Quelques applications de l'inégalité de Lojasiewicz à des discrétisations d'EDP

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Consider the **gradient flow**

$$U'(t) = -\nabla F(U(t)) \quad t \geq 0, \tag{1}$$

where $U = (u_1, \ldots, u_d)^t$, $F \in C^{1,1}_{loc}(\mathbf{R}^d, \mathbf{R})$. For every solution $U(t)$, we have

$$F(U(t)) + \int_0^t \|U'(s)\|^2 ds = F(U(0)), \quad t \geq 0.$$

If U is a solution of (1) which is bounded on $[0, +\infty)$, then

$$\omega(U(0)) := \{U^\star \ : \ \exists t_n \to +\infty, \ U(t_n) \to U^\star\}$$

is a **non-empty compact connected subset** of

$$\mathcal{S} = \{V \in \mathbf{R}^d \ : \ \nabla F(V) = 0\}.$$

Moreover, $d\big(U(t), \omega(U(0))\big) \to 0$ as $t \to +\infty$.

Does $U(t) \to U^\star$ as $t \to +\infty$?

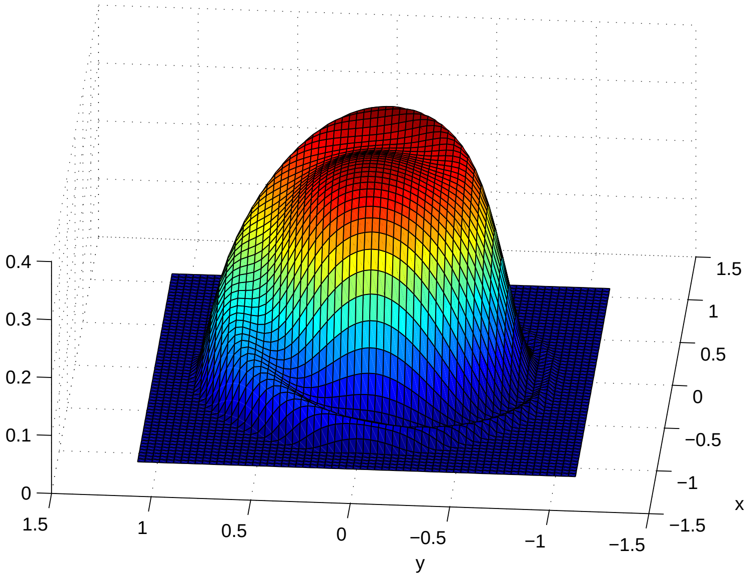
If $d = 1$, it is obvious by monotonicity.
If $d \geq 2$, it is obviously true if $\mathcal{S}$ is discrete, but it is no longer true in general: counterexample in Palis and De Melo'82.
The following counter-example is given in Absil, Mahony and Andrews'05 :

$$F(r,\theta) = e^{-1/(1-r^2)} \left[1 - \frac{4r^4}{4r^4 + (1-r^2)^4} \sin(\theta - \frac{1}{1-r^2}) \right],$$

if $r < 1$ and $F(r,\theta) = 0$ otherwise. We have $F \in C^\infty$, $F(r,\theta) > 0$ for $r < 1$ so every point on the circle $r = 1$ is a global minimizer.
We can check that the curve defined by

$$\theta = 1/(1-r^2)$$

is a trajectory.



"Mexican hat" function

Theorem (Lojasiewicz'65)

If $F : \mathbf{R}^d \to \mathbf{R}$ is real analytic in a neighbourhood of $\overline{U} \in \mathbf{R}^d$, there exist $\nu \in (0, 1/2]$, $\sigma > 0$ and $\gamma > 0$ s.t. for all $V \in \mathbf{R}^d$,

$$\|V - \overline{U}\| < \sigma \Rightarrow |F(V) - F(\overline{U})|^{1-\nu} \leq \gamma \|\nabla F(V)\|. \tag{2}$$

Theorem (Lojasiewicz'65)

If $F : \mathbf{R}^d \to \mathbf{R}$ *is real analytic in a neighbourhood of* $\overline{U} \in \mathbf{R}^d$*, there exist* $\nu \in (0, 1/2]$*,* $\sigma > 0$ *and* $\gamma > 0$ *s.t. for all* $V \in \mathbf{R}^d$*,*

$$\|V - \overline{U}\| < \sigma \Rightarrow |F(V) - F(\overline{U})|^{1-\nu} \leq \gamma \|\nabla F(V)\|. \tag{2}$$

Example: for $d = 1$ and $p \geq 2$, $x \mapsto |x|^p$ satisfies (2) at $x = 0$ with $\nu = 1/p$. Also true for $1 < p \leq 2$.
In the "generic case"' where $\nabla^2 F(U)$ inversible, $\nu = 1/2$.
Counter-examples: for $d = 1$, the C^∞ function $x \mapsto \exp(-1/x^2)$ satisfies (2) at $x = 0$ for $\nu = 0$ (too weak). The C^∞ function

$$x \mapsto \exp(-1/x^2)\sin(1/x)$$

does not satisfy (2) at $x = 0$.
NB: see the preprint of Michel Coste on his web page.

Corollary

If $F : \mathbf{R}^d \to \mathbf{R}$ is real analytic, then for any bounded semi-orbit of $U'(t) = -\nabla F(U(t))$, there exists $U^\infty \in \mathcal{S}$ s.t. $U(t) \to U^\infty$ as $t \to +\infty$.

Moreover, *let ν be a Lojasiewicz exponent of F at U^∞:*

- *if $\nu = 1/2$, then for t large enough,*

$$F(U(t)) \leq Ce^{-\alpha t} \text{ and } \|U(t) - U^\infty\| \leq C' e^{-\alpha t/2},$$

for some constants α, C and $C' > 0$;

- *if $\nu \in (0, 1/2)$, then for t large enough*

$$F(U(t)) \leq Ct^{-1/(1-2\nu)} \text{ and } \|U(t) - U^\infty\| \leq C' t^{-\nu/(1-2\nu)},$$

for some constants C and $C' > 0$.

NB : **optimal convergence** rates. If $F(x) = |x|^p$ ($p > 2$), then $\nu = 1/p$, $\nu/(1-2\nu) = 1/(p-2)$ and the solution of $x'(t) = -|x(t)|^{p-1}$ is $C_p(C+t)^{1/(2-p)}$.

A proof (convergence)

$$\begin{aligned}
-[F(U(t))^\nu]' &= -\nu U'(t)\cdot\nabla F(U(t))F(U(t))^{\nu-1}\\
&= \nu\|U'(t)\|\|\nabla F(U(t))\|F(U(t))^{\nu-1}\\
&\geq \nu\gamma^{-1}\|U'(t)\|,
\end{aligned}$$

so $$F(U(t_n))^\nu - F(U(t))^\nu \geq \nu\gamma^{-1}\int_{t_n}^{t}\|U'(s)\|ds.$$

A proof (convergence)

$F(U(t))$ is non increasing and so has a limit $F^\star(=0)$. Let $t_n \to +\infty$ s.t. $U(t_n) \to U^\star$. We have $F(U^\star) = F^\star$ and $U^\star \in \mathcal{S}$. Choose n large enough so that $\|U(t_n) - U^\star\| < \sigma/2$ and $\nu^{-1}\gamma F(U(t_n))^\nu < \sigma/2$, and define

$$t^+ = \sup\{t \geq t_n \mid \|U(s) - U^\star\| < \sigma \quad \forall s \in [t_n, t)\}.$$

For $t \in [t_n, t^+)$, we have

$$\begin{aligned} -[F(U(t))^\nu]' &= -\nu U'(t) \cdot \nabla F(U(t)) F(U(t))^{\nu-1} \\ &= \nu \|U'(t)\| \|\nabla F(U(t))\| F(U(t))^{\nu-1} \\ &\geq \nu\gamma^{-1}\|U'(t)\|, \end{aligned}$$

$$\text{so} \quad F(U(t_n))^\nu - F(U(t))^\nu \geq \nu\gamma^{-1} \int_{t_n}^{t} \|U'(s)\| ds.$$

Thus $\|U(t) - U(t_n)\| < \sigma/2$, $\forall t \in [t_n, t^+)$ and so $t^+ = +\infty$, otherwise $\|U(t^+) - U^\star\| = \sigma$ and

$$\|U(t^+) - U^\star\| \leq \|U(t^+) - U(t_n)\| + \|U(t_n) - U^\star\| < \sigma,$$

a contradiction. QED.

Questions:

- If we consider (stable) **time discretizations** of the gradient flow, can we obtain similar results of convergence to equilibrium ?
- In particular, what happens for the **backward Euler scheme** ?
- What **restriction on the time step** do we have ?
- Can we find a **unifying background** ?

The **backward Euler scheme** for (1) reads: let $U^0 \in \mathbf{R}^d$, and for $n \geq 0$, let U^{n+1} solve

$$\frac{U^{n+1} - U^n}{\Delta t} = -\nabla F(U^{n+1}), \tag{3}$$

where $\Delta t > 0$ is fixed and $F \in C^1(\mathbf{R}^d, \mathbf{R})$. Since existence is not obvious, we rewrite (3) in the form:

$$U^{n+1} \in \operatorname{argmin} \left\{ \frac{\|V - U^n\|^2}{2\Delta t} + F(V) \ : \ V \in \mathbf{R}^d \right\}. \tag{4}$$

In optimization, (4) is known as the **proximal algorithm**.
In particular, U^{n+1} satisfies

$$F(U^{n+1}) + \frac{1}{2\Delta t} \|U^{n+1} - U^n\|^2 \leq F(U^n).$$

By induction, any sequence defined by (4) satisfies

$$F(U^n) + \frac{1}{2\Delta t} \sum_{k=0}^{n-1} \|U^{k+1} - U^k\|^2 \leq F(U^0), \quad \forall n \geq 0 \qquad (5)$$

This is a **stability** result.
By (5), it is easy to prove that if $(U^n)_{n \in \mathbf{N}}$ is a bounded sequence defined by the proximal algorithm (4), then

$$\omega(U^0) := \left\{ U^\star \in \mathbf{R}^d \ : \ \exists n_k \to +\infty, \ U^{n_k} \to U^\star \right\}$$

is a non-empty compact connected subset of $\mathcal{S}$. Moreover, $d(U^n, \omega(U^0)) \to 0$ as $n \to +\infty$.

Question : does $U^n \to U^\star$ as $n \to +\infty$?

Theorem (Attouch and Bolte'09, Merlet and P.'10)

If $F : \mathbf{R}^d \to \mathbf{R}$ *is real analytic, and if* $(U^n)_n$ *is a bounded sequence defined by the proximal algorithm* (4)*, then there exists* $U^\infty \in \mathcal{S}$ *s.t.* $U^n \to U^\infty$ *as* $n \to +\infty$.

Theorem (Attouch and Bolte'09, Merlet and P.'10)

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Remark 1: If $\lim_{\|V\| \to +\infty} F(V) = +\infty$, then $(U^n)_n$ defined by (4) is bounded.

Remark 2: A more general version:

- variable stepsize $0 < \Delta t_\star \leq \Delta t_n \leq \Delta t^\star < +\infty$
- $F : \mathbf{R}^d \to \mathbf{R}$ real analytic replaced by $F : dom(F) \subset \mathbf{R}^d \to \mathbf{R}$ continuous and satisfies a Lojasiewicz property

Remark 3: in addition, (optimal) convergence rates

The proof of convergence extends to many situations:

- For any other scalar product on $\mathbf{R}^d$: $\boxed{AU'(t) = -\nabla F(U(t))}$, where A is positive definite (symmetric or not).
- Generalizations in infinite dimension (Simon, Jendoubi, Haraux, Chill,. . .)
- Semilinear heat equation: $\boxed{u_t = \Delta u - f(u), \quad t \geq 0, \ x \in \Omega}$
- Cahn-Hilliard equation (Hoffman, Rybka, Chill, Jendoubi): $\boxed{u_t = -\alpha\Delta^2 u + \Delta f'(u), \quad t \geq 0, \ x \in \Omega}$, with $f'(u) = u^3 - u$ typically, $\alpha > 0$, and Neumann or periodic BC. **Merlet and P.'10**
- Cahn-Hilliard equation with dynamic boundary conditions (Wu, Zheng, Chill, Fasangova, Pruss) **Cherfils, Petcu and P.'10**
- Cahn-Hilliard-Gurtin equations (Miranville and Rougirel): gradient-like flow **Injrou and P.'10**

- Generalization to second-order gradient-like asymptotically autonomous flows:

$$\epsilon U''(t) + U'(t) = -\nabla F(U(t)) + G(t), \quad t \geq 0,$$

where $\epsilon > 0$ and $G(t) \underset{\infty}{\longrightarrow} 0$ fast enough: Haraux and Jendoubi'98, Chill and Jendoubi'03, **Grasselli and P., to appear**

- Asymptotically autonomous damped wave equation

$$\epsilon u_{tt} + u_t = \Delta u - f(u) + g(t), \quad t \geq 0, \ x \in \Omega.$$

Haraux, Jendoubi, Chill,...

- Cahn-Hilliard equation with inertial term (Grasselli, Schimperna, Zelig, Miranville, Bonfoh) **Grasselli, Lecoq and P., to appear**
- (optimal) convergence rates for 1st and 2nd order

Application : Allen-Cahn equation

$$u_t(x,t) = \alpha\Delta u(x,t) - f'(u(x,t)), \quad t \geq 0,\ x \in \Omega,$$

where Ω is bounded with Lipschitz boundary, $\alpha > 0$, $f'(u) = u^3 - u$ and Neumann boundary condition. It is a $L^2(\Omega)$ gradient flow of the functional

$$E(u) = \int_\Omega \frac{\alpha}{2}|\nabla u(x)|^2 + f(u(x))dx.$$

NB: $+1$, -1 and 0 are steady states ; if $\Omega =$ unit disc and $\alpha > 0$ small, there is a **continuum of steady states**.

A space discretization by finite elements with a nodal basis $(\varphi_i)_i$ reads

$$MU'(t) = -AU(t) - \nabla F^h(U), \qquad (6)$$

where $M = (\varphi_i, \varphi_j)_{i,j}$ is the mass matrix,
$A = (\nabla\varphi_i, \nabla\varphi_j)_{i,j}$ is the discrete Laplacian, and

$$\nabla F^h(U)_i = \int_\Omega f'(\sum_i u_i\varphi_i(x))\varphi_i(x)dx,$$

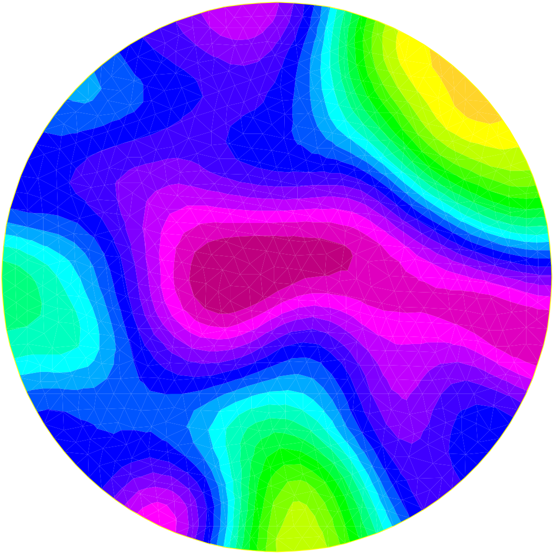
is the gradient of $F^h(U) = \int_\Omega f(\sum_i u_i\varphi_i(x))dx$.
(6) is a **gradient flow**, so we have **convergence to equilibrium for its time discretization** (by the backward Euler scheme).
A similar argument holds for the standard finite difference scheme.

The Cahn-Hilliard equation

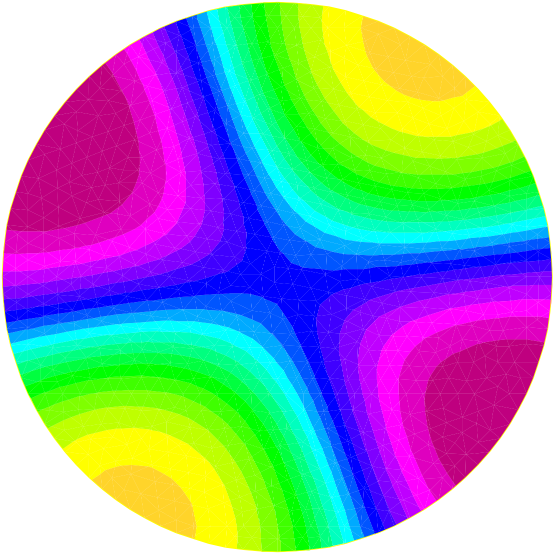
• Simulation on the "unit disc" for $f'(u) = u^3 - u$, $\alpha = 0.05$, Neumann boundary condition

- P1-P1 finite elements (splitting method for the bilaplacian)
- $\Delta t = 0.015$ and 600 iterations.

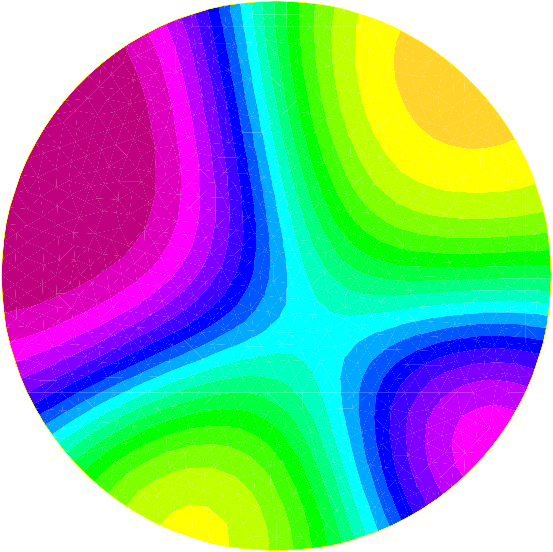
(FreeFem++ software)



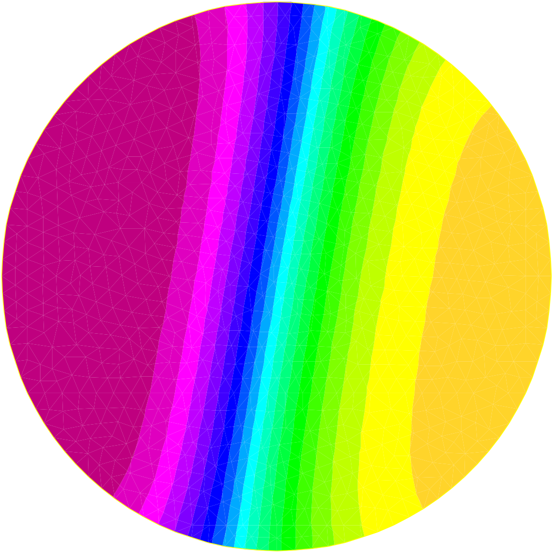
Initial state



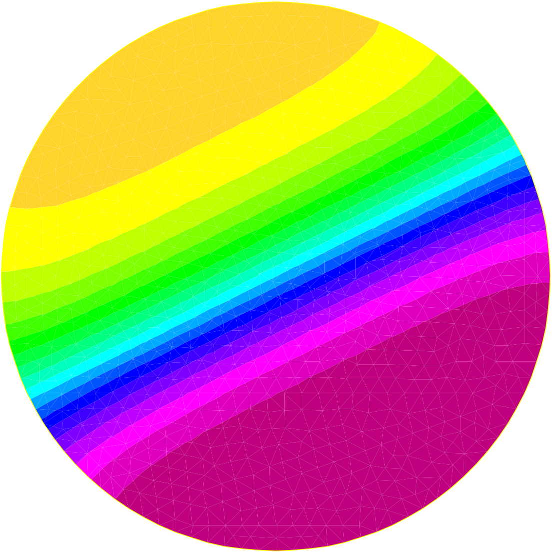
Iteration $n = 100$



Iteration $n = 400$



A steady state for the Cahn-Hilliard equation ($\alpha = 0.05$)



Another steady state for the Cahn-Hilliard equation ($\alpha = 0.05$)

Semilinear heat equation

$$\frac{u^{n+1} - u^n}{\delta t} = \Delta u^{n+1} - f(u^{n+1}) \text{ in } \Omega \tag{7}$$

$$\exists C > 0, \quad |f'(s)| \leq C(1 + |s|)^{p_1}, \quad \forall s \in \mathbf{R},$$

with $p_1 < 4/(d-2)$ if $d \geq 3$, $p_1 < \infty$ if $d = 2$,

$$\exists c_f \geq 0, \quad f'(s) \geq -c_f, \quad \forall s \in \mathbf{R}.$$

$$\liminf_{|s| \to +\infty} \frac{f(s)}{s} > -\lambda_1 \text{ where } \lambda_1 = \inf_{\|v\|_0 = 1} a(v, v).$$

Theorem (Merlet and P.'10)

If $f : \mathbf{R} \to \mathbf{R}$ is real analytic and $\delta t < 1/c_f$, then for all $u_0 \in L^2(\Omega)$, the sequence $(u^n)_n$ defined by (7) converges in $H^1_0(\Omega)$ to a stationary solution u^∞.

see also **Bolte, Daniilidis, Ley, Mazet'09**

Ongoing work and perspectives

- Replace "real analytic" by "Lojasiewicz inequality" : this allows explicit schemes or linearly explicit schemes
- Schemes with variable stepsize
- Multi-step schemes
- Asymptotically autonomous schemes
- Infinite dimension...

Phase-field crystal equation

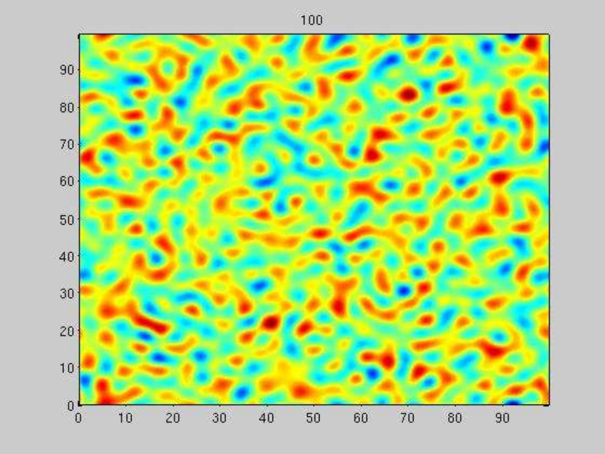
$$u_t = \Delta(u + 2\Delta u + \Delta^2 u + f'(u)) \text{ in } \Omega \times \mathbf{R}_+,$$

with periodic boundary conditions and $f'(u) = u^3 + ru$ $(r < 0)$.

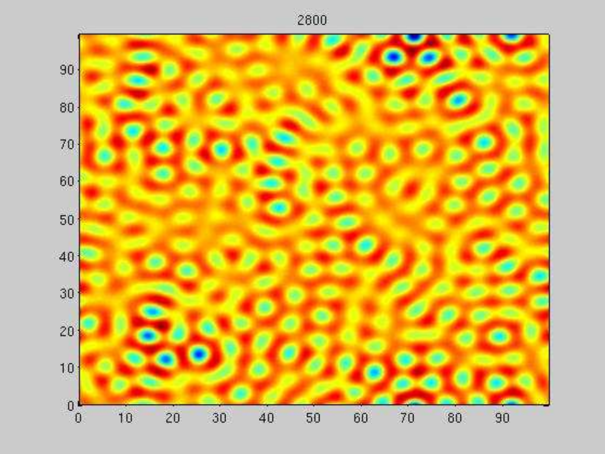
- Finite difference (FFT) in space : 256×256 grid
- linearly implicit Euler scheme in time: $\delta t = 0.01$
- $r = -0.9$, $\int_\Omega u_0 = 0.54|\Omega|$, 15000 iterations
- Matlab software

Rk: H^{-1} gradient flow for the Swift-Hohenberg functional

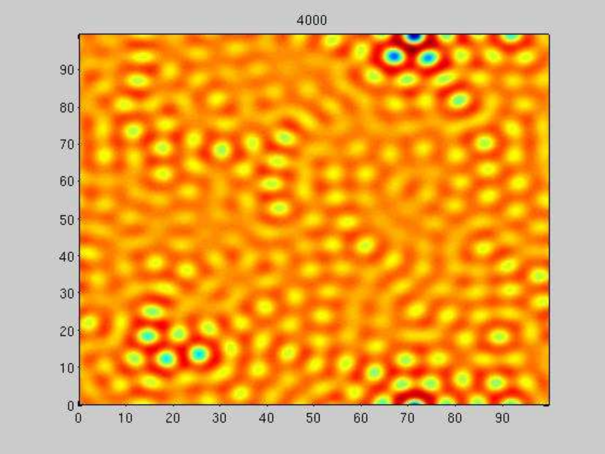
$$\mathcal{E}(u) = \int_\Omega \frac{1}{2}(u^2 - 2|\nabla u|^2 + |\Delta u|^2) + f(u).$$



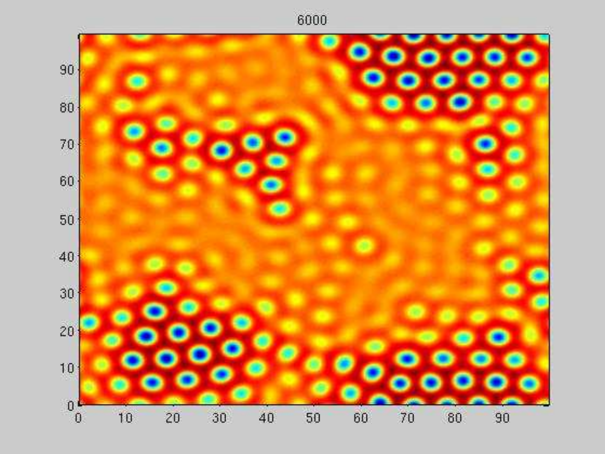
PFC, iteration $n = 100$



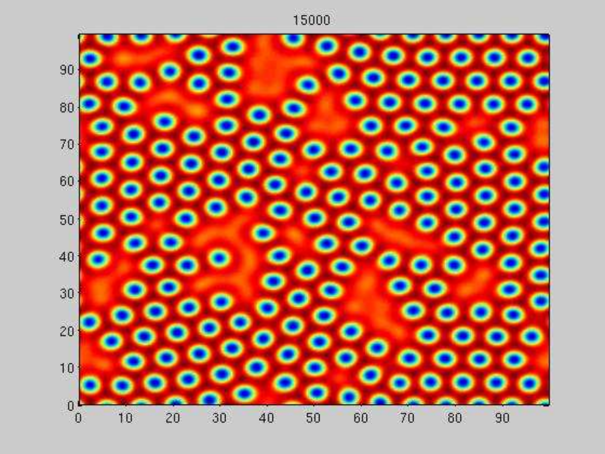
PFC, iteration $n = 2800$



PFC, iteration $n = 4000$



PFC, iteration $n = 6000$



PFC, iteration $n = 15000$

Some references

- Absil, Mahony and Andrews'05: *Convergence of iterates of descent methods for analytic cost functions*
- Attouch, Bolte'09: *On the convergence of the proximal algorithm...*
- Huang'06: *Gradient inequalities*