Méthode Eléments Finis multiéchelles pour les matériaux faiblement aléatoires

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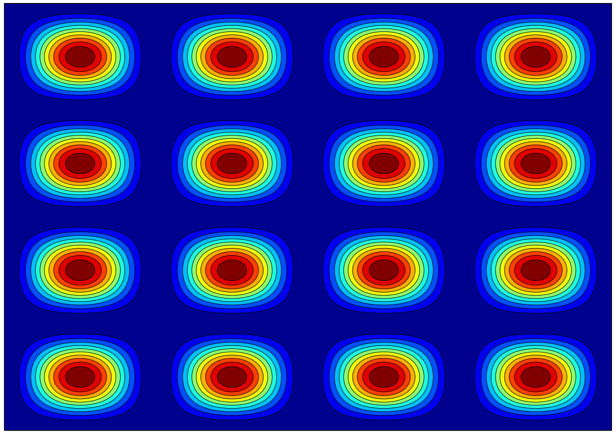
Congrès SMAI, Guidel

Motivation



- heterogeneous materials (*e.g.* composite materials).
- we consider the stationary heat equation: $-\text{div}\,[A\nabla u] = f$, where A is the conductivity matrix, u is the temperature and f is the source term.
- we want an efficient numerical method that provides accurate approximation of the temperature and its gradient.
- Difficulty: A varies at a small scale.

Multiscale FEM: the deterministic setting

$$-\text{div}\left[A^\varepsilon \nabla u^\varepsilon\right] = f \quad \text{in} \quad \mathcal{D}, \qquad u^\varepsilon \in H^1_0(\mathcal{D}),$$

where the matrix A^ε is symmetric, satisfies the standard coercivity and boundedness conditions and varies at the scale ε (*e.g.* $A^\varepsilon(x) = A\left(\frac{x}{\varepsilon}\right)$).

The variational formulation reads:

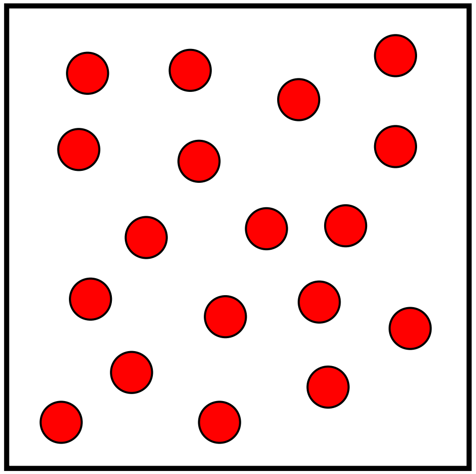
$$\text{Find } u^\varepsilon \in H^1_0(\mathcal{D}) \text{ such that,} \quad \forall v \in H^1_0(\mathcal{D}), \ \ \mathcal{A}_\varepsilon(u^\varepsilon, v) = b(v),$$

where

$$\mathcal{A}_\varepsilon(u, v) = \int_{\mathcal{D}} (\nabla v)^T A^\varepsilon \nabla u \quad \text{and} \quad b(v) = \int_{\mathcal{D}} f\, v.$$

The MsFEM approach: variational approximation where the basis functions are defined numerically and encode the fast oscillations. (see [Efendiev and Hou, 2009])

A three step method



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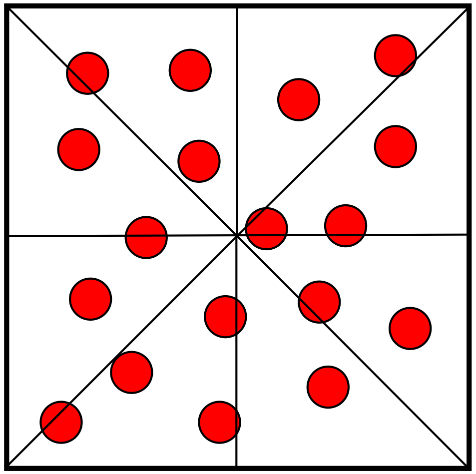
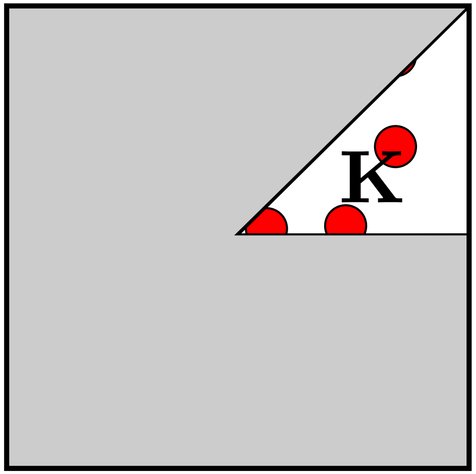
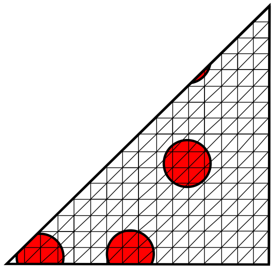


- Coarse mesh with a $\mathbf{P}1$ Finite Element basis $\phi_i^{0,\mathbf{K}}$.

A three step method



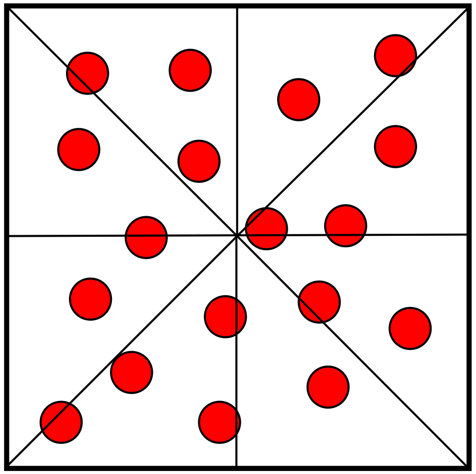
- Coarse mesh with a $\mathbf{P}1$ Finite Element basis $\phi_i^{0,\mathbf{K}}$.
- MsFEM basis



$$\begin{cases} -\text{div}(A^\varepsilon \nabla \phi_i^{\varepsilon,\mathbf{K}}) = 0 & \text{in } \mathbf{K} \\ \phi_i^{\varepsilon,\mathbf{K}} = \phi_i^{0,\mathbf{K}} & \text{on } \partial\mathbf{K} \end{cases}$$

The $\phi_i^{\varepsilon,\mathbf{K}}$ are computed independently (in parallel) over each $\mathbf{K}$.

A three step method



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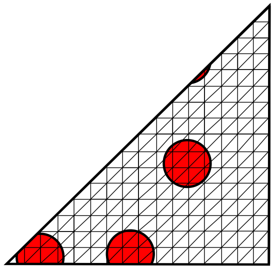

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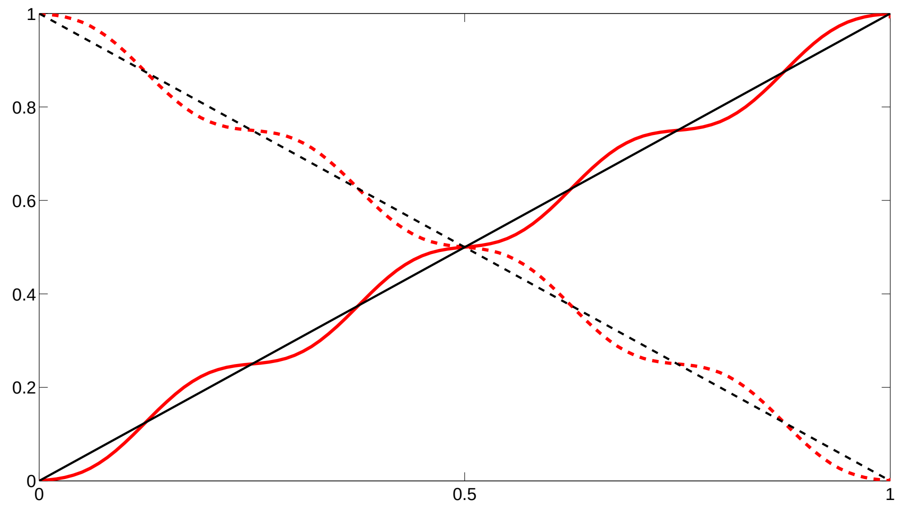
  The $\phi_i^{\varepsilon,\mathbf{K}}$ are computed independently (in parallel) over each $\mathbf{K}$.
- Galerkin approximation of the original problem with MsFEM basis $\phi_i^{\varepsilon,\mathbf{K}}$.

Numerical illustration



FEM Basis vs MsFEM basis

The MsFEM method is accurate even with a coarse mesh, because the basis functions encode the specific fast oscillations of the problem.

Natural adaptation to the stochastic setting

$$-\text{div}\left[A^{\varepsilon}\left(x,\omega\right)\nabla u^{\varepsilon}(x,\omega)\right]=f(x)\quad \text{in}\quad \mathcal{D},\qquad u^{\varepsilon}\in H_0^1(\mathcal{D}),$$

and assume that we wish to build an estimate of the mean $\mathbb{E}(u^{\varepsilon}(x,\cdot))$ using a Monte-Carlo simulation method.

Then, for each realization of $A^{\varepsilon,m}(x,\omega)$,

- first construct a (random) MsFEM basis $\phi_i^{\varepsilon,m}(x,\omega)$
- and next compute the Galerkin approximation $u_m^h(x,\omega)$

and approximate $\mathbb{E}(u^{\varepsilon}(x,\cdot)) \sim \frac{1}{M}\sum_{m=1}^{M} u_m^h(x,\omega)$.

This is extremly expensive.

Natural adaptation to the stochastic setting

$$-\text{div}\left[A^\varepsilon(x,\omega)\,\nabla u^\varepsilon(x,\omega)\right] = f(x) \quad \text{in} \quad \mathcal{D}, \qquad u^\varepsilon \in H^1_0(\mathcal{D}),$$

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This is extremly expensive.

Look for a specific setting, relevant from the application viewpoint, and where more affordable methods can be used.

Outline

- A weakly stochastic setting
- Numerical results
- Error bounds

A weakly stochastic setting

Model: we now consider the problem

$$-\text{div}\left[A_\eta^\varepsilon(x,\omega)\,\nabla u^\varepsilon(x,\omega)\right] = f(x) \quad \text{in} \quad \mathcal{D}, \qquad u^\varepsilon \in H_0^1(\mathcal{D})$$

with

$$A_\eta^\varepsilon(x,\omega) = A_0^\varepsilon(x) + \eta A_1^\varepsilon(x,\omega)$$

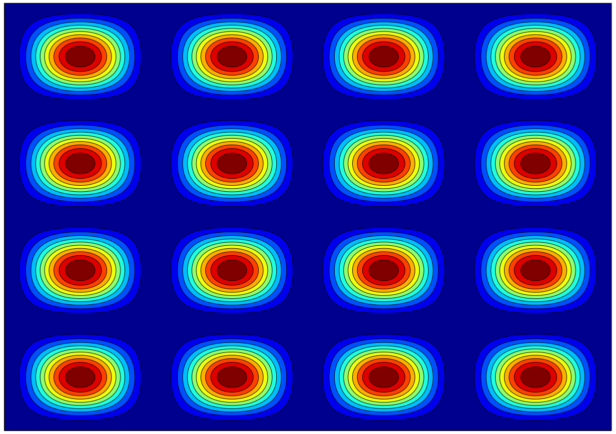
where $\eta \in \mathbb{R}$ is a small parameter, uniquely determined by $\left\| \frac{A_1^\varepsilon}{A_0^\varepsilon} \right\|_{L^\infty} = 1$.

- A_0^ε is a deterministic matrix uniformly elliptic;
- $A_1^\varepsilon(\cdot,\omega)$ is a bounded random matrix.

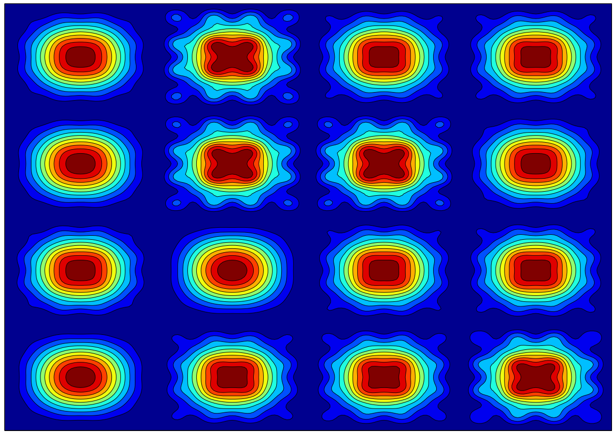
A_η^ε is a perturbation of the deterministic matrix A_0^ε.

For a review on various weakly stochastic settings see [C. Le Bris, Enumath 2009]

An example of a perturbation of a periodic material:



Deterministic matrix A_0^ε.



Stochastic perturbation A_η^ε, with $\eta = 0.1$.

A weakly stochastic MsFEM

$$A_\eta^\varepsilon(x,\omega) = A_0^\varepsilon(x) + \eta A_1^\varepsilon(x,\omega)$$

Adaptation of the MsFEM:

- Compute MsFEM basis only once, with the deterministic matrix A_0^ε;
- Perform Monte-Carlo realizations at the macro-scale level.

MsFEM basis:

$$-\text{div}\left[\mathbf{A}_\mathbf{0}^\varepsilon \nabla \phi_i^{\varepsilon,\mathbf{K}}\right] = 0 \quad \text{in} \quad \mathbf{K}, \qquad \phi_i^{\varepsilon,\mathbf{K}} = \phi_i^{0,\mathbf{K}} \quad \text{on} \quad \partial\mathbf{K}$$

build the finite dimentional space: $\mathcal{W}_h :=$ span $\{\phi_i^\varepsilon,\ i = 1,\ldots,L\} \subset H_0^1(\mathcal{D})$.

Macro scale problem: for each realization m compute $u_m^{ws,h} \in \mathcal{W}_h$ such that

$$\forall v \in \mathcal{W}_h, \quad \int_\mathcal{D} (\nabla v)^T \mathbf{A}_\eta^{\varepsilon,\mathbf{m}}(\cdot,\omega) \nabla u_m^{ws,h}(\cdot,\omega) = \int_\mathcal{D} f\, v.$$

Note that we have used different conductivity matrices for the MsFEM basis function problem and the macro scale problem

Numerical results

We compute:

- Reference solution u^ε (fine mesh FEM solution);
- Standard MsFEM u^h;
- Weakly stochastic MsFEM $u^{ws,h}$.

Our error estimator is

$$e(u_1, u_2) = \mathbb{E}\left(\frac{||u_1 - u_2||_{H^1}}{||u_2||_{H^1}}\right)$$

We estimate the expectation with an empirical mean μ_M. The Central Limit Theorem yields

$$|\mathbb{E}(X) - \mu_M(X)| \leq 1.96\frac{\sigma_M}{\sqrt{M}},$$

where σ_M denotes the empirical standard deviation.

A classical test case

$$
\begin{aligned}
A_\eta(x,y,\omega) \quad = \quad & \sum_{(k,l)\in\mathbb{Z}^2} \mathbf{1}_{(k,k+1]}(x)\mathbf{1}_{(l,l+1]}(y) \left(\frac{2+1.8\sin(2\pi x)}{2+1.8\sin(2\pi y)} \right. \\
& \left. + \frac{2+\sin(2\pi y)}{2+1.8\sin(2\pi x)} \right) (1+\eta X_{k,l}(\omega))\,\mathrm{Id}_2,
\end{aligned}
$$

where $X_{k,l}(\omega)$ are i.i.d. scalar random variables uniformly distributed over $[0,1]$.

- We compute u^ε solution to

$$
-\text{div}\left[A_\eta\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon},\omega\right)\nabla u^\varepsilon\right] = 1 \quad \text{in} \quad \mathcal{D}, \qquad u^\varepsilon \in H^1_0(\mathcal{D})
$$

  in the domain $\mathcal{D}=(0,1)^2$ with $\varepsilon = 0.025$.

- We next proceed with the two MsFEM methods, using a coarse mesh size $H=1/30$ and a fine mesh $h=\varepsilon/80$. We consider $M=30$ realizations.

Error (u^h: General MsFEM; $u^{ws,h}$: weakly-stochastic MsFEM)

$H^1(\mathcal{D})$ relative error (in $\%$).

η	$e(u^h, u^\varepsilon)$	$e(u^{ws,h}, u^\varepsilon)$	$e(u^{ws,h}, u^h)$
1	8.12 ± 0.19	17.37 ± 0.78	15.51 ± 0.87
0.1	7.17 ± 0.02	7.62 ± 0.07	2.56 ± 0.10
0.01	7.15 ± 0.002	7.28 ± 0.007	1.39 ± 0.002

- provided that η is small (here $\eta \leq 0.1$), $u^{ws,h}$ is an approximation of u^ε as accurate as u^h
- a large computational gain, of order M, is observed, as the MsFEM basis function is only computed once.
- same conclusion if we look at the energy defined by

$$E = \mathbb{E}\left(\int_{\mathcal{D}} A_\eta^\varepsilon(x,\cdot)\nabla u(x,\cdot) \cdot \nabla u(x,\cdot) dx \right)$$

Error bounds: the deterministic case ($\eta = 0$)

$$\text{Assume} \quad A^{\varepsilon}_{\eta=0}(x) := A_{\mathrm{per}}\left(\frac{x}{\varepsilon}\right)$$

where A_{per} is a deterministic periodic matrix.

Theorem 1: *Consider u^h, solution of MsFEM problem. There exists C, independent of h and ε such that*

$$\begin{aligned} \|u^{\varepsilon} - u^h\|_{H^1(\mathcal{D})} &\leq C\left(h + \frac{\varepsilon}{h} + \sqrt{\varepsilon}\right). \\ \|u^{\varepsilon} - u^h\|_{L^2(\mathcal{D})} &\leq C\left(h^2 + \frac{\varepsilon}{h} + \varepsilon\right). \end{aligned}$$

The two main arguments of the proof are

- Homogenization result
- Interpolation / Finite Element estimate

Recall that $\mathcal{W}_h :=$ span $\{\phi_i^\varepsilon,\ i = 1, \ldots, L\}$.

$$\begin{aligned} \|u^\varepsilon - u^h\|_{H^1(\mathcal{D})} &\leq \inf_{v_h \in \mathcal{W}_h} \|u^\varepsilon - v^h\|_{H^1(\mathcal{D})} \\ &\leq \|u^\varepsilon - \widetilde{u}^\varepsilon\|_{H^1(\mathcal{D})} + \inf_{v_h \in \mathcal{W}_h} \|\widetilde{u}^\varepsilon - v^h\|_{H^1(\mathcal{D})} \end{aligned}$$

where $\widetilde{u}^\varepsilon$ is the two-scale expansion of u^ε (homogenization setting).

- Bound on the first term: periodic homogenization result

$$\|u^\varepsilon - \widetilde{u}^\varepsilon\|_{H^1(\mathcal{D})} \leq C\sqrt{\varepsilon}$$

- Bound on the second term: (standard) interpolation result.

Error bounds: the weakly stochastic case ($\eta \neq 0$)

Assume $\quad A^{\varepsilon}_{\eta}(x,\omega) := A_{\mathrm{per}}\left(\frac{x}{\varepsilon}\right) + \eta \sum_{k\in\mathbb{Z}^2} \mathbf{1}_{Q+k}\left(\frac{x}{\varepsilon}\right) B_{\mathrm{per}}\left(\frac{x}{\varepsilon}\right) X_k(\omega)$

where A_{per} and B_{per} are deterministic periodic matrices and $(X_k(\omega))_{k\in\mathbb{Z}^2}$ is a sequence of i.i.d. scalar random variables.

Theorem 2: *Consider $u^{ws,h}$, solution of the weak stochastic MsFEM problem. There exists C, deterministic, independent of h, ε and η such that*

$$\sqrt{\mathbb{E}\left(\|u^{\varepsilon} - u^{ws,h}(\cdot,\omega)\|^2_{H^1(\mathcal{D})}\right)} \leq C\left(h + \frac{\varepsilon}{h} + \sqrt{\varepsilon} + \eta\right)$$

$$\sqrt{\mathbb{E}\left(\|u^{\varepsilon} - u^{ws,h}(\cdot,\omega)\|^2_{L^2(\mathcal{D})}\right)} \leq C\left(h^2 + \frac{\varepsilon}{h} + \varepsilon + \eta\frac{\varepsilon}{h}\ln(h^{-1}) + \eta^2\right)$$

If $\eta = 0$, we recover the standard MsFEM bound.

Conclusions

The approach has been

- tested in several 2D situations.
- is as accurate as the standard MsFEM provided that η is small.
- a computational gain of order M, the number of realizations, is observed.

A. Anantharaman, R. Costaouec, C. Le Bris, F. Legoll and FT, *Introduction to numerical stochastic homogenization and the related computational challenges: some recent developments*, Lecture Notes Series, Institute for Mathematical Sciences, National University of Singapore, in press.

C. Le Bris, F. Legoll and FT, *Multiscale FEM for weakly random problems and related issues*, in preparation.

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