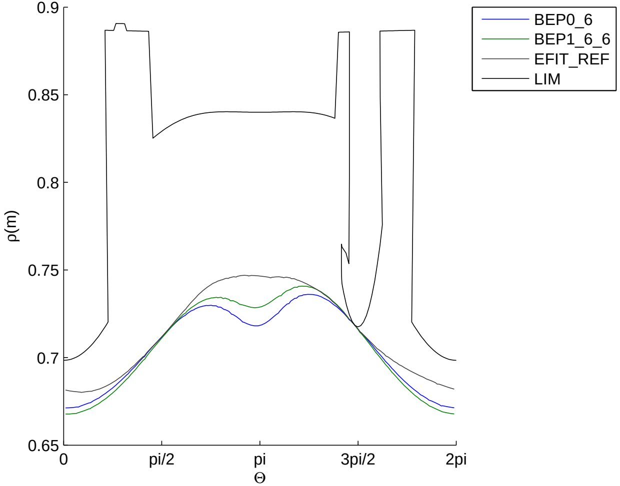


FIG.: Graphe de $\theta \mapsto \rho(\theta)$ pour les frontières plasma et le limiteur.

Simulations



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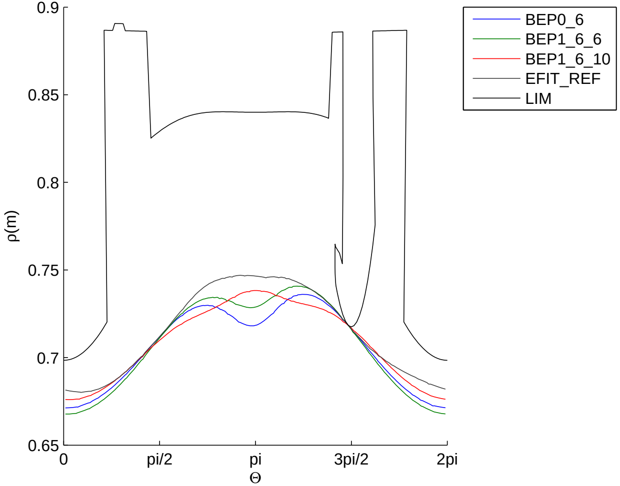

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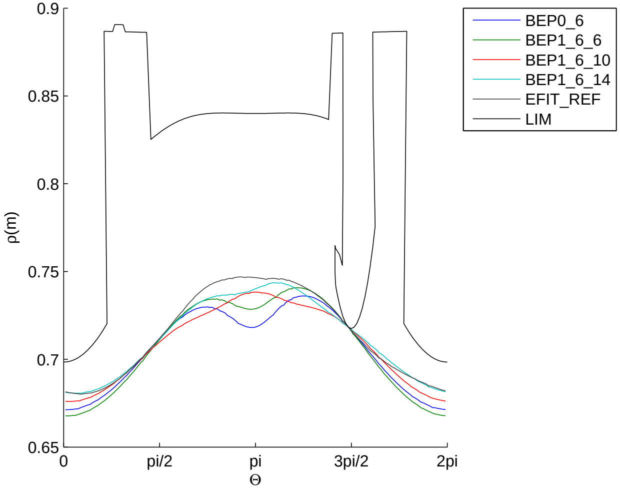

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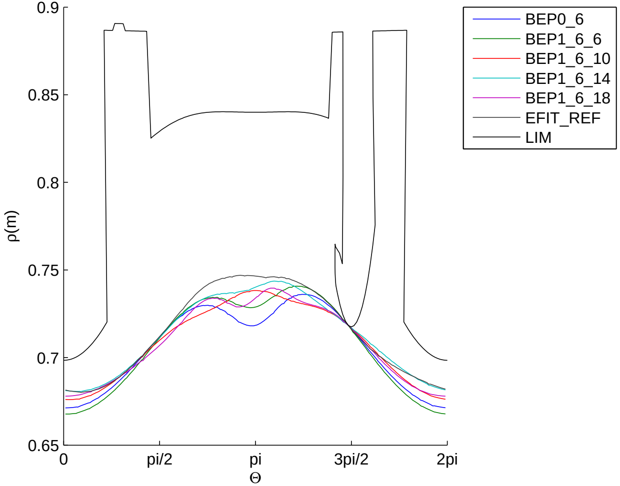


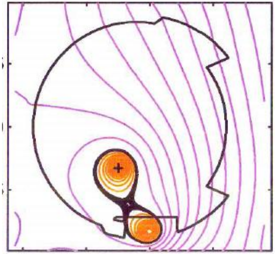
FIG.: Graphe de $\theta \mapsto \rho(\theta)$ pour les frontières plasma et le limiteur.

Conclusion

- Fast method (no mesh)
- compact representation with solutions of the equation
- evaluation of the poloidal flux between the plasma and the outside boundary

Work in progress :

- Optimization of the Lagrange parameter λ
- pathological cases



Thank you for your attention