Revue historique de méthodes de couplage

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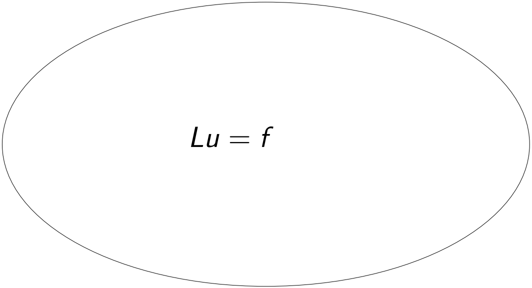
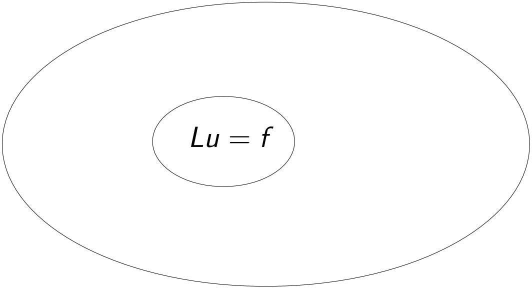
En collaboration avec Martin Gander.

Coupling Equations



Coupling Equations



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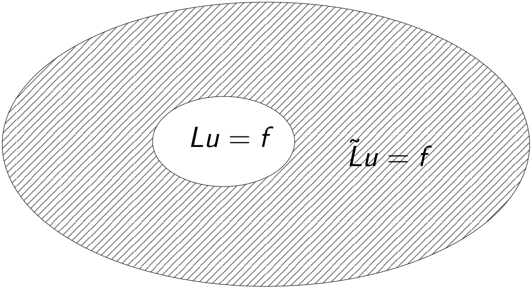

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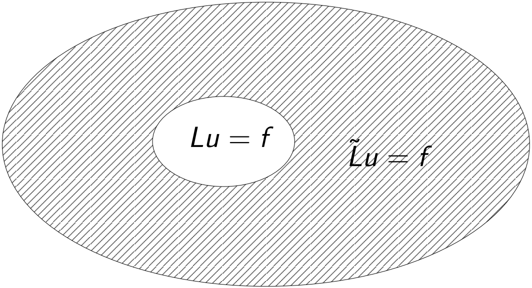


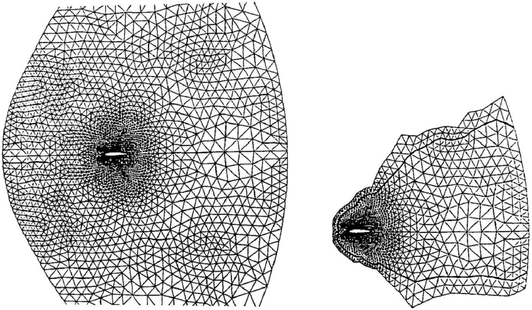
- The position of the interface is known a priori or is determined by the model itself.
- The domains can overlap or not.

Outline

Outline

Overlapping Method

Dinh, Glowinski, Periaux, Terrasson (1988): On the Coupling of Viscous and Inviscid Models for Incompressible Fluid Flows Via Domain Decomposition.

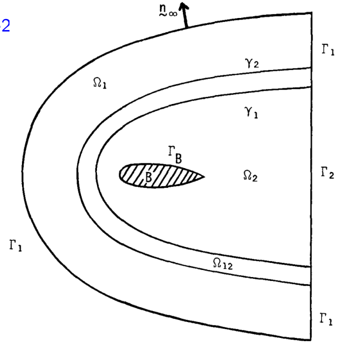


"The main goal of this paper is to present a computational method for the coupling of two distinct mathematical models describing the same physical phenomenon, namely the flow of an incompressible viscous fluid. The basic idea is to replace the Navier-Stokes equations by the potential one in those regions where we can neglect the viscous effects and where the vorticity is small".

Overlapping Method

Navier-Stokes equations:

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu \triangle u + (u \cdot \nabla) u + \nabla p = 0 \text{ in } \Omega_2 \\ \nabla \cdot u = 0 \text{ in } \Omega_2 \\ +\text{Boundary Conditions} \end{cases}$$



Laplace equation (if the vorticity is small : $u = \nabla\phi$):

$$\begin{cases} \triangle \phi = 0 \text{ in } \Omega_1 \\ +\text{Boundary Conditions on } \gamma_1 \end{cases}$$

How to couple the two problems?

Overlapping Coupling

Dinh, Glowinski, Periaux, Terrasson (1988): *"To couple the two models, we use a least squares approach in which we minimize over the overlapping region some distance between* **u** *and* $\nabla\phi$*"*

Navier-Stokes equations

$$\begin{cases} \frac{\partial u}{\partial t} - \nu \triangle u + (u \cdot \nabla) u + \nabla p = 0 \text{ in } \Omega_2 \\ \nabla \cdot u = 0 \text{ in } \Omega_2 \\ u = v \text{ on } \gamma_2 \end{cases}$$

Laplace equations

$$\begin{cases} \triangle \phi = 0 & \text{in } \Omega_1 \\ \phi = \psi & \text{on } \gamma_1 \end{cases}$$

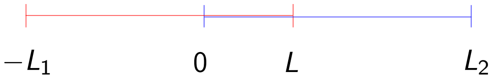
The coupling condition is :

$$\mathrm{Inf}_{(v,\psi)} \frac{1}{2} \int_{\Omega_1 \cap \Omega_2} |u - \nabla\phi|^2.$$

Illustration on a Model Problem

Advection diffusion equation

$$
\begin{aligned}
\mathcal{L}_{ad} u := -\nu u'' + a u' + c u &= f \quad \text{on } (-L_1, L_2) \\
u &= g_1 \quad \text{on } x = -L_1 \\
\mathcal{B} u &= 0 \quad \text{on } x = L_2
\end{aligned}
$$



Advection diffusion equation

$$
\begin{aligned}
\mathcal{L}_{ad} u_{ad} &= f \quad \text{on } (-L_1, L) \\
u_{ad} &= g_1 \text{ on } x = -L_1 \\
u_{ad} &= \psi \quad \text{on } x = L
\end{aligned}
$$

Advection equation

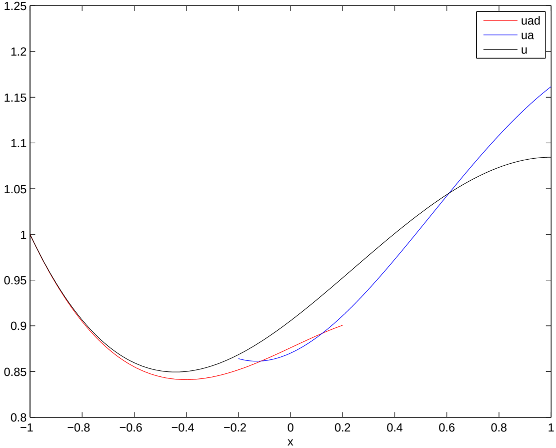
$$
\begin{aligned}
\mathcal{L}_a u_a := a u_a' + c u_a &= f \text{ on } (0, L_2) \\
u_a &= \tau \text{ on } x = 0 \\
\mathcal{B} u_a &= 0 \text{ on } x = L
\end{aligned}
$$

How to determine ψ and τ ?

A Least Squares Approach

Idea in Dinh, Glowinski, Periaux, Terrasson (1988):

$$||u_{ad}(\psi) - u_a(\tau)||_{(0,L)} \longrightarrow \min$$



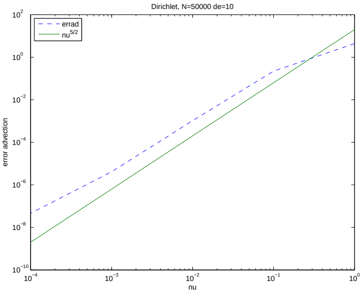
$\nu = 0.5,\ N = 1000,\ L = 100h$

Quality of this Coupling

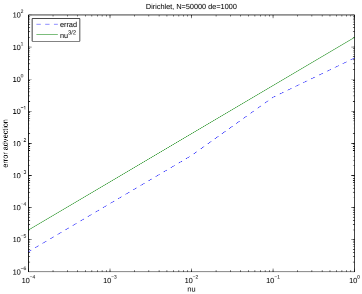
We evaluate this coupling technique for ν small by comparing to the fully viscous solution on the entire domain.

$$f = \cos x + \sin x, a = 1, c = 1, L_1 = 1, L_2 = 1, g_1 = 1, \mathcal{B} = \partial_x$$

Discretization with centered finite difference, $N = 100000$ points:



$L = 10h$, $||e_{ad}|| \sim \nu^{5/2}$



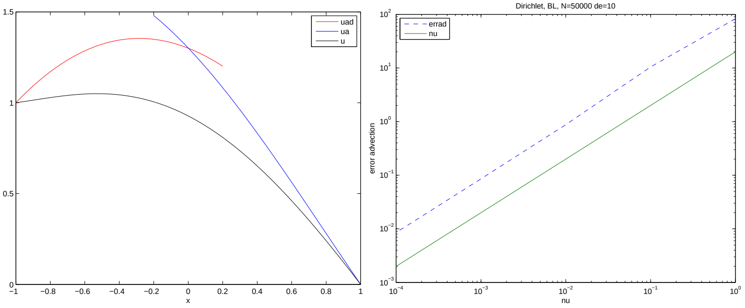
$L = 1000h$, $||e_{ad}|| \sim \nu^{3/2}$

Quality of this Coupling, Boundary Layer

Experiments for ν small with a boundary layer

$$f = \cos x + \sin x, a = -1, c = 0, L_1 = 1, L_2 = 1, g_1 = 1, \mathcal{B} = Id$$

Discretization with centered finite difference, $N = 100000$ points:



$$L = 10h, \; ||e_{ad}|| \sim \nu \text{ (independent of } L\text{).}$$

Literature

Q. V. Dinh, R. Glowinski, J. Periaux, G. Terrasson (1988): On the Coupling of Viscous and Inviscid Models for Incompressible Fluid Flows Via Domain Decomposition, DD1.

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P. Gervasio, J.-L. Lions, A. Quarteroni (2001): Domain Decomposition and Virtual Control for Fourth Order Problems.

V. Agoshkov, P. Gervasio, and A. Quarteroni (2006). Optimal control in heterogeneous domain decomposition methods for advection-diffusion equations.

Outline

Introduction

Overlapping Method

Boundary Conditions as Coupling Conditions
- Singular Perturbation Method
- Boundary Layer Correction
- Method Based on the Factorization of the Operator

χ-Formulation

Outline

Non-Overlapping Coupling

Gastaldi, Quarteroni, Sacchi Landriani (1989): On the Coupling of Two Dimensional Hyperbolic and Elliptic Equations: Analytical and Numerical Approach.

"Physical evidence suggests that viscosity effects are negligible apart from a small region close to the rigid body. This is one instance where the mathematical model of the problem may lead to the use of equations of different character in separate regions, just by dropping the viscous terms when they are very small."

Gastaldi, Quarteroni (1989): On the Coupling of Hyperbolic and Parabolic systems: Analytical and Numerical Approach.

"The justification of the interface conditions is based on a singular perturbation analysis, that is the hyperbolic system is rendered parabolic by adding a small artificial 'viscosity'. As this goes to zero, the coupled parabolic-parabolic problem degenerates into the original one, yielding some conditions at the interface."

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Coupling Hyperbolic and Parabolic Problems

Gastaldi, Quarteroni (1989): Start with two different problems to be coupled:

$$
\begin{aligned}
-\nu u''_{ad} + a u'_{ad} + c u_{ad} &= f \quad \text{on } (-L_1, 0) \\
a u'_a + c u_a &= f \quad \text{on } (0, L_2)
\end{aligned}
$$

Introduce for regularization a small artificial viscosity ϵ:

$$
\begin{aligned}
-\nu w''_\epsilon + a w'_\epsilon + c w_\epsilon &= f \quad \text{on } (-L_1, 0) \\
-\epsilon v''_\epsilon + a v'_\epsilon + c v_\epsilon &= f \quad \text{on } (0, L_2)
\end{aligned}
$$

Two types of boundary conditions are possible :

Variational Conditions	*Non Variational Conditions*
$w_\epsilon(0) = v_\epsilon(0)$	$w_\epsilon(0) = v_\epsilon(0)$
$\nu {w_\epsilon}'(0) = \epsilon {v_\epsilon}'(0)$	${w_\epsilon}'(0) = {v_\epsilon}'(0)$

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Two types of boundary conditions are possible :

Variational Conditions

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Non Variational Conditions

$$\begin{aligned} w_\epsilon(0) &= v_\epsilon(0) \\ {w_\epsilon}'(0) &= {v_\epsilon}'(0) \end{aligned}$$

What Happens in the Limit $\epsilon \to 0$?

Quarteroni et al. proved rigorously that as $\epsilon \to 0$, we have $w_\epsilon \to u_{ad}$ in $(-L_1, 0)$ and $v_\epsilon \to u_a$ in $(0, L_2)$ with the B.C. :

Variational coupling conditions :

$a > 0$	$a < 0$
$u_{ad}(0) = u_a(0)$, $u'_{ad}(0) = 0$,	$-\nu u'_{ad}(0) + a u_{ad}(0) = a u_a(0)$.

Non-variational coupling conditions :

$$\begin{aligned} u_{ad}(0) &= u_a(0), \\ u'_{ad}(0) &= u'_a(0), \end{aligned} \qquad u_{ad}(0) = u_a(0),$$

Gastaldi, Quarteroni, Sacchi Landriani (1989): *"Among all allowed choices, we make the most natural one, namely we take those interface conditions which are generated by a limit procedure on 'globally viscous problems"'.*

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Rigorous Error Estimates

Theorem (Gander, Halpern, Japhet, Martin (2008))

For $a > 0$, the error between the fully viscous solution u and the coupled solution satisfies for the **variational** *coupling conditions*

$$||u - u_{ad}||_2 = O(\nu^{3/2}), \quad ||u - u_a||_2 = O(\nu),$$

and for the **non-variational** *coupling conditions*

$$||u - u_{ad}||_2 = O(\nu^{5/2}), \quad ||u - u_a||_2 = O(\nu).$$

For $a < 0$, both choices give

$$||u - u_{ad}||_2 = O(\nu), \quad ||u - u_a||_2 = O(\nu).$$

Literature

F. Gastaldi, A. Quarteroni (1989): On the Coupling of Hyperbolic and Parabolic Systems: Analytical and Numerical Approach.

F. Gastaldi, A. Quarteroni, G. Sacchi Landriani (1990): On the Coupling of Two Dimensional Hyperbolic and Elliptic Equations: Analytical and Numerical Approach.

A. Quarteroni and F. Pasquarelli and A. Valli (1992): Heterogeneous domain decomposition principles, algorithms, applications.

Lie, Bourgat, Le Tallec Tidriri Qiu, Schenk Hebeker, Alonso Valli . . .

Outline

A Posteriori Correction Technique

Case $a < 0$

Coclici, Morosanu, Wendland (2000): The Coupling of Hyperbolic and Elliptic Boundary Value Problems with Variable Coefficients.

Consider the coupling model proposed by Quarteroni et al. :

$$-\nu u_{ad}'' + a u_{ad}' + c u_{ad} = f \text{ on } (-L_1, 0) \qquad a u_a' + c u_a = f \text{ on } (-L_1, 0)$$

$$-\nu u_{ad}'(0) + a u_{ad}(0) = a u_a(0)$$

Coclici et al: *"But this transmission condition implies that solutions of the coupled hyperbolic-elliptic problem exhibit jumps at the interface..."*

Different point of view :the original problem is

$$-\nu w_\epsilon'' + a w_\epsilon' + c w_\epsilon = f \qquad -\epsilon v_\epsilon'' + a v_\epsilon' + c v_\epsilon = f$$
$$\epsilon v_\epsilon'(0) = \nu w_\epsilon'(0)$$
$$v_\epsilon(0) = w_\epsilon(0)$$

Asymptotic Correction

Coclici et al: *"(...) we can affirm that the approximate solution to the heterogeneous coupled Navier-Stokes/Euler problem is a first approximation of exterior viscous flows taking into account viscosity as well as far field behaviour. This coupled solution needs to be corrected by special terms (...)"*

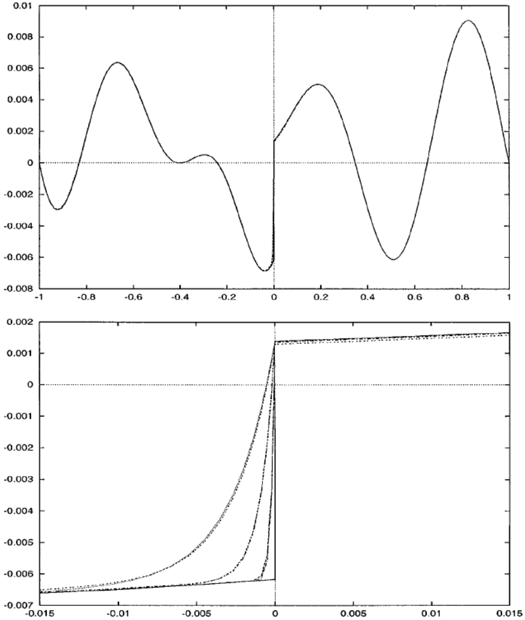
The coupled solution obtained represents only the first term in an asymptotic expansion:

$$\begin{aligned} w_\epsilon(x) &= u_{ad}(x) + r_\epsilon(x) \\ v_\epsilon(x) &= u_a(x) + l_\epsilon(x) + s_\epsilon(x) \end{aligned}$$

where l_ϵ represent the boundary layer term missing for continuity, and r_ϵ and s_ϵ are small for ϵ small.

The correction term is computed l_ϵ analytically.

Numerical Experiment, Coclici et al.



Outline

Idea of the method on the stationary case

Gander, Halpern, Japhet, Martin (2008): Viscous Problems with Inviscid Approximations in Subregions: a New Approach Based on Operator Factorization.

$\hookrightarrow$ Can we have a better error estimate than $\mathcal{O}(\nu^{5/2})$?

Factorization of the operator :

$$\mathcal{L}u = (\partial_x - \lambda^+)\mathcal{L}^- u = f.$$

We can prove that if $(\partial_x - \lambda^+)\tilde{u}_a = f$, then

$$\mathcal{L}^- u(0) = \tilde{u}_a(0) + (\mathcal{L}^- u(L_2) - \tilde{u}_a(L_2))e^{-\lambda^+ L_2},$$

So that the exact transmission condition is :

$$\mathcal{L}^- u_{ad}(0) = \tilde{u}_a(0) + (\mathcal{L}^- u(L_2) - \tilde{u}_a(L_2))e^{-\lambda^+ L_2}.$$

New approach : the procedure

1. We solve the modified advection equation

$$\widetilde{\mathcal{L}_a}\tilde{u}_a := \tilde{u}_a' - \lambda^+\tilde{u}_a = f \text{ on } (0, L_2),$$
$$\tilde{u}_a(L_2) = \alpha_1\nu + \alpha_2\nu^2 + \cdots + \alpha_m\nu^m.$$

2. We solve the advection-diffusion equation

$$\mathcal{L}_{ad}u_{ad} := -\nu u_{ad}'' + au_{ad}' + cu_{ad} = f \text{ on } (-L_1, 0),$$
$$-\nu u_{ad}'(0) + \nu\lambda^- u_{ad}(0) = \tilde{u}_a(0).$$

3. We solve the advection equation (optional)

$$\mathcal{L}_a u_a := au_a' + cu_a = f \text{ on } (0, L_2),$$
$$u_a(0) = u_{ad}(0).$$

Error Estimates

$a > 0$			
	Factorization	Variational	Non-Variational
$\|e_{ad}\|_{\Omega^-}$	$\mathcal{O}(e^{-\frac{a}{\nu}})$	$\mathcal{O}(\nu^{3/2})$	$\mathcal{O}(\nu^{5/2})$
$\|e_a\|_{\Omega^+}$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$

$a < 0$			
	Factorization	Variational	Non-Variational
$\|e_{ad}\|_{\Omega^-}$	$\mathcal{O}(\nu^m),\ m = 1, 2, \ldots$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$
$\|e_a\|_{\Omega^+}$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$	$\mathcal{O}(\nu)$

Outline

The χ-Formulation

Brezzi, Canuto, Russo (1989): A Self- Adaptative Formulation for the Euler/Navier-Stokes Coupling

"This means, assuming again that ν is constant and small everywhere, that the diffusion effects are negligible in the region where Δu is not too large. We, therefore, propose to introduce an additional nonlinearity, by considering, instead of Δu, a function $\chi(\Delta u)$ which coincides with Δu when $|\Delta u| \geq \delta$ (δ to be chosen) and vanishes otherwise"

$$\begin{aligned} -\nu\chi(u'') + au' + cu &= f && \text{on } (-L_1, L_2) \\ u &= g_1 && \text{on } x = -L_1 \\ \mathcal{B}u &= 0 && \text{on } x = L_2 \end{aligned}$$

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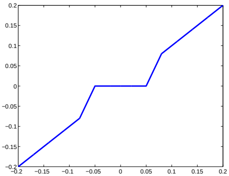
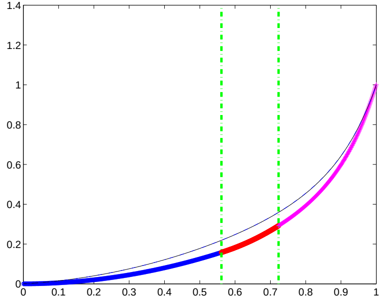


$$\chi(s) = \begin{cases} 0, & 0 \leq s < \delta - \sigma \\ (s - \delta + \sigma)\frac{\delta}{\sigma}, & \delta - \sigma \leq s \leq \delta \\ s, & s > \delta \end{cases}$$

The χ-solution

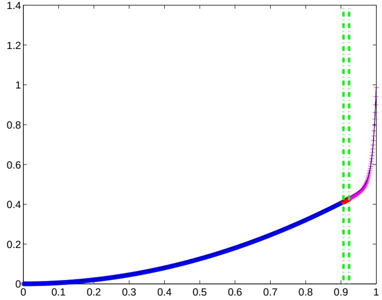
Advective solution
Intermediate solution
Viscous solution



$$\nu = 0.1$$

The χ-solution

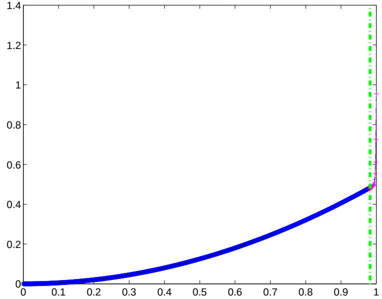
Advective solution
Intermediate solution
Viscous solution



$\nu = 0.01$

The χ-solution

Advective solution
Intermediate solution
Viscous solution



$$\nu = 0.001$$

Theoretical Results

For the model problem (in 2d) :

- Brezzi, Canuto and Russo (1989)
  - proved for each (δ, σ) pair existence of a solution in C^1.
  - gave the following estimate :

$$\sqrt{\nu}\|u - u_\chi\|_{H^1} + \|u - u_\chi\|_{L^2} \leq C\frac{\nu\delta}{\alpha}.$$

- Canuto and Russo (1993)
  - proved uniqueness of the solution.
  - proved convergence of the iterative procedure.

Literature

F. Brezzi, C. Canuto, A. Russo (1989): A Self-Adaptative Formulation for the Euler/Navier-Stokes Coupling.

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R. Arina, C. Canuto (1994): A chi-formulation of the viscous-inviscid domain decomposition for the Euler/Navier-Stokes equations.

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Summary

Overlapping Method

$\mathcal{L}(\nu\triangle u, u) = f$ $\quad$ $\mathcal{L}(0, u) = f$

Singular Perturbation Strategy

$\mathcal{L}(\nu\triangle u, u) = f$ $\quad$ $\mathcal{L}(0, u) = f$

Boundary Layer Correction

$\mathcal{L}(\nu\triangle u, u) = f$ $\quad$ $\mathcal{L}(\epsilon\triangle u, u) = f$

Factorization

$\mathcal{L}(\nu\triangle u, u) = f$ $\quad$ $\tilde{\mathcal{L}}(0, u) = f$

χ-Formulation

$\mathcal{L}(\nu\chi(\triangle u), u) = f$ $\quad$ $\mathcal{L}(0, u) = f$

Summary : error estimates

Overlapping Method

$\mathcal{O}(\nu^{5/2}) - -\mathcal{O}(\nu^{3/2})$ $\mathcal{O}(\nu)$

Singular Perturbation Strategy

$\mathcal{O}(\nu^{5/2})$ $\mathcal{O}(\nu)$

Boundary Layer Correction

? $\mathcal{O}(\nu)$

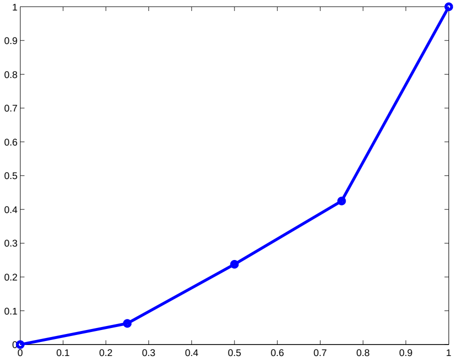
Factorization

$\mathcal{O}(e^{-a/\nu})$ $\mathcal{O}(\nu)$

χ-Formulation

$\mathcal{O}(\nu^{5/2})$ $\mathcal{O}(\nu)$

Convergence of the Newton Algorithm for the χ-formulation



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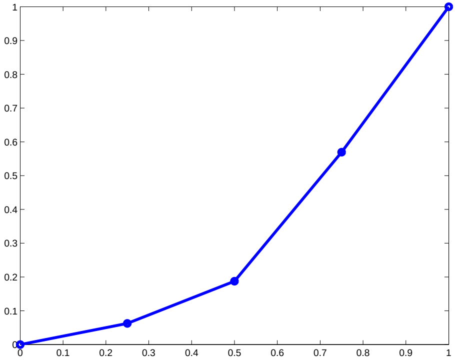

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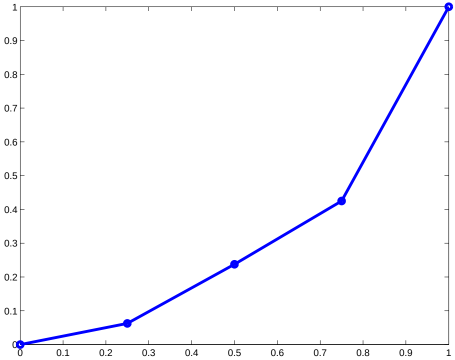

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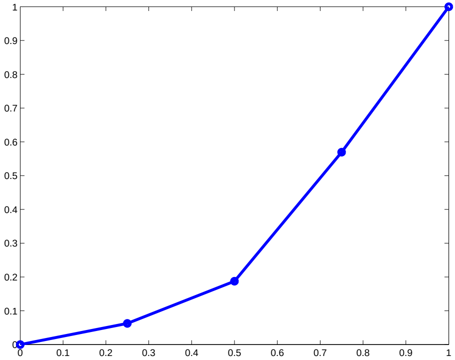

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