Optimized Schwartz algorithms for the time-harmonic Maxwell equations discretized by a discontinuous Galerkin method

M. EL BOUAJAJI

NACHOS project-team
INRIA Sophia Antipolis-Méditerranée research center, France

Collaboration with V. Dolean, M. Gander and S. Lanteri

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE INRIA
centre de recherche SOPHIA ANTIPOLIS - MÉDITERRANÉE

AGENCE NATIONALE DE LA RECHERCHE ANR

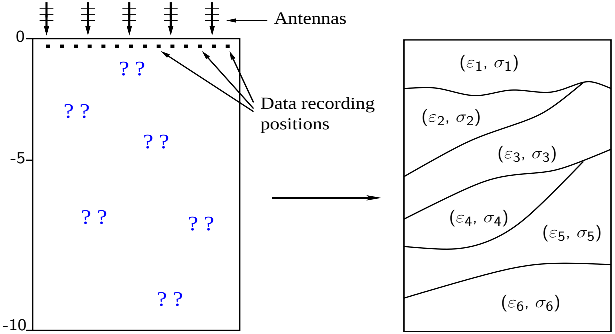
Université Nice SOPHIA ANTIPOLIS Laboratoire J. A. Dieudonné

SMAI 2011, Guidel, Bretagne
23 mai-27 mai 2011

- French National Research Agency (ANR) "MAXWELL" project :

> Objective : development of a complete microwave imaging system of subsurfaces

- Imaging of a subsurface



- Imaging of a subsurface : solve an inverse problem based in minimizing an objective function
- The minimization = solution of sereval direct problems at each optimization iteration
- Numerical modeling of electromagnetic wave propagation in heterogeneous media

$\Rightarrow$ Solution of first order time-harmonic Maxwell equations with damping

Discontinuous Galerkin method for the Time-Harmonic Maxwell equations (DGTH) + Domain decomposition method

- Time-harmonic Maxwell equations in $\Omega \subset \mathbb{R}^d$ $(d = 2,\ 3)$:

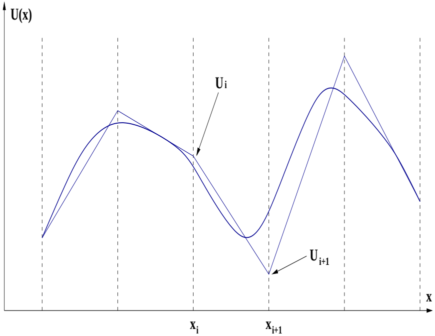
$$\left\{ \begin{array}{rcl} (\mathbf{i}\omega\varepsilon + \sigma)\mathbf{E} - \mathrm{curl}\,(\mathbf{H}) & = & -\mathbf{J} \\ \mathbf{i}\omega\mu\mathbf{H} + \mathrm{curl}\,(\mathbf{E}) & = & \mathbf{0} \end{array} \right. \Leftrightarrow i\omega Q\mathbf{W} + \nabla \cdot F(\mathbf{W}) = \mathbf{S}$$

  where :

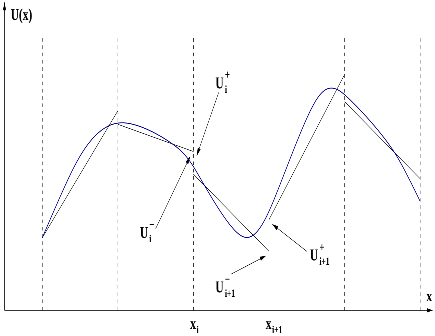
  - $\mathbf{E}$: the electric field, $\mathbf{H}$: the magnetic field
  - ε the *electric permittivity*, μ the *magnetic permeability*
  - σ is the *electric conductivity*, $\mathbf{J}$ is the applied current density
  - $\mathbf{W} = {}^t(\mathbf{E},\ \mathbf{H})$, $\mathbf{S} = {}^t(-\mathbf{J},\ \mathbf{0})$

- Boundary conditions on $\partial\Omega = \Gamma = \Gamma^a \cup \Gamma^m$:
  - On Γ^m : $\mathbf{n} \times \mathbf{E} = 0$
  - On Γ^a : $\mathbf{n} \times (\mathbf{E} - \mathbf{E}^{inc}) + z\mathbf{n} \times (\mathbf{n} \times (\mathbf{H} - \mathbf{H}^{inc})) = 0$
    - $\mathbf{n}$: unitary outwards normal, $z = \sqrt{\mu/\epsilon}$
    - $\mathbf{E}^{inc}$, $\mathbf{H}^{inc}$ incident fields

Discontinuous Galerkin method



Continuous P1 interpolation



Discontinuous P1 interpolation

Formulation

- Triangulation : $\mathcal{T}_h = \bigcup_{i=1}^{N} \tau_i$
  - Approximation space :

$$V_h = \{\mathbf{W} \in L^2(\Omega)^3 \mid \mathbf{W}_{/\tau_i} \in \{\mathcal{P}^{p_i}[\tau_i]^3, \forall \tau_i \in \mathcal{T}_h\}$$

$$\mathcal{P}^{p_i}[\tau_i] = \{\text{polynomial function on } \tau_i \text{ of degree } \leq p_i\}$$

- Variational formulation :

$$\int_{\tau_i} \mathbf{i}\omega Q\mathbf{W}\varphi d\mathbf{x} + \int_{\tau_i} (\nabla \cdot F(\mathbf{W}))\,\varphi d\mathbf{x} = \int_{\tau_i} \mathbf{S}\varphi d\mathbf{x}$$

$$\Leftrightarrow \quad \int_{\tau_i} \mathbf{i}\omega Q\mathbf{W}\varphi d\mathbf{x} - \int_{\tau_i} \nabla\varphi \cdot F(\mathbf{W}) d\mathbf{x} + \int_{\partial\tau_i} (F(\mathbf{W}) \cdot \mathbf{n})\,\varphi d\sigma = \int_{\tau_i} \mathbf{S}\varphi d\mathbf{x}$$

- Calculation of the boundary term on $\partial\tau_i$: centered or upwind numerical flux

- $(\varphi_{i1}, \varphi_{i2}, \cdots, \varphi_{id_i})$ a basis of $\mathcal{P}^{m_i}(\tau_i)$: $\mathbf{W}_i(\mathbf{x}) = \sum_{j=1}^{d_i} \mathbf{W}_{ij}\varphi_{ij}(\mathbf{x})$
- $\mathbf{W}_{ij} \in \mathbb{C}^6$ are the degrees of freedom on τ_i and $d_i = dim(\mathcal{P}^{m_i}(\tau_i))$

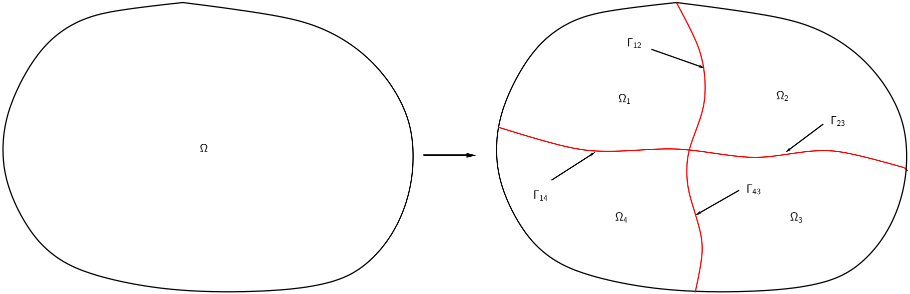
$$\begin{aligned} \mathbf{i}\omega Q_i \int_{\tau_i} \mathbf{W}_i\varphi_{ij}d\mathbf{x} \quad &+ \quad \frac{1}{2}\int_{\tau_i} \left(\left(\nabla \cdot F(\mathbf{W}_i) \right) \varphi_{ij} - \nabla\varphi_{ij} \cdot F(\mathbf{W}_i) \right) d\mathbf{x} \\ &+ \quad \frac{1}{2}\sum_{j\in\mathcal{V}_i} \int_{a_{ij}} \left(F(\mathbf{W}_j) \cdot \mathbf{n}_{ij} \right) \varphi_{ij} d\sigma = \int_{\tau_i} \mathbf{S}\varphi_{ij} d\mathbf{x} \end{aligned}$$

$$\downarrow$$

$$\boxed{\text{Linear system : } \mathcal{A}\mathbf{X} = b}$$

- $\mathcal{A}$ is of very large dimension, with complex coefficients, sparse and non-hermitian

Domain decomposition method : Motivations

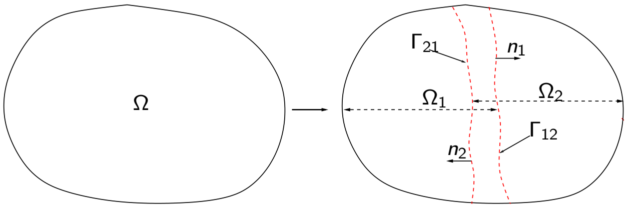


- Each subdomains is assigned to a processor.
- Local problems on the subdomains are solved in parallel : $\mathcal{L}(X^i) = b_i$ on Ω_i
- Condition on Γ_{ij} ?
- Schwarz method (iterative process) :

$$\{\mathbf{X}^n = (X^{1,n}, ..., X^{N,n})\} | \left\{ \begin{array}{l} X^{i,n} \text{ solution on } \Omega_i \\ \mathcal{B}_i(X^{i,n}) = \mathcal{B}_i(X^{j,n-1}) \text{ on } \Gamma_{ij} \end{array} \right.$$

Classical Schwarz Method for Maxwell's Equations

☞B. Després, P. Joly and J.E. Roberts, *A domain decomposition method for the harmonic Maxwell equations*, Iterative methods in linear algebra, *1992*



Perform for $n = 1, 2\dots$ *the following subdomain iteration*

$$
\begin{array}{rcll}
-i\omega\varepsilon \mathbf{E}^{1,n} + \mathrm{curl}\ \mathbf{H}^{1,n} - \sigma\mathbf{E}^{1,n} & = & \mathbf{J} & \textit{in } \Omega_1 \\
i\omega\mu \mathbf{H}^{1,n} + \mathrm{curl}\ \mathbf{E}^{1,n} & = & \mathbf{0} & \textit{in } \Omega_1 \\
\mathcal{B}_{\mathbf{n_1}}(\mathbf{E}^{1,n}, \mathbf{H}^{1,n}) & = & \mathcal{B}_{\mathbf{n_1}}(\mathbf{E}^{2,n-1}, \mathbf{H}^{2,n-1}) & \textit{on } \Gamma_{12} \\
-i\omega\varepsilon \mathbf{E}^{2,n} + \mathrm{curl}\ \mathbf{H}^{2,n} - \sigma\mathbf{E}^{2,n} & = & \mathbf{J} & \textit{in } \Omega_2 \\
i\omega\mu \mathbf{H}^{2,n} + \mathrm{curl}\ \mathbf{E}^{2,n} & = & \mathbf{0} & \textit{in } \Omega_2 \\
\mathcal{B}_{\mathbf{n_2}}(\mathbf{E}^{2,n}, \mathbf{H}^{2,n}) & = & \mathcal{B}_{\mathbf{n_2}}(\mathbf{E}^{1,n-1}, \mathbf{H}^{1,n-1}) & \textit{on } \Gamma_{21}
\end{array}
$$

with : $\mathcal{B}_{\mathbf{n}}(\mathbf{E}, \mathbf{H}) := \mathbf{n} \times \dfrac{\mathbf{E}}{\mathbf{Z}} + \mathbf{n} \times (\mathbf{H} \times \mathbf{n})$, $\mathbf{n}$ *: unitary outwards normal,* $z = \sqrt{\mu/\epsilon}$

Convergence analysis

- We consider the domain $\Omega = \mathbb{R}^2$ with the Silver-Muller radiation condition

$$\lim_{r\to\infty} r\left(\mathbf{H}\times\mathbf{n}-\mathbf{E}\right)=0,\ r=|\mathbf{x}|,\ \mathbf{n}=\mathbf{x}/|\mathbf{x}|$$

- Decomposition :

$$\Omega_1 =]-\infty, L]\times\mathbb{R}, \qquad \Omega_2=[0,\infty[\times\mathbb{R}.$$

- Taking a Fourier transform in the y variable of

$$\widehat{\mathbf{W}}(x,k)=(\mathcal{F}\,\mathbf{W})(x,k)=\int_{\mathbb{R}}\mathbf{W}(x,y)e^{-iky}dy$$

we obtain :

$$\partial_x\begin{pmatrix}\hat{E}_z^{j,n}\\ \hat{H}_y^{j,n}\end{pmatrix}=\begin{pmatrix}0 & i\omega\mu\\ \frac{k^2-\omega^2\varepsilon\mu+i\omega\mu\sigma}{i\omega\mu} & 0\end{pmatrix}\begin{pmatrix}\hat{E}_z^{j,n}\\ \hat{H}_y^{j,n}\end{pmatrix}=:M\begin{pmatrix}\hat{E}_z^{j,n}\\ \hat{H}_y^{j,n}\end{pmatrix},\ j=1,2$$

- The eigenvalues of the matrix M, and their corresponding eigenvectors are

$$\begin{array}{lcl}\lambda_1 & = & \sqrt{k^2-\tilde{\omega}^2+i\tilde{\omega}\sigma Z}:=\lambda\\ \lambda_2 & = & -\sqrt{k^2-\tilde{\omega}^2+i\tilde{\omega}\sigma Z}\end{array},\quad \mathbf{v_1}=\begin{pmatrix}\frac{-i\omega\mu}{\lambda}\\ 1\end{pmatrix},\quad \mathbf{v_2}=\begin{pmatrix}\frac{i\omega\mu}{\lambda}\\ 1\end{pmatrix}.$$

Convergence analysis

- The solutions of ordinary differential systems are given by :

$$\left(\hat{E}_z^{1,n}, \hat{H}_y^{1,n}\right) = \alpha_1^n \mathbf{v}_1 e^{\lambda(x-L)} + \alpha_2^n \mathbf{v}_2 e^{-\lambda x}, \ \left(\hat{E}_z^{2,n}, \hat{H}_y^{2,n}\right) = \beta_1^n \mathbf{v}_1 e^{\lambda x} + \beta_2^n \mathbf{v}_2 e^{-\lambda x}.$$

- Using the Silver-Muller radiation condition we have $\alpha_2^n = \beta_1^n = 0$, and inserting the solutions into the interface conditions in we get

$$\alpha_1^n = \frac{\lambda - i\tilde{\omega}}{\lambda + i\tilde{\omega}} \beta_2^{n-1} e^{-\lambda L}, \quad \beta_2^n = \frac{\lambda - i\tilde{\omega}}{\lambda + i\tilde{\omega}} \alpha_1^{n-1} e^{-\lambda L}.$$

Therefore the convergence factor is :

$$\rho(k, \tilde{\omega}, \sigma, L) = \left| \frac{\alpha_1^n}{\alpha_1^{n-2}} \right|^{\frac{1}{2}} = \left| \frac{\lambda - i\tilde{\omega}}{\lambda + i\tilde{\omega}} \frac{\lambda - i\tilde{\omega}}{\lambda + i\tilde{\omega}} \, e^{-2\lambda L} \right|^{\frac{1}{2}},$$

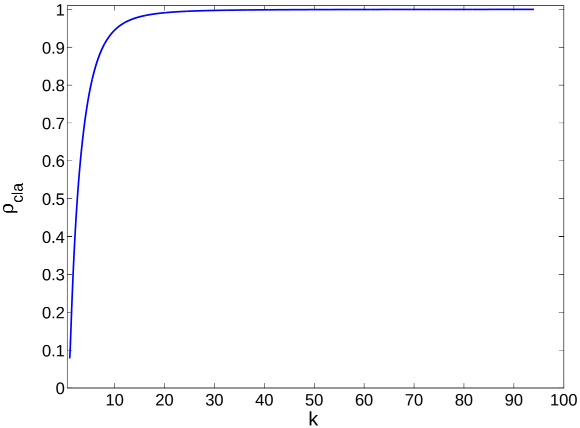


FIG.: Convergence factor ρ_{cla} of the classical Schwarz method as a function of k, for $L = 0$, $\omega = 2\,\pi$, $\sigma = 2$ and $\mu = \varepsilon = 1$

Better Transmission Conditions Between Subdomains

☞V. Dolean, L. Gerardo-Giorda and M. J. Gander, *Optimized Schwarz methods for Maxwell equations*, SIAM J. Scient. Comp., *2009*

- *We propose to modify the algorithm in one aspect only :*

$$
\begin{array}{rll}
-i\omega\varepsilon \mathbf{E}^{1,n} + \mathrm{curl}\ \mathbf{H}^{1,n} - \sigma \mathbf{E}^{1,n} = \mathbf{J} & & \text{in } \Omega_1 \\
i\omega\mu \mathbf{H}^{1,n} + \mathrm{curl}\ \mathbf{E}^{1,n} = \mathbf{0} & & \text{in } \Omega_1 \\
(\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1 \mathcal{B}_{\mathbf{n}_2})(\mathbf{E}^{1,n}, \mathbf{H}^{1,n}) = (\mathcal{B}_{\mathbf{n}_1} + \mathcal{S}_1 \mathcal{B}_{\mathbf{n}_2})(\mathbf{E}^{2,n-1}, \mathbf{H}^{2,n-1}) & & \text{on } \Gamma_{12} \\
-i\omega\varepsilon \vec{E}^{2,n} + \mathrm{curl}\ \mathbf{H}^{2,n} - \sigma \mathbf{E}^{2,n} = \mathbf{J} & & \text{in } \Omega_2 \\
i\omega\mu \mathbf{H}^{2,n} + \mathrm{curl}\ \mathbf{E}^{2,n} = \mathbf{0} & & \text{in } \Omega_2 \\
(\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2 \mathcal{B}_{\mathbf{n}_1})(\mathbf{E}^{2,n}, \mathbf{H}^{2,n}) = (\mathcal{B}_{\mathbf{n}_2} + \mathcal{S}_2 \mathcal{B}_{\mathbf{n}_1})(\mathbf{E}^{1,n-1}, \mathbf{H}^{1,n-1}) & & \text{on } \Gamma_{21},
\end{array}
$$

where $\mathcal{S}_j$, $j = 1, 2$ are tangential, possibly pseudo-differential operators.

- *How to Choose the operators $\mathcal{S}_j$?*

Optimal convergence

- Taking a Fourier transform in the y variable, we get :

$$\rho(k,\tilde{\omega},\sigma,L) = \left| \frac{\lambda - i\tilde{\omega} + \mathcal{F}(\mathcal{S}_1)(\lambda + i\tilde{\omega})}{\lambda + i\tilde{\omega} + \mathcal{F}(\mathcal{S}_1)(\lambda - i\tilde{\omega})} \frac{\lambda - i\tilde{\omega} + \mathcal{F}(\mathcal{S}_2)(\lambda + i\tilde{\omega})}{\lambda + i\tilde{\omega} + \mathcal{F}(\mathcal{S}_2)(\lambda - i\tilde{\omega})} \, e^{-2\,\lambda L} \right|^{\frac{1}{2}}.$$

- Note that if we choose $\mathcal{F}(\mathcal{S}_1) = \mathcal{F}(\mathcal{S}_2) = -\frac{\lambda - i\tilde{\omega}}{\lambda + i\tilde{\omega}}$, then $\rho = 0$ and the algorithm converges in two iterations.
- Unfortunately, it is difficult to use these operators in practice. The symbols $\mathcal{F}(\mathcal{S}_j)$ contain square roots $\left(\lambda = \sqrt{k^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z}\right)$
- This choice of $\mathcal{F}(\mathcal{S}_j)$ corresponds to nonlocal operators $\mathcal{S}_j$.

Approximation of optimal conditions

- Local operators : if we approximate λ by polynomial in ik, we will get differential operators in physical space.
- The symbols $\mathcal{F}(\mathcal{S}_j)$ can be written in several forms :

$$\mathcal{F}(\mathcal{S}_j) = -\frac{\lambda - i\tilde{\omega}}{\lambda + i\tilde{\omega}} \quad = \quad -\frac{k^2 + i\tilde{\omega}\sigma Z}{k^2 - 2\tilde{\omega}^2 + i\tilde{\omega}\sigma Z + 2i\tilde{\omega}\lambda}$$

$$= \quad -\frac{k^2 - 2\tilde{\omega}^2 + i\tilde{\omega}\sigma Z - 2\,i\,\tilde{\omega}\lambda}{k^2 + i\tilde{\omega}\sigma Z}$$

- Approximation of λ by a constant polynomial $s \in \mathbb{C}$.

Lemma

- *If $\mathcal{S}_j$, $j = 1, 2$, have the Fourier symbols $\sigma_j = \mathcal{F}(\mathcal{S}_j) = -\dfrac{s - i\tilde{\omega}}{s + i\tilde{\omega}}$, $s \in \mathbb{C}$, then the Schwarz algorithm has the convergence factor*

$$\rho_2(k, \tilde{\omega}, \sigma, Z, L, s) = \left| \left(\frac{\sqrt{k^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} - s}{\sqrt{k^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} + s} \right) e^{-\sqrt{k^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} L} \right|.$$

- *If $\mathcal{S}_j$, $j = 1, 2$ have the Fourier symbols $\sigma_j = -\dfrac{k^2 + i\tilde{\omega}\sigma Z}{k^2 - 2\tilde{\omega}^2 + i\tilde{\omega}\sigma Z + 2i\tilde{\omega}s}$, then the Schwarz algorithm has the convergence factor*

$$\rho_3(k, \tilde{\omega}, \sigma, Z, L, s) = \left| \left(\frac{\sqrt{k^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} - i\tilde{\omega}}{\sqrt{k^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} + i\tilde{\omega}} \right) \right| \rho_2(k, \tilde{\omega}, \sigma, Z, L, s).$$

- *If $\mathcal{S}_j$, $j = 1, 2$, , have the Fourier symbols $\sigma_j = \mathcal{F}(\mathcal{S}_j) = -\dfrac{s_j - i\tilde{\omega}}{s_j + i\tilde{\omega}}$, $s_2 \in \mathbb{C}$, then the Schwarz algorithm has the convergence factor*

$$\rho_4(k, \tilde{\omega}, \sigma, Z, L, s_1, s_2) = \left| \prod_{l=1}^{2} \frac{\sqrt{k^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} - s_l}{\sqrt{k^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} + s_l} e^{-2\sqrt{k^2 - \tilde{\omega}^2 + i\tilde{\omega}\sigma Z} L} \right|^{\frac{1}{2}}.$$

Hierarchy of Schwarz algorithms

Algorithm 1	$\mathcal{F}(\mathcal{S}_1) = \mathcal{F}(\mathcal{S}_2) = 0$
Algorithm 2	$\mathcal{F}(\mathcal{S}_1) = \mathcal{F}(\mathcal{S}_2) = \dfrac{s - i\tilde{\omega}}{s + i\tilde{\omega}}$
Algorithm 3	$\mathcal{F}(\mathcal{S}_1) = \mathcal{F}(\mathcal{S}_2) - \dfrac{k^2 + i\tilde{\omega}\sigma Z}{k^2 - 2\tilde{\omega}^2 + i\tilde{\omega}\sigma Z + 2i\tilde{\omega}s}$
Algorithm 4	$\mathcal{F}(\mathcal{S}_1) = \dfrac{s_1 - i\tilde{\omega}}{s_1 + i\tilde{\omega}}$ $\mathcal{F}(\mathcal{S}_2) = \dfrac{s_2 - i\tilde{\omega}}{s_2 + i\tilde{\omega}}$
Algorithm 5	$\mathcal{F}(\mathcal{S}_1) = -\dfrac{k^2 + i\tilde{\omega}\sigma Z}{k^2 - 2\tilde{\omega}^2 + i\tilde{\omega}\sigma Z + 2i\tilde{\omega}s_1}$ $\mathcal{F}(\mathcal{S}_2) = -\dfrac{k^2 + i\tilde{\omega}\sigma Z}{k^2 - 2\tilde{\omega}^2 + i\tilde{\omega}\sigma Z + 2i\tilde{\omega}s_2}$

How to Choose the Complex Constants s, s_1 and s_2

- In order to obtain some effecient algorithms, we choose σ_j, $j = 1, 2$, such that ρ, is minimum over a range of frequencies
- We need to solve some min-max problem

$$\min_{s\in\mathbb{C}} \max_{k\in K} \rho(k, \tilde{\omega}, \sigma, Z, L, s), \qquad \min_{s_1,\, s_2\in\mathbb{C}} \max_{k\in K} \rho(k, \tilde{\omega}, \sigma, Z, L, s_1, s_2), K = [k_{min}, k_{max}]$$

- Solution of min-max problem is based on equioscillation

$$\rho(\overline{k_i}, \tilde{\omega}, \sigma, Z, L, s) = \rho(\overline{k_l}, \tilde{\omega}, \sigma, Z, L, s)$$

How to Choose the Complex Constants s, s_1 and s_2

Theorem

For $\sigma > 0$, treating all frequencies k, we find for the optimal $s = p(1+i)$ based on equioscillation :

- *Non-overlapping case*

$$p^* = \frac{(\omega\sigma\mu)^{\frac{1}{4}}\sqrt{C}}{2^{\frac{1}{4}}\sqrt{h}} \quad \longrightarrow \quad \rho_2^* = 1 - \frac{2^{\frac{3}{4}}(\omega\sigma\mu)^{\frac{1}{4}}\sqrt{h}}{\sqrt{C}}, \quad L = 0$$

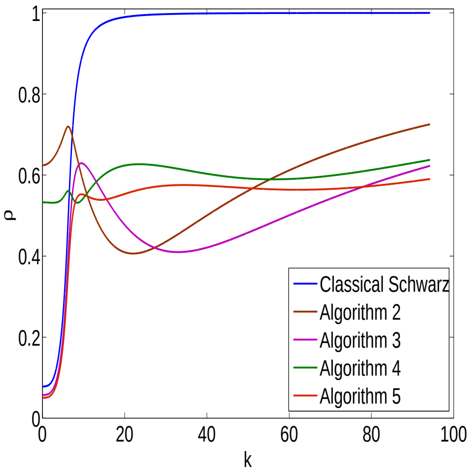
- *Overlapping case*

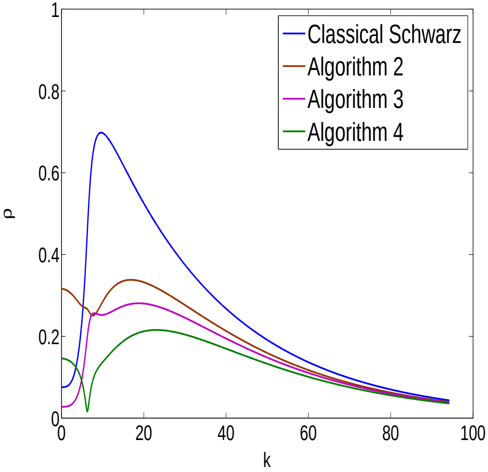
$$p^* = \frac{(2\,\omega\,\sigma\,\mu)^{1/3}}{2\,h^{\frac{1}{3}}} \quad \longrightarrow \quad \rho_2^* = 1 - 2^{\frac{7}{6}}\,(\omega\,\sigma\,\mu)^{1/6}\;h^{\frac{1}{3}}, \quad L = h$$

where $k_{max} = \frac{C}{h}$

	with overlap, $L = h$	
Algorithm	ρ	parameters
1	$1-\frac{4}{3}\left(9\omega^4\sigma^2\mu^3\varepsilon\right)^{\frac{1}{8}} h^{\frac{3}{4}}$	none
2	$1-2^{\frac{7}{6}}\left(\omega\,\sigma\,\mu\right)^{1/6}\ h^{\frac{1}{3}}$	$p=\frac{(2\,\omega\,\sigma\,\mu)^{1/3}}{2\,h^{\frac{1}{3}}}$
3	$1-\frac{2^{\frac{17}{10}}(\omega^4\varepsilon\mu^3\sigma^2)^{\frac{1}{20}}\,h^{\frac{3}{10}}}{3^{\frac{3}{10}}}$	$p=\frac{2^{\frac{2}{5}}(\omega^4\varepsilon\mu^3\sigma^2)^{\frac{1}{10}}}{3^{\frac{3}{5}}\,h^{\frac{2}{5}}}$
4	$1-4\sqrt{2}\left(\omega\,\sigma\,\mu\right)^{\frac{1}{10}}\ h^{\frac{1}{5}}$	$p_1=\frac{(\omega\,\sigma\,\mu)^{\frac{1}{5}}}{2\,h^{\frac{3}{5}}},\ p_2=\frac{(\omega\,\sigma\,\mu)^{\frac{2}{5}}}{2\,h^{\frac{1}{5}}}$
5	$1-\frac{2^{\frac{23}{8}}(\omega^4\varepsilon\mu^3\sigma^2)^{\frac{1}{32}}\,h^{\frac{3}{16}}}{3^{\frac{3}{16}}}$	$p_1=\frac{(\omega^4\varepsilon\mu^3\sigma^2)^{\frac{1}{16}}}{2^{\frac{1}{4}}\,3^{\frac{3}{8}}\,h^{\frac{5}{8}}},p_2=\frac{\sqrt{2}(\omega^4\varepsilon\mu^3\sigma^2)^{\frac{1}{8}}}{3^{\frac{3}{4}}\,h^{\frac{1}{4}}}$
	without overlap, $L = 0$	
1	$1-\frac{\omega^2\sigma\sqrt{\mu^3\varepsilon}}{C^3}h^3$	none
2	$1-\frac{2^{\frac{3}{4}}(\omega\sigma\mu)^{\frac{1}{4}}\sqrt{h}}{\sqrt{C}}$	$p=\frac{(\omega\sigma\mu)^{\frac{1}{4}}\sqrt{C}}{2^{\frac{1}{4}}\sqrt{h}}$
3	$1-\frac{2^{\frac{11}{7}}(\omega^4\varepsilon\mu^3\sigma^2)^{\frac{1}{14}}\,h^{\frac{3}{7}}}{3^{\frac{3}{7}}\,C^{\frac{3}{7}}}$	$p=\frac{2^{\frac{4}{7}}(\omega^4\varepsilon\mu^3\sigma^2)^{\frac{1}{14}}\,C^{\frac{4}{7}}}{3^{\frac{3}{7}}\,h^{\frac{4}{7}}}$
4	$1-\frac{(2\,\omega\,\sigma\,\mu)^{\frac{1}{8}}\,h^{\frac{1}{4}}}{C^{\frac{1}{4}}}$	$p_1=\frac{(2\,\omega\,\sigma\,\mu)^{\frac{1}{8}}\,C^{\frac{3}{4}}}{h^{\frac{3}{4}}}\ ,\ p_2=\frac{(2\,\omega\,\sigma\,\mu)^{3/8}C^{1/4}}{2\,h^{\frac{1}{4}}}$
5	$1-\frac{2^{\frac{34}{13}}(\omega^4\varepsilon\mu^3\sigma^2)^{\frac{1}{26}}\,h^{\frac{3}{13}}}{3^{\frac{3}{13}}\,C^{\frac{3}{13}}}$	$p_1=\frac{2^{\frac{8}{13}}(\omega^4\varepsilon\mu^3\sigma^2)^{\frac{1}{26}}\,C^{\frac{10}{13}}}{3^{\frac{3}{13}}\,h^{\frac{10}{13}}},p_2=\frac{2^{\frac{11}{13}}(\omega^4\varepsilon\mu^3\sigma^2)^{\frac{3}{26}}\,C^{\frac{4}{13}}}{3^{\frac{9}{13}}\,h^{\frac{4}{13}}}$

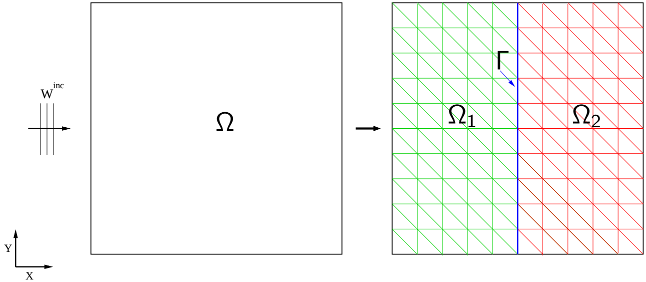
Convergence factor for the optimazed Schwarz algorithms



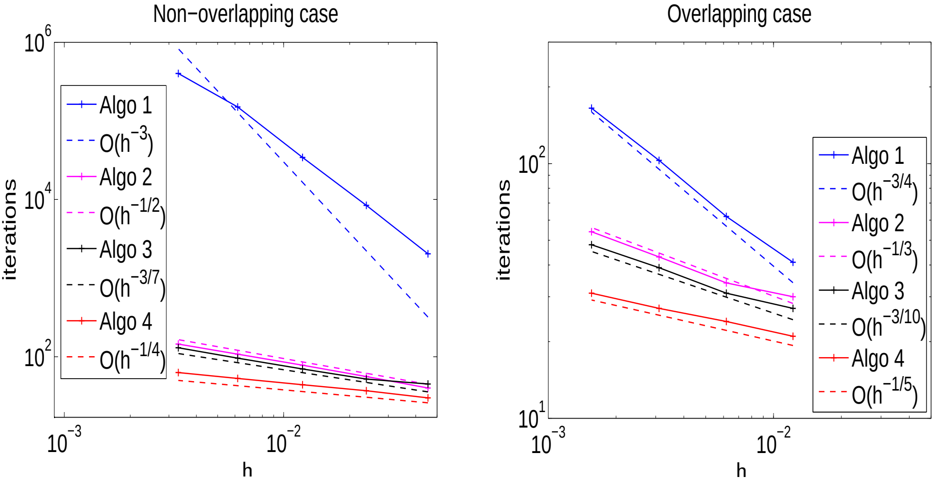


Performance for a Two Subdomain Decomposition

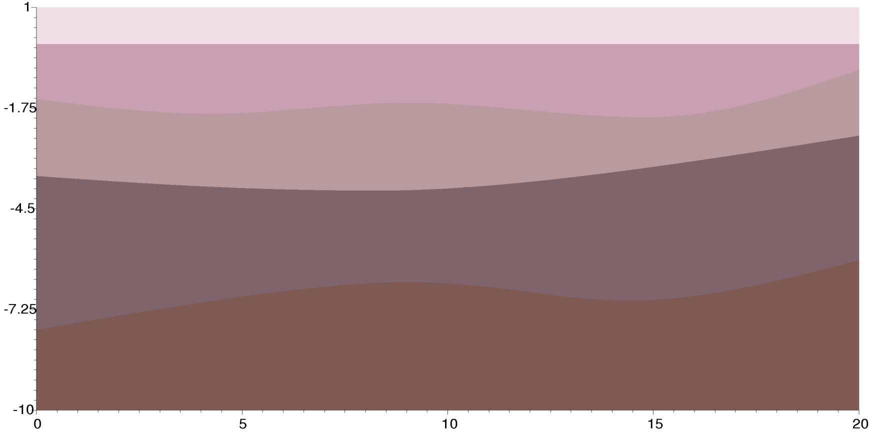
$$\begin{cases} \Omega = (0,1)^2, \varepsilon = \mu = 1,\ \sigma = 5,\ \omega = 2\pi \\ \mathbf{W}^{inc} = \left(H_x^{inc}, H_y^{inc}, E_z^{inc}\right) = \left(\frac{k_y}{\mu\omega}, \frac{-k_x}{\mu\omega}, 1\right) e^{-i\mathbf{k}\cdot\mathbf{x}} \\ \mathbf{k} = (k_x, k_y) = (\omega\sqrt{\varepsilon - i\frac{\sigma}{\omega}}, 0),\ \mathbf{x} = (x, y) \\ \Omega_1 = (0, \frac{1}{2}) \times (0,1), \Omega_2 = (\frac{1}{2}, 1) \times (0,1), \Gamma = \Omega_1 \cap \Omega_2 \end{cases}$$



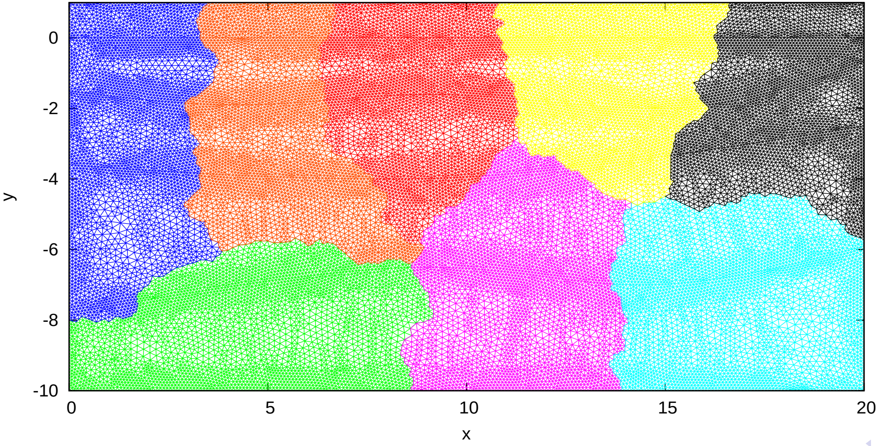
Performance for a Two Subdomain Decomposition



Electromagnetic wave propagation in heterogeneous subsurface



Electromagnetic wave propagation in heterogeneous subsurface



Electromagnetic wave propagation in heterogeneous subsurface

# processor	2	4	8	16
Algorithm 1	59	72	80	85
Algorithm 2	39	47	53	61
Algorithm 4	35	41	49	57

- Imaging of a subsurface : solve an inverse problem based in minimizing an objective function

- The minimization = solution of sereval direct problems at each optimization iteration