Long time behavior of oscillatory solutions to the Maxwell-Landau-Lifshitz equations in one-dimension

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The equations

Maxwell-Landau-Lifshitz

$$(MLL)\begin{cases} \partial_t E - \nabla \times H = 0 \\ \partial_t H + \nabla \times E = -\partial_t M \\ \partial_t M \qquad\qquad = -M \times H. \end{cases}$$

Constant solutions

$$(E, H, M) = (0, \alpha M_0, M_0),\ \ \alpha > 0.$$

Slow variable perturbations

$$\begin{cases} E(t,x) = \varepsilon \widetilde{E}(\varepsilon t, \varepsilon x) \\ H(t,x) = \alpha M_0 + \varepsilon \widetilde{H}(\varepsilon t, \varepsilon x) \\ M(t,x) = M_0 + \varepsilon \widetilde{M}(\varepsilon t, \varepsilon x). \end{cases}$$

A symmetric hyperbolic system

Let

$$u = (\widetilde{E}, \widetilde{H}, \alpha^{\frac{1}{2}} \widetilde{M}),$$

We get a symmetric hyperbolic equation in u,

$$\partial_t u + A(\partial_x) u + \frac{1}{\varepsilon} L_0 u = B(u, u),$$

where

$$A(\partial_x) = \begin{pmatrix} 0 & -\partial_x \times & 0 \\ \partial_x \times & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -M_0 \times & \alpha^{\frac{1}{2}} M_0 \times \\ 0 & \alpha^{\frac{1}{2}} M_0 \times & -\alpha M_0 \times \end{pmatrix},$$

$$B(u, v) = \frac{1}{2} \begin{pmatrix} 0 \\ \alpha^{-\frac{1}{2}} (u^3 \times v^2 + v^3 \times u^2) \\ -(u^3 \times v^2 + v^3 \times u^2) \end{pmatrix}.$$

Cauchy problem and long time stability

1d setting and highly oscillatory initial data

We consider the problem in a 1d setting, with highly oscillatory initial data,

$$\begin{cases} \partial_t v + A(e_1)\partial_y v + \dfrac{1}{\varepsilon} L_0 v = B(v, v), \\ v(0, y) = a(y)e^{iky/\varepsilon} + \overline{a(y)}e^{-iky/\varepsilon} + \varepsilon a_1(y) + \varepsilon^2 a_2(y, iky/\varepsilon). \end{cases} \tag{1}$$

where $e_1 = (1, 0, 0)$, $a, a_1 \in H^s_y, a_2(y, \theta) \in H^1_\theta(H^s_y)$, $s > 1/2 + 3$.
We study the solution on large time interval $O(1/\varepsilon)$. Under appropriate assumptions to a and a_1, we prove that

Theorem

The Cauchy problem (1) admits a unique solution on large time $[0, T/\varepsilon]$ *where* $T > 0$ *is independent of* ε*. Moreover, for such time, the solution is well approximated by a WKB approximate solution of which the leading terms satisfy cubic Schrödinger equations .*

The first issue: Oscillatory initial data

High oscillation

For general $a(y) \in H^s$, we have the estimate

$$|v(0)|_{H^{1/2}} = |a(y)e^{iky/\varepsilon} + \ldots|_{H^{1/2}} = O(\varepsilon^{-1/2}) \to \infty.$$

which is not uniformly bounded with respect to ε.

Profile

We remark that this problem can be overcome by considering the solution of the form

$$v(t, y) = V(t, y, \theta)|_{\theta = \frac{ky - \omega t}{\varepsilon}},$$

The new variable $\theta \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$.

The second issue

Energy estimate

For (1), the classical H^s energy estimate for semilinear symmetric hyperbolic operators then yields

$$|v(t)|_{H^s} \leq |v(0)|_{H^s} + C \int_0^t |v(t')|_{H^s} dt',$$

where the constant C depend on $|v|_{L^\infty}$. Then, with Gronwalls lemma, the bound

$$|v(t)|_{H^s} \leq |v(0)|_{H^s} e^{Ct}.$$

In time $[0, T/\varepsilon]$, the solution is unbounded with respect to ε.

Idea

Main idea

We will study the problem in two steps,

First, Construct an approximate solution in time $O(1/\varepsilon)$,

Second, Show uniform well-posedness with respect to ε in time $O(1/\varepsilon)$ for the perturbation equations about the approximate solution.

Ansatz

Since we want to describe the solution in time $O(1/\varepsilon)$, we look for solutions of the form

$$v(t,x) = V(\tau, t, x, \theta)\big|_{\tau=\varepsilon t, \theta=\frac{kx-\omega t}{\varepsilon}}.$$

Then the equation in V is,

$$\begin{cases} \partial_\tau V + \frac{1}{\varepsilon}\{\partial_t + A(e_1)\partial_y\}V + \frac{1}{\varepsilon^2}\{-\omega\partial_\theta + A(e_1)k\partial_\theta + L_0\}V = \frac{1}{\varepsilon}B(V,V) \\ V(0,0,y,\theta) = V_0(y,\theta) := a(y)e^{i\theta} + \overline{a(y)}e^{-i\theta} + \varepsilon a_1(y) + \varepsilon^2 a_2(y,\theta). \end{cases} \tag{2}$$

Approximate solution

WKB expansion

Firstly, we construct an approximate solution to (2), initiating from prepared initial data on time interval $[0, T]_\tau \times [0, T/\varepsilon]_t$ for some $T > 0$ independent of ε.
It can be done by standard WKB expansion. An approximate solution V^a is constructed, and it takes the form

$$V^a = (V_{0,1}e^{i\theta} + \overline{V}_{0,1}e^{-i\theta}) + \varepsilon(V_{10} + ...) + \varepsilon^2(...),$$

where the components of the leading term V_{01} satisfy cubic Schrödinger equations

Error estimate

Error

We look for exact solution of (1) as a perturbation of the approximate solution $v^a = V^a(\varepsilon t, t, y, (ky - \omega t)/\varepsilon)$. We define the error

$$W(\tau, y, \theta) = \frac{V - V^a}{\varepsilon}(\tau, \frac{\tau}{\varepsilon}, y, \theta),$$

Then the equation in W is

$$\begin{cases} \partial_\tau W + \frac{i}{\varepsilon^2}\mathcal{A}W + \frac{\omega\partial_\theta}{\varepsilon^2}W = \frac{2}{\varepsilon}B(V^a, W) + R \\ W(0, y, \theta) = b(y, \theta) \end{cases} \tag{3}$$

Where the differential operator $\mathcal{A}$ is

$$\mathcal{A} = A(e_1)(\varepsilon D_y + kD_\theta) + L_0/i, \;\; D = \frac{1}{i}\partial$$

We now show the wellposedness of (3) on $[0, T]_\tau$.

Error estimate

Exponential cancellations

We observed that equation (3), together with an initial datum of size $O(1)$, can be likened to an ordinary differential equation,

$$y' + \frac{i\alpha}{\varepsilon^2} y = \frac{1}{\varepsilon} y^2, \tag{4}$$

with an initial data $y(0) = y_0$. The singular source term in the right-hand side may cause the solution to blow-up in small time $O(\varepsilon)$, but exponential cancellations are expected to happen because of the rapid oscillations created by the term in $1/\varepsilon^2$.

Error estimate

Projection

To investigate these exponential cancellations, it is natural to project the source term B/ε over the eigendirections of $\mathcal{A}$. We write the spectral decomposition of $\mathcal{A}$ as follows:

$$\mathcal{A} = \sum_{nz} \lambda_{nz} \Pi_{nz} + \sum_{z} \lambda_z \Pi_z.$$

The real eigenvalues λ_{nz} are called non-zero modes, while the real eigenvalues λ_z are called zero modes. Let the total eigenprojectors,

$$\Pi_0 = \sum_{nz} \Pi_{nz}, \quad \Pi_s = \sum_{z} \Pi_z.$$

Error estimate

Change of variable

We consider the change of variable,

$$W_1 = (\Pi_0 W, \Pi_s W)$$

Then the equation in W_1 is

$$\partial_\tau W_1 + \frac{i}{\varepsilon^2}\mathcal{A} W_1 + \frac{\omega\partial_\theta}{\varepsilon^2} W_1 = \frac{2}{\varepsilon} B_1 W_1 + \widetilde{R}.$$

Where

$$B_1 = \begin{pmatrix} \Pi_0 B(V^a)\Pi_0 & \Pi_0 B(V^a)\Pi_s \\ \Pi_s B(V^a)\Pi_0 & \Pi_s B(V^a)\Pi_s \end{pmatrix} = \begin{pmatrix} O(\varepsilon) & O(1) \\ B_{s0} & O(\varepsilon) \end{pmatrix}, \ \widetilde{R} = \begin{pmatrix} \Pi_0 R \\ \Pi_s R \end{pmatrix}.$$

We show that the term B_{s0} in the left block is transparent, it can be eliminated by normal form reduction. While the $O(1)$ term in the right block do not satisfy the transparency condition.

Error estimate

Normal form reduction

To eliminate the transparent singular term B_{s0}/ε, we consider a change of variable in the form

$$W_2 := (\mathrm{Id} + \varepsilon N)^{-1} W_1,$$

where the differential operator N is determined later, and is of the form

$$N := \begin{pmatrix} 0 & 0 \\ N_{s0} & 0 \end{pmatrix}.$$

Then the equation in W_2 is

$$\begin{aligned} \partial_\tau W_2 + \frac{i}{\varepsilon^2}\mathcal{A}W_2 - \frac{1}{\varepsilon^2}\omega\partial_\theta W_2 &= \frac{1}{\varepsilon}\{-\varepsilon^2\partial_\tau N - i[\mathcal{A}, N] + \omega\partial_\theta N + \begin{pmatrix} 0 \\ 2B_{s0} \end{pmatrix} \\ &\quad + \frac{2}{\varepsilon}\begin{pmatrix} 0 & O(1) \\ 0 & 0 \end{pmatrix} W_2 + \widetilde{R} + O(\varepsilon). \end{aligned}$$

Error estimate

Resonance and transparency

We look for N solution of the homological equation,

$$-\varepsilon^2\partial_\tau N - i[\mathcal{A}, N] + \omega\partial_\theta N + 2\begin{pmatrix} 0 & 0 \\ B_{s0} & 0 \end{pmatrix} = O(\varepsilon^2).$$

The equation takes the form $\Phi N_{s0} = B_{s0} + O(\varepsilon^2)$, where $\Phi = \lambda_{j'} - \lambda_j - p\omega$. The equation $\Phi = 0$ is the resonance equation. The crucial transparency assumption states that the interaction coefficient B_{s0} is sufficiently small at the resonances, that is,

$$|B_{s0}| \leq C|\Phi|.$$

Error estimate

New system after normal form reduction

The transparency is satisfied for (MLL), the above homological equation can be solved. Then the equation in W_2 becomes

$$\partial_\tau W_2 + \frac{i}{\varepsilon^2}\mathcal{A}W_2 - \frac{1}{\varepsilon^2}\omega\partial_\theta W_2 = \widetilde{B}W_2 + \widetilde{R} + O(\varepsilon).$$

where the linear operator $\widetilde{B}$ is of the form

$$\widetilde{B} = \begin{pmatrix} O(1) & O(1/\varepsilon) \\ O(\varepsilon) & 0 \end{pmatrix}.$$

Error estimate

Rescaling

We now rescale the solution,

$$W_3 = \begin{pmatrix} W_{30} \\ W_{3s} \end{pmatrix} := \begin{pmatrix} W_{20} \\ \varepsilon^{-1} W_{2s} \end{pmatrix}.$$

Then the equation in W_3 is

$$\partial_\tau W_3 + \frac{i}{\varepsilon^2}\mathcal{A} W_3 - \frac{1}{\varepsilon^2}\omega\partial_\theta W_3 = \widetilde{B_0} W_3 + \widetilde{R} + O(1).$$

with the notation

$$\widetilde{B_0} = \begin{pmatrix} O(1) & O(1) \\ O(1) & 0 \end{pmatrix}.$$

Then local well-posedness follows, with existence time independent of ε.