Multilayer approximation of the Navier-Stokes system

Beyond the Saint-Venant system

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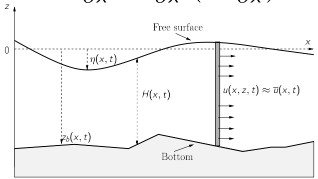
Shallow Water approximation of Navier-Stokes

$$(NS)\begin{cases} \operatorname{div}\underline{\mathbf{u}} = 0, \\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)\underline{\mathbf{u}} + \nabla p = \mathbf{G} + \operatorname{div}\underline{\underline{\Sigma}}, \end{cases}$$

- small parameter $\varepsilon = \frac{H_0}{L_0}$, expansion in $\mathcal{O}(\varepsilon^2)$
- Saint-Venant 1872, Gerbeau 2001, Saleri 2004, Marche 2007
- 2 assumptions : hydrostatic, $u = \bar{u}$

$$(SV)\begin{cases} \frac{\partial H}{\partial t} + \frac{\partial(H\bar{u})}{\partial x} = 0, \\ \frac{\partial(H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} + \frac{\partial}{\partial x}\left(4\mu\frac{\partial \bar{u}}{\partial x}\right) - \kappa(\bar{u}) \end{cases}$$

- Reduced complexity
- Many applications
- Hyperbolic CL



NOTATIONS: free surface η, bottom z_b, water height $H = \eta - z_b$,
velocities $\underline{\mathbf{u}} = (u, w)$, $\bar{u}(x,t) = \frac{1}{H}\int_{z_b}^{\eta} u(x,z,t)dz$

Navier-Stokes vs. Saint-Venant

2D Navier-Stokes $H, \mathbf{u} = (u, w)$

$$\begin{cases} \operatorname{div} \underline{\mathbf{u}} = 0, \\ \dot{\underline{\mathbf{u}}} + (\underline{\mathbf{u}}.\nabla)\underline{\mathbf{u}} + \nabla p = \mathbf{G} + \operatorname{div} \underline{\underline{\Sigma}}, \end{cases}$$

Saint-Venant $H, \bar{u}$

$$\begin{cases} \frac{\partial H}{\partial t} + \frac{\partial (H\bar{u})}{\partial x} = 0, \\ \frac{\partial (H\bar{u})}{\partial t} + \frac{\partial}{\partial x}\left(H\bar{u}^2 + \frac{g}{2}H^2\right) = -gH\frac{\partial z_b}{\partial x} - \kappa(\bar{u}) \end{cases}$$

Complexity of Navier-Stokes

- functional analysis
- numerical schemes
- geometrical models
- computational costs and use

Advantages of Saint-Venant

- fixed meshes
- reduced computational costs
- efficient numerical methods (finite volumes)

Main ideas & Outline

- The Saint-Venant system is widely used
  but too simple for complex flows
  Various approximations of Navier-Stokes
  with a Saint-Venant-like approach

- Importance of the discretization, especially for the source terms
  Advantages of the kinetic description

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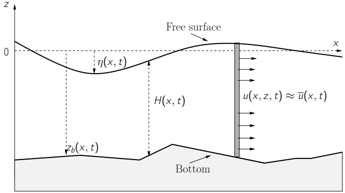
1.1 Beyond the Saint-Venant system

Objective



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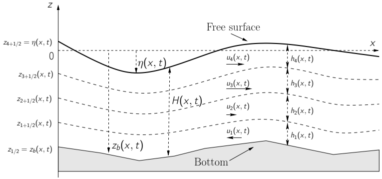


- Many contributions (Audusse, Bouchut, LeVeque, Pares, . . .)
- Valid for non-miscible fluids
- Pb. with underlying physics,. . .

Key idea

Saint-Venant ⇒ Multilayer Saint-Venant

$u(x,z,t) \approx \bar{u}(x,t)$ ⇒ $u(x,z,t) \approx \sum_{\alpha=1}^{N} \mathbb{I}_{z\in L_\alpha(x,t)} u_\alpha(x,t)$

1.1 Model derivation

Starting point

$$\text{(Euler hydro)} \quad \begin{cases} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial p}{\partial z} = -g \end{cases}$$

Obtained model

- Weak form ($\mathbb{P}_0^t$) of the continuity equation
  with $\mathbb{P}_0^t = \{\mathbb{I}_{z\in L_\alpha(x,t)},\ 1 \leq \alpha \leq N\}$

$$\frac{\partial h_\alpha}{\partial t} + \frac{\partial h_\alpha u_\alpha}{\partial x} = G_{\alpha+1/2} - G_{\alpha-1/2}$$

$$G_{\alpha+1/2} = \frac{\partial z_{\alpha+1/2}}{\partial t} + u_{\alpha+1/2}\frac{\partial z_{\alpha+1/2}}{\partial x} - w_{\alpha+1/2}$$

$G_{1/2} = G_{N+1/2} = 0$ (kinematic boundary conditions)

1.1 Model derivation

Starting point

$$\text{(Euler hydro)} \quad \begin{cases} \frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \\ \frac{\partial u}{\partial t} + \frac{\partial u^2}{\partial x} + \frac{\partial uw}{\partial z} + \frac{\partial p}{\partial x} = 0 \\ \frac{\partial p}{\partial z} = -g \end{cases}$$

Obtained model

- Weak form $(\mathbb{P}_0^t)$ of the Euler system $\mathbb{P}_0^t = \{\mathbb{I}_{z \in L_\alpha(x,t)}, \ 1 \leq \alpha \leq N\}$

$$\begin{cases} \frac{\partial H}{\partial t} + \sum_{\alpha=1}^{N} \frac{\partial}{\partial x}(h_\alpha u_\alpha) = 0 \\ \frac{\partial (h_\alpha u_\alpha)}{\partial t} + \frac{\partial}{\partial x}\left(h_\alpha u_\alpha^2 + \frac{g}{2} h_\alpha f(\{h_j\}_{j \geq \alpha})\right) = F_{\alpha+1/2} - F_{\alpha-1/2} \\ \frac{\partial E_\alpha}{\partial t} + \frac{\partial}{\partial x}\left(u_\alpha \left(E_\alpha + \frac{g}{2} h_\alpha H\right)\right) = E_{\alpha+1/2} - E_{\alpha-1/2} \end{cases}$$

- Only one "global" continuity equation, $H = \sum h_\alpha$
- Exchange terms $G_{\alpha+1/2}$, $F_{\alpha+1/2} = u_{\alpha+1/2} G_{\alpha+1/2} + P_{\alpha+1/2}$,

$$E_{\alpha+1/2} = \frac{u_{\alpha+1/2}^2}{2} G_{\alpha+1/2} + u_{\alpha+1/2} P_{\alpha+1/2}$$

- If $G_{\alpha+1/2} \equiv 0$ non-miscible fluids, N cont. equations

1.2 Properties of the model

- **Hyperbolicity ?**
  quasi-linear form $X = (H, u_1, \ldots, u_N, E_1, \ldots, E_N)^T$

$$M(X)\dot{X} + F(X)\frac{\partial X}{\partial x} = 0$$

  - "often" hyperbolic
  - strictly hyperbolic for $N = 2$
  - for $N > 2$, "arrow matrices" and interlacing of eigenvalues

$$\frac{1}{N}\sum_{1}^{N} u_i^2 \leq gH \qquad (generalized\ Froude\ number)$$

  - family of entropies

- More complex approximation

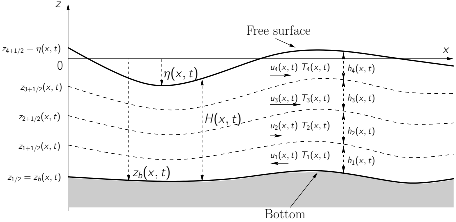
$$u(x,z,t) \approx \sum_{\alpha=1}^{N} P_\alpha(z)\, u_\alpha(x,t), \quad \text{with} \quad deg(P_\alpha) = k$$

Hydrostatic Euler (NS) with varying density (Aud., Brist., JSM, JCP 2010)

Starting point

$$\left\{\begin{array}{l}\dot{\rho} + \mathrm{div}\,(\rho\underline{\mathbf{u}}) = 0,\\ \dot{\overline{\rho\underline{\mathbf{u}}}} + (\underline{\mathbf{u}}.\nabla)(\rho\underline{\mathbf{u}}) + \nabla p = \rho\mathbf{G},\\ \dot{\overline{\rho T}} + \mathrm{div}\,(\rho T\underline{\mathbf{u}}) = \mu_T \Delta T,\end{array}\right.$$

with $\rho = \rho(\{T, S\})$ $\quad(= \rho(T, S, H))$



Obtained model

$$u(x,z,t) \approx \sum_{\alpha=1}^{N} \mathbb{I}_{z\in L_\alpha} u_\alpha(x,t)$$

$$\left\{\begin{array}{l}\frac{\partial}{\partial t}\sum_{\alpha=1}^{N}(\rho_\alpha h_\alpha) + \sum_{\alpha=1}^{N}\frac{\partial}{\partial x}(\rho_\alpha h_\alpha u_\alpha) = 0,\\ \frac{\partial(\rho_\alpha h_\alpha u_\alpha)}{\partial t} + \frac{\partial}{\partial x}\left(\rho_\alpha h_\alpha u_\alpha^2 + \frac{g}{2}h_\alpha f(\{\rho_j h_j\}_{j\geq\alpha})\right) =\\ \qquad\qquad +u_{\alpha+1/2}G_{\alpha+1/2} - u_{\alpha-1/2}G_{\alpha-1/2} + \textit{Sc. Terms},\\ \frac{\partial(\rho_\alpha h_\alpha T_\alpha)}{\partial t} + \frac{\partial}{\partial x}(\rho_\alpha h_\alpha u_\alpha T_\alpha) = T_{\alpha+1/2}G_{\alpha+1/2} - T_{\alpha-1/2}G_{\alpha-1/2},\end{array}\right.$$

What to do with such models ?

- A family of models between Saint-Venant and Navier-Stokes
- Also other models (non-hydrostatic Saint-Venant system, section-averaged)
- Good candidates
  - rigourous derivation process
  - energy balance, entropies
- Simpler than the corresponding Navier-Stokes system
  - independant of z, **fixed meshes**
- But not so simple to analyse & discretize !

Kinetic approach for conservation laws

- A fantastic tool for
  - physical understanding
  - mathematical analysis
  - numerical analysis and schemes
- Basis : adopt a microscopic description (Boltzmann)

$$\textit{Cont. model} \quad \Leftrightarrow \quad \frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} - g \frac{\partial z_b}{\partial x} \frac{\partial M}{\partial \xi} = Q(x,t,\xi)$$

  - $M(x,t,\xi)$ particle density, $Q(x,t,\xi)$ collision term ($= 0$ a.e.)
  - $\int_{\mathbb{R}} \xi^p M \, d\xi$ gives the macroscopic variables
  - linear transport equation + Vlasoz
- Only kinetic representations and not kinetic formulations

Kinetic representation of the Saint-Venant system

- Gibbs equilibrium $M(x,t,\xi) = \frac{H}{c}\chi\left(\frac{\xi-\bar{u}}{c}\right)$ with $c = \sqrt{gH/2}$

where $\chi(\omega) = \chi(-\omega) \geq 0$, $\text{supp}\,(\chi) \subset \Omega$, $\int_{\mathbb{R}} \chi(\omega) = \int_{\mathbb{R}} \omega^2 \chi(\omega) = 1$

Proposition (Audusse, Bristeau, Perthame 2004)

The functions $(H, \bar{u}, E)(t,x)$ are strong solutions of the Saint-Venant system if and only if $M(x,t,\xi)$ is solution of the kinetic equation

$$(\mathcal{B}), \qquad \frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} - g\frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi} = Q(x,t,\xi)$$

where $Q(t,x,\xi)$ is a "collision term".

- Macroscopic variables $(H, \bar{u}, E) = \int_{\mathbb{R}} (1, \xi, \xi^2/2) M \, d\xi$
- A linear transport equation . . . easy to upwind

Kinetic interpretation for the multilayer system

- Gibbs equilibria
  - $M_\alpha(x,t,\xi) = \frac{h_\alpha}{c_\alpha}\chi\left(\frac{\xi-u_\alpha}{c_\alpha}\right)$, with $c_\alpha = \sqrt{gf(\{h_j\}_{j\geq\alpha})}$
  - $N_{\alpha+1/2}(x,t,\xi) = G_{\alpha+1/2}\,\delta\left(\xi - u_{\alpha+1/2}\right)$,

Proposition (Audusse, Bristeau, Perthame, JSM 2009)

The functions $(h_\alpha, u_\alpha, E_\alpha)(t,x)$ are strong solutions of the multilayer Saint-Venant system if and only if the set $\{M_j(x,t,\xi)\}_{j=1}^N$ is solution of the kinetic equations

$$\frac{\partial M_\alpha}{\partial t} + \xi\frac{\partial M_\alpha}{\partial x} - N_{\alpha+1/2} + N_{\alpha-1/2} = Q_\alpha(x,t,\xi)$$

- source terms (exchanges, pressure)
- also for variable density case

Main properties of the schemes

- $f_i^{n+1}(\xi) = M_i^n(\xi) - \xi\sigma_i^n \left(M_{i+1/2}^n(\xi) - M_{i-1/2}^n(\xi) \right) + \Delta t^n S_i^n(\xi)$
- Positive schemes (CFL=1 but more complex)
- 2^{nd} order schemes (space & time)
- Well balanced (with hydrostatic reconstruction)
- (Discrete entropy)
- Maximum principle (tracer)
- The source terms (pressure)

$$gH\frac{\partial z_b}{\partial x} \quad \Rightarrow \quad g\frac{\partial z_b}{\partial x}\frac{\partial M}{\partial \xi}$$

  - kinetic interpretation useful for discretization
- Numerical cost $\mathcal{O}(N \times 1)$, possibly mesh adaptation

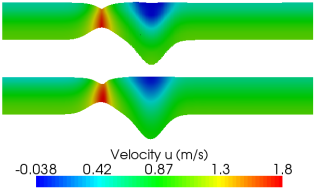
Analytical validation (I) (Boulanger, JSM 2011)

- Analytical solutions to the Euler system
  - 2D and 3D solutions, for any bottom topography $z_b(x, y)$
  - with entropic shocks
  - limit of Navier-Stokes solutions at vanishing viscosity
  - not necessarily free surface flows
  - In 2D (continuous solutions), u and H characterized by

$$u = \alpha\beta \frac{\cos\beta(z - z_b)}{\sin\beta H}, \quad \left[\!\left[\frac{gH^2}{2} + \frac{\alpha^2\beta^2 H}{2\sin(\beta H)^2} + \frac{\alpha^2\beta\cos(\beta H)}{2\sin(\beta H)} \right]\!\right] = 0$$

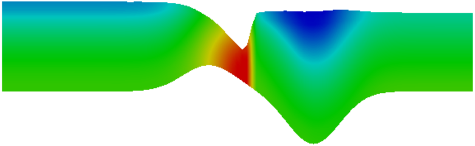
- Recovered by the 2D and 3D codes

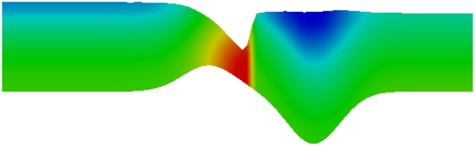
  - analytic
  - simulated



Analytical validation (II)

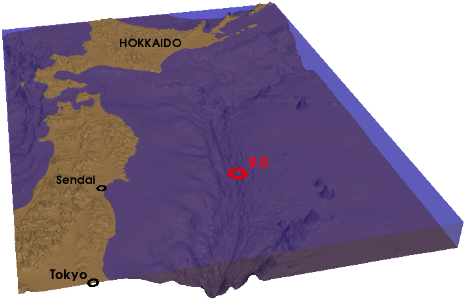
- With shocks

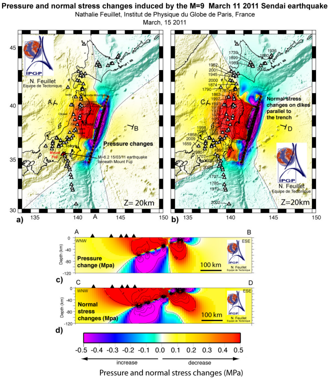




- Also without free surface

Honshu seism

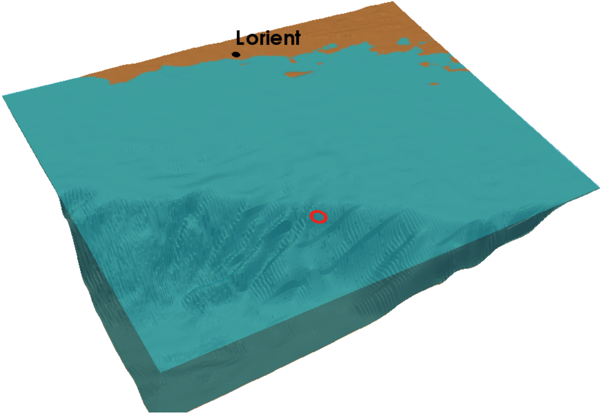




seism

source IPGP (A. Mangeney)

The next tsunami : Guidel 6.0 !



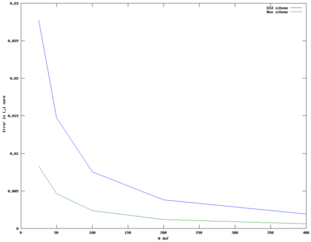
seism

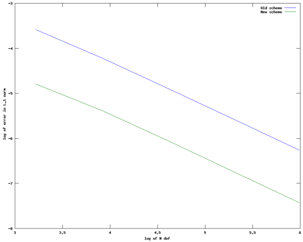
New scheme without hydrostatic reconstruction

- Well balanced scheme using discretization of the Vlasov term

$$\frac{\partial M}{\partial t} + \xi \frac{\partial M}{\partial x} - g \frac{\partial z_b}{\partial x} \frac{\partial M}{\partial \xi} = Q(x,t,\xi)$$

- Convergence test





For the future

- Numerical analysis & schemes
  - for the full NS system
  - hyperbolicity, entropies,. . .
  - stability of the schemes
  - towards industrial codes
- Interaction with structures
  - erosion, carriage and associated problems
  - CEMRACS 2011
- Hydrodynamics-biology coupling
  - water quality management
- Control, data assimilation,. . .
  - everything is more simple at the kinetic level