Entropy-based artificial viscosity

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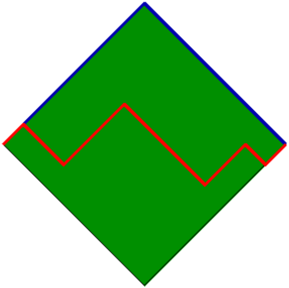
Introduction

Why L1 for PDEs?

- Solve 1D eikonal

$$|u'(x)| = 1, \quad u(0) = 0, \; u(1) = 0$$

- Exists infinitely many weak solutions



Why L1 for PDEs?

- Exists a unique (positive) viscosity solution, u

$$|u_\epsilon'| - \epsilon u_\epsilon'' = 1, \quad u_\epsilon(0) = 0, \ u_\epsilon(1) = 0.$$

- $\|u - u_\epsilon\|_{H^1} \leq c\epsilon^{\frac{1}{2}}$,
- Sloppy approximation.

Why L1 for PDEs?

One can do better with L^1 (of course)

- Define mesh $\mathcal{T}_h = \cup_{i=0}^{N}[x_i, x_{i+1}]$, $h = x_{i+1} - x_i$.
- Use continuous finite elements of degree 1.

$$V = \{v \in \mathcal{C}^0[0,1];\ v_{|[x_i,x_{i+1}]} \in \mathbb{P}_1,\ \ v(0) = v(1) = 0\}.$$

Why L1 for PDEs?

- Consider $p > 1$ and set

$$J(v) = \underbrace{\int_0^1 \big||v'| - 1\big| \, \mathrm{d}x}_{L^1\text{-norm of residual}} + \underbrace{h^{2-p} \sum_1^N (v'(x_i^+) - v'(x_i^-))_+^p}_{\text{Entropy}}$$

- Define $u_h \in V$

$$\boxed{u_h = \arg\min_{v \in V} J(v)}$$

Why L1 for PDEs?

- Implementation: use mid-point quadrature

$$J_h(v) = \underbrace{\sum_{i=0}^{N} h \left| |v'(x_{i+\frac{1}{2}})| - 1 \right|}_{\ell^1\text{-norm of residual}} + \text{Entropy}.$$

- Define

$$\boxed{\widetilde{u}_h = \arg\min_{v \in V} J_h(v)}$$

Why L1 for PDEs?

Theorem (J.-L. G.&B. Popov (2008))

$u_h \longrightarrow u$ and $\widetilde{u}_h \longrightarrow u$ strongly in $W^{1,1}(0,1) \cap \mathcal{C}^0[0,1]$.

- Fast solution in 1D (JLG&BP 2010) and in higher dimension (fast-marching/fast sweeping, Osher/Sethian) to compute $\widetilde{u}_h$.
- Similar results in 2D for convex Hamiltonians (JLG&BP 2008).

A new idea based on L^1 minimization

Some provable properties of minimizer $\tilde{u}_h$
(JLG&BP 2008, 2009, 2010). Minimizer $\tilde{u}_h$ is such that:

- Residual is SPARSE:

$$|\tilde{u}_h'(x_{i+\frac{1}{2}})| - 1 = 0, \quad \forall i \text{ such that } \frac{1}{2} \notin [x_i, x_{i+1}].$$

- Entropy makes it so that graph of $\tilde{u}_h'(x)$ is concave down in $[x_i, x_{i+1}] \ni \frac{1}{2}$.

A new idea based on L^1 minimization

Conclusion:

- Residual is SPARSE: PDE solved almost everywhere. Entropy does not play role in those cells.
- Entropy plays a key role only in cell where PDE is not solved.

Can L1 help anyway?

New idea:

- Go back to the notion of viscosity solution
- Add smart viscosity to the PDE:

$$|u_\epsilon'| - \partial_x(\epsilon(u_\epsilon)\partial_x u_\epsilon) = 1$$

- Make ϵ depend on the entropy production
  1. Viscosity large (order h) where entropy production is large
  2. Viscosity vanish when no entropy production
- Entropy plays a key role in cell where PDE is not solved.

NONLINEAR SCALAR CONSERVATION EQUATIONS

Transport, mixing

The PDE

- Solve the transport equation

$$\partial_t u + \beta\cdot\nabla u = 0, \quad u|_{t=0} = u_0, \quad +\text{BCs}$$

- Use standard discretizations (ex: continuous finite elements)
- Deviate as little possible from Galerkin.

The idea

Entropy for linear transport?

- Notion of renormalized solution (DiPerna/Lions (1989))
  Good framework for non-smooth transport.
- $\forall E \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ is an entropy
- If solution is smooth $\Rightarrow$ $E(u)$ solves PDE, $\forall E \in \mathcal{C}^1(\mathbb{R};\mathbb{R})$ (multiply PDE by $E'(u)$ and apply chain rule)

$$\underbrace{\partial_t E(u) + \beta\cdot\nabla E(u) = 0}_{\text{Entropy residual}}$$

The idea

Key idea 1:

Use entropy residual to construct viscosity

The idea

viscosity $\sim$ entropy residual

The idea

viscosity $\sim$ entropy residual

- Viscosity $\sim$ residual (Hughes-Mallet (1986) Johnson-Szepessy (1990))
- Entropy Residual $\sim$ a posteriori estimator (Puppo (2003))
- Add entropy to formulation (For Hamilton-Jacobi equations Guermond-Popov (2007))
- Application to nonlinear conservation equations (Guermond-Pasquetti (2008))

The algorithm + time discretization

- Numerical analysis 101:

  Up-winding=centered approx + $\frac{1}{2}|\beta|h$ viscosity

- Proof:

$$\beta_i \frac{u_i - u_{i-1}}{h_i} = \beta_i \frac{u_{i+1} - u_{i-1}}{2h_i} - \frac{1}{2}\beta_i h_i \frac{u_{i+1} - 2u_i + u_{i-1}}{h_i}$$

The algorithm + time discretization

Key idea 2:

Entropy viscosity should not exceed $\frac{1}{2}|\beta|h$

The algorithm

- Choose one entropy functional.
  EX1: $E(u) = |u - \overline{u_0}|$,
  EX2: $E(u) = (u - \overline{u_0})^2$, etc.
- Define entropy residual $D_h := \partial_t E(u_h) + \beta \cdot \nabla E(u_h)$,
- Define local mesh size of cell K: $h_K = \text{diam}(K)/p^2$
- Construct a wave speed associated with this residual on each mesh cell K:

$$v_K := h_K \|D_h\|_{\infty,K} / \overline{E(u_h)}$$

- Define entropy viscosity on each mesh cell K:

$$\nu_K := h_K \min(\frac{1}{2}\|\beta\|_{\infty,K}, v_K)$$

Summary

- Space approximation: Galerkin + entropy viscosity:

$$\underbrace{\int_\Omega (\partial_t u_h + \beta\cdot\nabla u_h) v_h \mathrm{d}\mathbf{x}}_{\text{Galerkin(centered approximation)}} + \underbrace{\sum_K \int_K \nu_K \nabla u_h \nabla v_h \mathrm{d}\mathbf{x}}_{\text{Entropy viscosity}} = 0, \quad \forall v_h$$

- Time approximation: Use an explicit time stepping: BDF2, RK3, RK4, etc.
- Idea: make the viscosity explicit $\Rightarrow$ Stability under CFL condition.

Space + time discretization

- EX: 2nd-order centered finite differences 1D
- Compute the entropy residual D_i on each cell (x_i, x_{i+1})

$$D_i := \max\left(\left| \frac{E(u_i^n) - E(u_i^{n-1})}{\Delta t} + \beta_{i+\frac{1}{2}} \frac{E(u_{i+1}^n) - E(u_i^n)}{h_i} \right|, \left| \frac{E(u_{i+1}^n) - E(u_{i+1}^{n-1})}{\Delta t} + \beta_{i+\frac{1}{2}} \frac{E(u_{i+1}^n) - E(u_i^n)}{h_i} \right| \right)$$

- Compute the entropy viscosity

$$\nu_i^n := h_i \min\left(\frac{1}{2} |\beta_{i+\frac{1}{2}}|, \frac{1}{2} \frac{D_i}{\overline{E(u^n)}} h_i \right)$$

Space + time discretization

- Use RK to solve on next time interval $[t^n, t^n + \Delta t]$

$$u_i(t = t^n) = u_i^n$$

$$\partial_t u_i + \underbrace{\beta_{i+\frac{1}{2}} \frac{u_{i+1} - u_{i-1}}{2\overline{h_i}}}_{\text{Centered approximation}} - \underbrace{\left(\nu_i^n \frac{u_{i+1} - u_i}{h_i} - \nu_{i-1}^n \frac{u_i - u_{i-1}}{h_{i-1}} \right)}_{\text{Centered viscous fluxes}} = 0$$

- The entropy viscosity can be computed on the fly for some RK techniques.

Space + time discretization: RK2 midpoint

- Advance half time step to get w^n

$$w_i^n = u_i^n - \frac{1}{2}\Delta t \beta_{i+\frac{1}{2}} \frac{u_{i+1}^n - u_{i-1}^n}{2\overline{h_i}}$$

- Compute entropy viscosity on the fly

$$D_i := \max\left(\left| \frac{E(w_i^n) - E(u_i^n)}{\Delta t/2} + \beta_{i+\frac{1}{2}} \frac{E(w_{i+1}^n) - E(w_i^n)}{h_i} \right|, \left| \frac{E(w_{i+1}^n) - E(u_{i+1}^n)}{\Delta t/2} + \beta_{i+\frac{1}{2}} \frac{E(w_{i+1}^n) - E(w_i^n)}{h_i} \right| \right)$$

- Compute u^{n+1}

$$u_i^{n+1} = u_i^n - \Delta t \beta_{i+\frac{1}{2}} \frac{w_{i+1}^n - w_{i-1}^n}{2\overline{h_i}}$$

$$+ \left(\nu_i^n \frac{w_{i+1}^n - w_i^n}{} - \nu_{i-1}^n \frac{w_i^n - w_{i-1}^n}{} \right)$$

Theory for linear steady equations

- Consider

$$\partial_t u + \beta \cdot \nabla u = f, \quad u|_{\Gamma^-} = 0.$$

Theory for linear steady equations

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$$\partial_t u + \beta \cdot \nabla u = f, \quad u|_{\Gamma^-} = 0.$$

Theorem

Let u_h be the finite element approximation with Euler time approximation and u^2 entropy viscosity, then u_h converges to u.

Theory for linear steady equations

- Consider

$$\partial_t u + \beta \cdot \nabla u = f, \quad u|_{\Gamma^-} = 0.$$

Theorem

Let u_h be the finite element approximation with Euler time approximation and u^2 entropy viscosity, then u_h converges to u.

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Theorem

Let u_h be the $\mathbb{P}_1$ finite element approximation with RK2 time approximation and u^2 entropy then u_h converges to u.

Conjecture

The results should hold for nonlinear scalar conservation laws with convex Lipschitz flux

Theory for linear steady equations

Why convergence is so difficult to prove?

- Key a priori estimate

$$\int_0^T \nu(u)|\nabla u|^2 \mathrm{d}\mathbf{x} \leq c$$

- Ok in $\{\nu(u)(\mathbf{x},t) = \frac{1}{2}\|\beta\| h\}$ (non-smooth region)
- The estimate is useless in smooth region.
- Explicit time stepping makes the viscosity depend on the past.

1D Numerical tests, BV solution

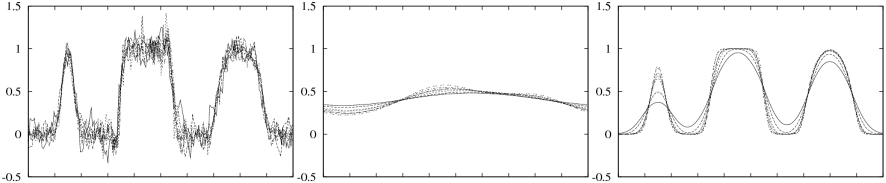
- linear transport

$$\partial_t u + \partial_x u = 0, \quad u_0(x) = \begin{cases} e^{-300(2x-0.3)^2} & \text{if } |2x-0.3| \leq 0.25, \\ 1 & \text{if } |2x-0.9| \leq 0.2, \\ \left(1 - \left(\frac{2x-1.6}{0.2}\right)^2\right)^{\frac{1}{2}} & \text{if } |2x-1.6| \leq 0.2, \\ 0 & \text{otherwise.} \end{cases}$$

- Periodic boundary conditions.

1D Numerical tests, BV solution, Spectral elements

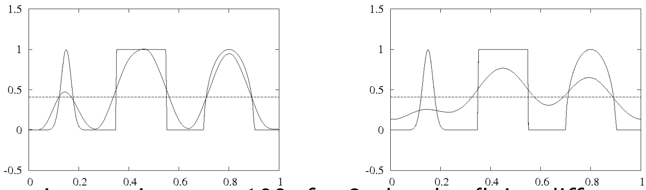
- Spectral elements in 1D on random meshes.
- Long time integration, 100 periods.



Long time integration, $t = 100$, for polynomial degrees $k = 2, \ldots 8$, #d.o.f.=200. Galerkin (left); Constant viscosity (center); Entropy viscosity (right).

1D Numerical tests, BV solution, Finite differences

- Second-order finite differences in 1D on uniform and random meshes.
- Long time integration, 100 periods.



Long time integration, $t = 100$, for 2nd order finite differences #d.o.f.=200. Uniform mesh (left); Random mesh (right).

Numerical tests, smooth solution

- $\Omega = \{(x, y) \in \mathbb{R}^2, \sqrt{x^2 + y^2} \leq 1\} := B(0, 1)$,
- Speed: rotation about origin, angular speed 2π
- $u(x, y) = \frac{1}{2}\left(1 - \tanh\left(\frac{(x - r_0 \cos(2\pi t))^2 + (y - r_0 \sin(2\pi t))^2}{a^2} - 1\right) + 1\right)$,
- $a = 0.3,\ r_0 = 0.4$

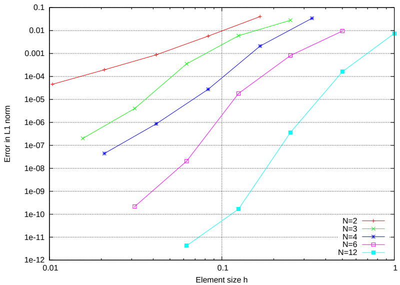
2D numerical tests, smooth solution, $\mathbb{P}_1$ FE

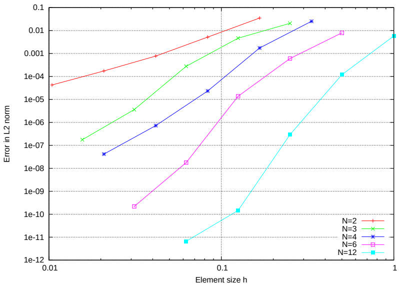
- $\mathbb{P}_1$ finite elements

h	$\mathbb{P}_1$ Stab.			
	L^2	rate	L^1	rate
2.00E-1	2.5893E-1	-	3.6139E-1	-
1.00E-1	9.7934E-2	1.403	1.3208E-1	1.452
5.00E-2	1.9619E-3	2.320	2.7310E-3	2.274
2.50E-2	3.5360E-4	2.472	5.1335E-3	2.411
1.25E-2	6.4959E-4	2.445	1.0061E-3	2.351
1.00E-2	3.9226E-4	2.261	6.3555E-4	2.058
6.25E-3	1.4042E-4	2.186	2.3829E-4	2.087

Table: $\mathbb{P}_1$ approximation.

2D numerical tests, smooth solution, spectral elements





Linear transport problem with smooth initial condition. Errors in L^1 (at left) and L^2 (at right) norms vs h for $N = 2, 4, 6, 8, 12$.

2D Numerical tests, BV solution

- $\Omega = \{(x, y) \in \mathbb{R}^2, \sqrt{x^2 + y^2} \leq 1\} := B(0, 1)$,
- Speed: rotation about origin, angular speed 2π
- $u(x, y) = \chi_{B(0,a)}(\sqrt{(x - r_0 \cos(2\pi t))^2 + (y - r_0 \sin(2\pi t))^2})$,
- $a = 0.3$, $r_0 = 0.4$

2D Numerical tests, BV solution, $\mathbb{P}_2$ FE

- $\mathbb{P}_2$ finite elements

h	$\mathbb{P}_2$ Stab.			
	L^2	rate	L^1	rate
2.00E-1	1.0930E-1	-	4.3373E-2	-
1.00E-1	7.3222E-2	0.578	2.3771E-2	0.868
5.00E-2	5.5707E-2	0.394	1.3704E-2	0.795
2.50E-2	4.2522E-2	0.389	8.0365E-3	0.770
1.25E-2	3.2409E-2	0.392	4.6749E-3	0.782
1.00E-2	2.9812E-2	0.374	3.9421E-3	0.764
6.25E-3	2.4771E-2	0.394	2.7200E-3	0.790

Table: $\mathbb{P}_2$ approximation.

NONLINEAR SCALAR CONSERVATION EQUATIONS

Johannes Martinus Burgers

2D Nonlinear scalar conservation laws

- Solve

$$\partial_t u + \partial_x f(u) + \partial_y g(u) = 0 \quad u|_{t=0} = u_0, \quad +\text{BCs.}$$

- The unique entropy solution satisfies

$$\partial_t E(u) + \partial_x F(u) + \partial_y G(u) \leq 0$$

for all entropy pair $E(u)$, $F(u) = \int E'(u)f'(u)\mathrm{d}u$, $G(u) = \int E'(u)g'(u)\mathrm{d}u$

2D scalar nonlinear conservation laws

- Choose one entropy $E(u)$
- Define entropy residual, $D_h(u) := \partial_t E(u) + \partial_x F(u) + \partial_y G(u)$
- Define local mesh size of cell K: $h_K = \text{diam}(K)/p^2$
- Construct a speed associated with residual on each cell K:

$$v_K := h_K \|D_h\|_{\infty,K} / \overline{E(u_h)}$$

- Compute maximum local wave speed: $\beta_K = \|\sqrt{f'(u)^2 + g'(u)^2}\|_{\infty,K}$
- Define entropy viscosity on each mesh cell K:

$$\nu_K := h_K \min(\frac{1}{2}\beta_K, v_K)$$

Summary

- Space approximation: Galerkin + entropy viscosity:

$$\underbrace{\int_\Omega (\partial_t u_h + \partial_x f(u_h) + \partial_y g(u_h)) v_h \mathrm{d}\mathbf{x}}_{\text{Galerkin (centered approximation)}} + \underbrace{\sum_K \int_K \nu_K \nabla u_h \nabla v_h \mathrm{d}\mathbf{x}}_{\text{Entropy viscosity}} = 0, \quad \forall v$$

- Time approximation: explicit RK

The algorithm + time discretization

EX: 2nd-order centered finite differences 1D

- Compute local speed on on each cell (x_i, x_{i+1})

$$\beta_{i+\frac{1}{2}} := \frac{1}{2}(f'(u_i) + f'(u_{i+1}))$$

- Compute the entropy residual D_i on each cell (x_i, x_{i+1})

$$D_i := \max\left(\left|\frac{E(u_i^n) - E(u_i^{n-1})}{\Delta t} + \beta_{i+\frac{1}{2}}\frac{E(u_{i+1}^n) - E(u_i^{n-1})}{h_i}\right|,\right.$$
$$\left.\left|\frac{E(u_{i+1}^n) - E(u_{i+1}^{n-1})}{\Delta t} + \beta_{i+\frac{1}{2}}\frac{E(u_{i+1}^n) - E(u_i^{n-1})}{h_i}\right|\right)$$

The algorithm + time discretization

- Compute the entropy viscosity

$$\nu_i^n := h_i \min\left(\frac{1}{2}|\beta_{i+\frac{1}{2}}|, \frac{1}{2}\frac{D_i}{\overline{\overline{E(u^n)}}} h_i\right)$$

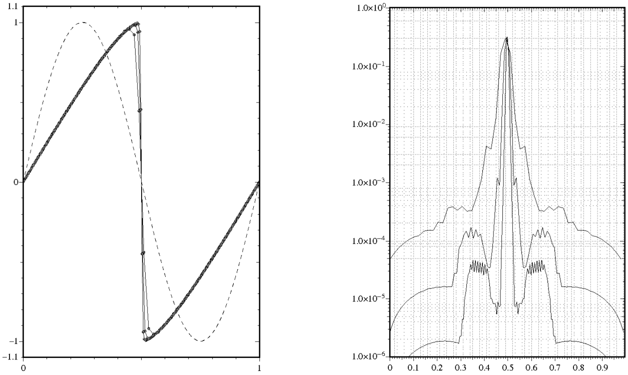
- Use RK to solve on next time interval $[t^n, t^n + \Delta t]$

$$u_i(t = t^n) = u_i^n$$

$$\partial_t u_i + \frac{f(u_{i+1}) - f(u_{i-1})}{2\overline{h_i}} - \left(\nu_i^n \frac{u_{i+1} - u_i}{h_i} - \nu_{i-1}^n \frac{u_i - u_{i-1}}{h_{i-1}}\right) = 0$$

EX: 1D burgers + 2nd-order Finite Differences

- Second-order Finite Differences + RK2/RK3/RK4

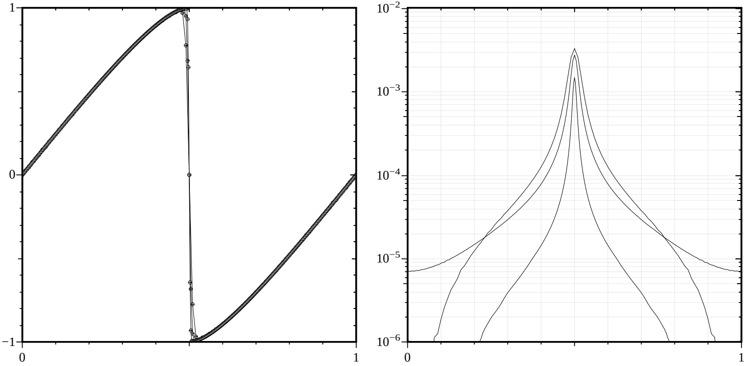


u_h $\qquad$ $\nu_h(u_h)|\partial_x u_h|$

Burgers, $t = 0.25$, $N = 50$, 100, and 200 grid points.

EX: 1D burgers + Fourier

- Solution method: Fourier + RK4 + entropy viscosity



u_N $\qquad$ $\nu_N(u_N)$

Burgers at $t = 0.25$ with $N = 50$, 100, and 200.

EX: 1D Nonconvex flux + Fourier (WENO5 + SuperBee (or minmod 2) fails)

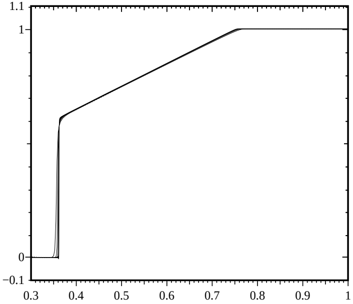
- Consider $\partial_t + \partial_x f(u) = 0$, $u(x,0) = u_0(x)$

$$f(u) = \begin{cases} \frac{1}{4}u(1-u) & \text{if } u < \frac{1}{2}, \\ \frac{1}{2}u(u-1) + \frac{3}{16} & \text{if } u \geq \frac{1}{2}, \end{cases} \qquad u_0(x) = \begin{cases} 0, & x \in (0, 0.25], \\ 1, & x \in (0.25, 1] \end{cases}$$

EX: 1D Nonconvex flux + Fourier (WENO5 + SuperBee (or minmod 2) fails)

- Consider $\partial_t + \partial_x f(u) = 0$, $u(x, 0) = u_0(x)$

$$f(u) = \begin{cases} \frac{1}{4}u(1-u) & \text{if } u < \frac{1}{2}, \\ \frac{1}{2}u(u-1) + \frac{3}{16} & \text{if } u \geq \frac{1}{2}, \end{cases} \qquad u_0(x) = \begin{cases} 0, & x \in (0, 0.25], \\ 1, & x \in (0.25, 1] \end{cases}$$



Non-convex flux problem u_N at $t = 1$ with $N = 200$, 400, 800, and 1600.

Convergence tests, 2D Burgers

- Solve 2D Burgers

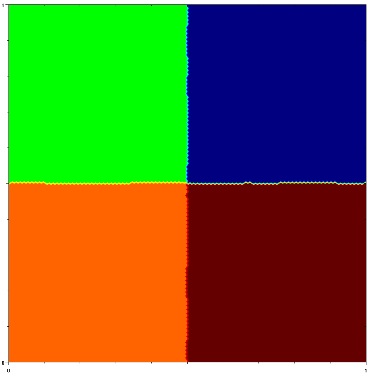
$$\partial_t u + \partial_x(\frac{1}{2}u^2) + \partial_y(\frac{1}{2}u^2) = 0$$

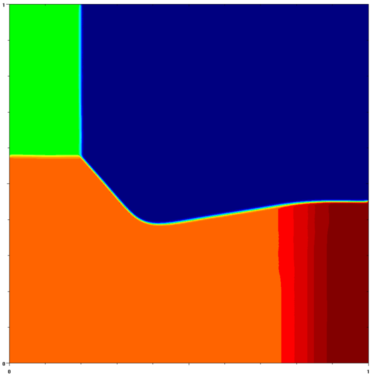
- Subject to the following initial condition

$$u(x,y,0) = u^0(x,y) = \begin{cases} -0.2 & \text{if } \; x < 0.5 \text{ and } y > 0.5 \\ -1 & \text{if } \; x > 0.5 \text{ and } y > 0.5 \\ 0.5 & \text{if } \; x < 0.5 \text{ and } y < 0.5 \\ 0.8 & \text{if } \; x > 0.5 \text{ and } y < 0.5 \end{cases}$$

- Compute solution in $(0,1)^2$ at $t = \frac{1}{2}$.

Convergence tests, 2D Burgers





Initial data

$\mathbb{P}_1$ FE, $3\,10^4$ nodes

Convergence tests, 2D Burgers

h	$\mathbb{P}_1$			
	L^2	rate	L^1	rate
5.00E-2	2.3651E-1	–	9.3661E-2	–
2.50E-2	1.7653E-1	0.422	4.9934E-2	0.907
1.25E-2	1.2788E-1	0.465	2.5990E-2	0.942
6.25E-3	9.3631E-2	0.449	1.3583E-2	0.936
3.12E-3	6.7498E-2	0.472	6.9797E-3	0.961

Table: Burgers, $\mathbb{P}_1$ approximation.

Convergence tests, 2D Burgers

h	$\mathbb{P}_2$			
	L^2	rate	L^1	rate
5.00E-2	1.8068E-1	–	5.2531E-2	–
2.50E-2	1.2956E-1	0.480	2.7212E-2	0.949
1.25E-2	9.5508E-2	0.440	1.4588E-2	0.899
6.25E-3	6.8806E-2	0.473	7.6435E-3	0.932

Table: Burgers, $\mathbb{P}_2$ approximation.

Buckley Leverett, $\mathbb{P}_2$ FE

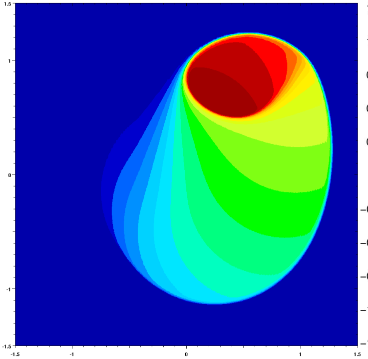
- Solve $\partial_t u + \partial_x f(u) + \partial_y g(u) = 0$.

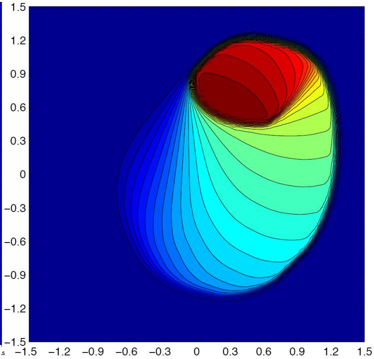
$$f(u) = \frac{u^2}{u^2+(1-u)^2}, \qquad g(u) = f(u)(1-5(1-u)^2)$$

Non-convex fluxes (composite waves)

$$u(x,y,0) = \begin{cases} 1, & \sqrt{x^2+y^2} \le 0.5 \\ 0, & \text{else} \end{cases}$$

Buckley Leverett, $\mathbb{P}_2$ FE





KPP (WENO + superbee limiter fails), $\mathbb{P}_2$ FE

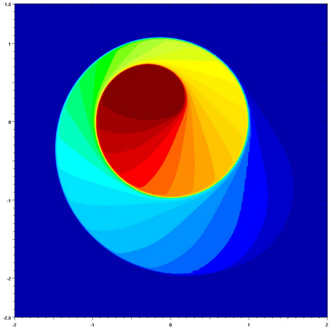
- Solve $\partial_t u + \partial_x f(u) + \partial_y g(u) = 0$.

$$f(u) = \sin(u), \qquad g(u) = \cos(u)$$

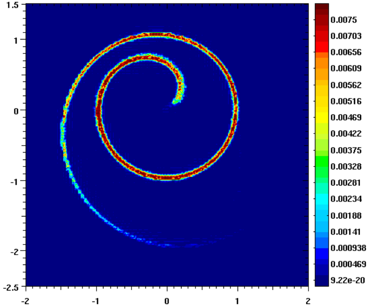
Non-convex fluxes (composite waves)

$$u(x,y,0) = \begin{cases} \frac{7}{2}\pi, & \sqrt{x^2+y^2} \leq 1 \\ \frac{1}{4}\pi, & \text{else} \end{cases}$$

KPP (WENO + superbee limiter fails)



$\mathbb{P}_2$ approx



$\mathbb{Q}_4$ entrop visco.

NONLINEAR SCALAR CONSERVATION EQUATIONS

Leonhard Euler

Euler flows

- Solve compressible Euler equations

$$\partial_t \rho + \nabla\cdot(\rho\mathbf{u}) = 0$$
$$\partial_t(\rho\mathbf{u}) + \nabla\cdot(\rho\mathbf{u}\otimes\mathbf{u} + p\mathbb{I}) = 0$$
$$\partial_t(E) + \nabla\cdot(\mathbf{u}(E+p)) = 0$$
$$\rho e = E - \frac{1}{2}\rho\mathbf{u}^2, \quad T = (\gamma-1)e \quad T = \frac{p}{\rho}$$

Initial data + BCs

- Use continuous finite elements of degree p.
- Deviate as little possible from Galerkin.

The algorithm

- Compute the entropy $S_h = \frac{\rho_h}{\gamma-1}\log(p_h/\rho_h^\gamma)$
- Define entropy residual, $D_h := \partial_t S_h + \nabla\cdot(\mathbf{u}_h S_h)$
- Define local mesh size of cell K: $h_K = \mathrm{diam}(K)/p^2$
- Construct a speed associated with residual on each cell K:

$$v_K := h_K \|D_h\|_{\infty,K}$$

- Compute maximum local wave speed:
  $\beta_K = \| \|\mathbf{u}\| + (\gamma T)^{\frac{1}{2}} \|_{\infty,K}$

The algorithm

- Use Navier-Stokes regularization: define μ_K and κ_K.
- Entropy viscosity and thermal conductivity on each mesh cell K:

$$\mu_K := h_K \min(\frac{1}{2}\beta_K \|\rho_h\|_{\infty,K}, v_K), \quad \kappa_K = \mathcal{P}\mu_K$$

- In practice use $\boxed{\mathcal{P} = \frac{1}{10}}$, .
- Solution method: Galerkin + entropy viscosity + thermal conductivity

1D Euler flows + Fourier

- Solution method: Fourier + RK4 + entropy viscosity

1D Euler flows + Fourier

- Solution method: Fourier + RK4 + entropy viscosity

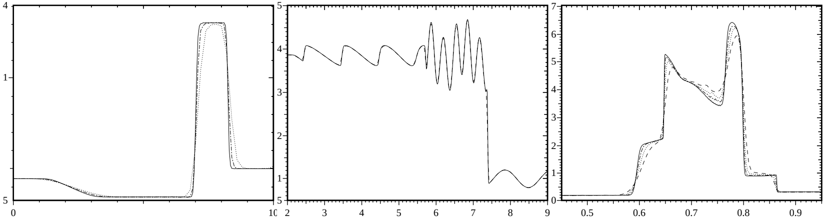


Figure: Lax shock tube, $t = 1.3$, 50, 100, 200 points. Shu-Osher shock tube, $t = 1.8$, 400, 800 points. Right: Woodward-Collela blast wave, $t = 0.038$, 200, 400, 800, 1600 points.

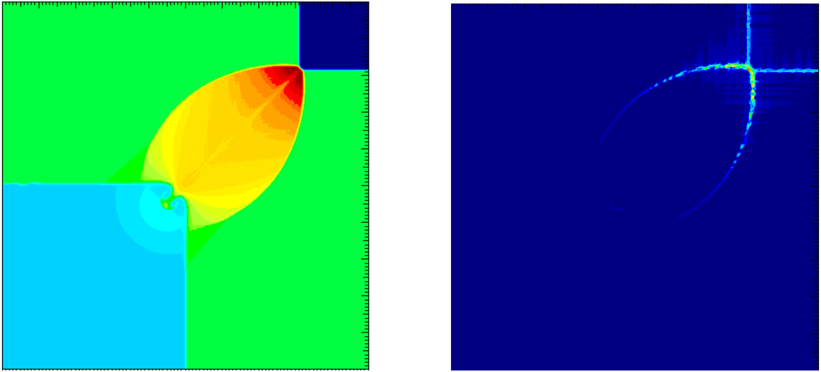
2D Euler flows + Fourier

- Domain $\Omega = (-1,1)^2$
- Rieman problem with the initial condition:

$$\begin{aligned}
&0 < x < 0.5 \text{ and } 0 < y < 0.5, \quad p = 1, \rho = 0.8, \mathbf{u} = (0,0),\\
&0 < x < 0.5 \text{ and } 0.5 < y < 1, \quad p = 1, \rho = 1, \mathbf{u} = (0.7276,0),\\
&0.5 < x < 1 \text{ and } 0 < y < 0.5, \quad p = 1, \rho = 1, \mathbf{u} = (0,0.7276),\\
&0 < x < 0.5 \text{ and } 0.5 < y < 1, \quad p = 0.4, \rho = 0.5313, \mathbf{u} = (0,0).
\end{aligned}$$

- Solution at time $t = 0.2$.

2D Euler flows + Fourier (Riemann test case 12)

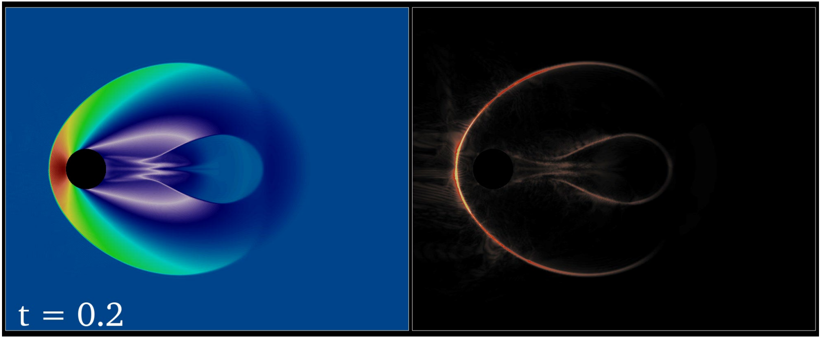


Euler benchmark, Fourier approximation: Density (at left), $0.528 < \rho_N < 1.707$ and viscosity (at right), $0 < \mu_N < 3.410^{-3}$, at $t = 0.2$, $N = 400$.

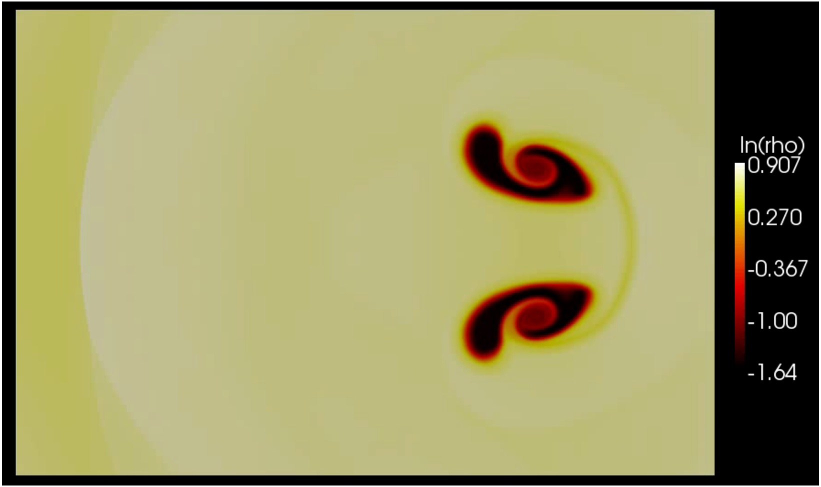
Riemann problem test case no 12, $\mathbb{P}_1$ FE

movie, Riemann no 12

Cylinder in a channel, Mach 2, $\mathbb{P}_1$ FE (By M. Nazarov, KTH)



Bubble, density ratio 10^{-1}, Mach 1.65, $\mathbb{P}_1$ FE (by M. Nazarov, KTH)



Mach 3 Wind Tunnel with a Step, $\mathbb{P}_1$ finite elements

- Mach 3 Wind Tunnel with a Step (Standard Benchmark since Woodward and Colella (1984))
- Inflow boundary, density 1.4, pressure 1, and x-velocity 3, (Mach =3)

Mach 3 Wind Tunnel with a Step, $\mathbb{P}_1$ finite elements

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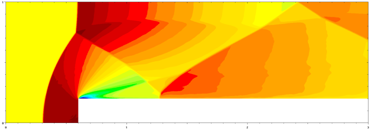
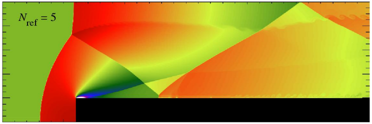


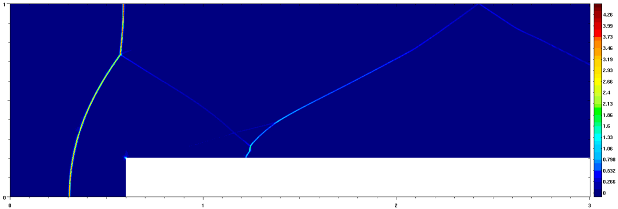


$\mathbb{P}_1$ FE, $1.3\,10^5$ nodes
Log(density)

Flash Code, adaptive *PPM*, $\sim 4.9\,10^6$ nodes

Movie, density field (entropy visc.) Movie, density field (viscous)

Mach 3 Wind Tunnel with a Step, $\mathbb{P}_1$ finite elements



Viscous flux of entropy Viscosity.

Mach 10 Double Mach reflection, $\mathbb{P}_1$ finite elements

- Right-moving Mach 10 shock makes 60^o angle with x-axis (Standard Benchmark, Woodward and Colella (1984))
- Shock interacts with flat plate $x \in (\frac{1}{6}, +\infty)$.
- The un-shocked fluid $\rho = 1.4$, $p = 1$, and $\mathbf{u} = 0$

Mach 10 Double Mach reflection, $\mathbb{P}_1$ finite elements

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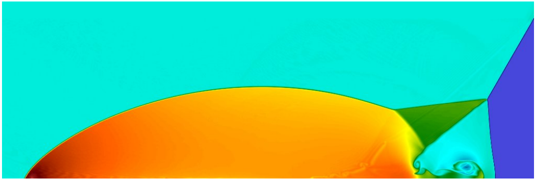
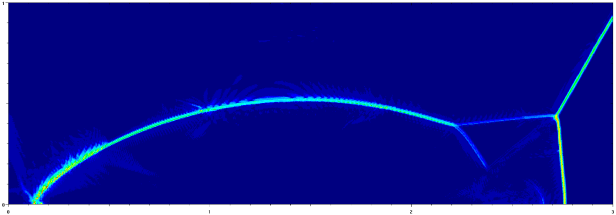


$\mathbb{P}_1$ FE, $4.5\,10^5$ nodes, $t = 0.2$

Movie, density field

Mach 10 Double Mach reflection



Entropy Viscosity

NONLINEAR SCALAR CONSERVATION EQUATIONS

Leonhard Euler

EULER IN LAGRANGIAN COORDINATES

- Solve compressible Euler equations in Lagrangian form

$$
\begin{aligned}
&\rho\partial_t \mathbf{u} + \nabla p = 0\\
&\rho\partial_t e + p\nabla\cdot\mathbf{u} = 0\\
&J\rho = \rho_0\\
&\partial_t \mathbf{x} = \mathbf{u}(\mathbf{x}, t)\\
&T = (\gamma - 1)e \quad T = \frac{p}{\rho}\\
&\text{Initial data} + \text{BCs}
\end{aligned}
$$

- Work with ρ and nonconservative variables $\mathbf{u}$, e.

EULER IN LAGRANGIAN COORDINATES

- Weak forms

$$\int_{\Omega_0} \rho_0 \partial_t \mathbf{u}(\phi_t(\mathbf{x}_0))\psi(\phi_t(\mathbf{x}_0))\,\mathrm{d}\mathbf{x}_0 = -\int_{\Omega_t} \psi(\mathbf{x})\nabla p(\mathbf{x},t)\,\mathrm{d}\mathbf{x} - \int_{\Omega_t} \nu(\mathbf{x},t)\nabla\psi(\mathbf{x})\nabla\mathbf{u}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}$$

$$\int_{\Omega_0} \rho_0 \partial_t e(\phi_t(\mathbf{x}_0))\psi(\phi_t(\mathbf{x}_0))\,\mathrm{d}\mathbf{x}_0 = -\int_{\Omega_t} \psi(\mathbf{x})p(\mathbf{x},t)\nabla\cdot\mathbf{u}(\mathbf{x},t)\,\mathrm{d}\mathbf{x} - \int_{\Omega_t} \frac{1}{2}\nu(\mathbf{x},t)\nabla\psi(\mathbf{x})\nabla|\mathbf{u}(\mathbf{x},t)|^2\,\mathrm{d}\mathbf{x} - \int_{\Omega_t} \kappa(\mathbf{x},t)\nabla\psi(\mathbf{x})\nabla T(\mathbf{x},t$$

$$\int_{\Omega_t} \rho(\mathbf{x})\psi(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_{\Omega_0} \rho_0(\mathbf{x}_0)\psi(\phi_t(\mathbf{x}_0))\,\mathrm{d}\mathbf{x}_0 \quad \partial_t\mathbf{x} = \mathbf{u}(\mathbf{x},t)$$

$$T = (\gamma-1)e = \frac{p}{\rho}$$

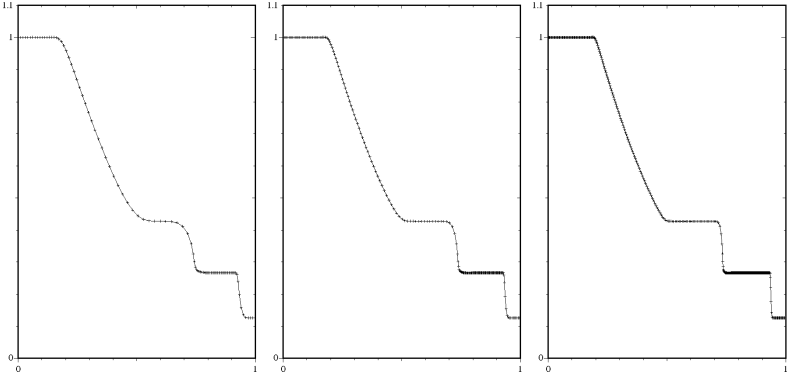
EULER IN LAGRANGIAN COORDINATES

- Specific entropy $s = \frac{1}{\gamma-1}\log(p/\rho^\gamma)$
- Entropy residual

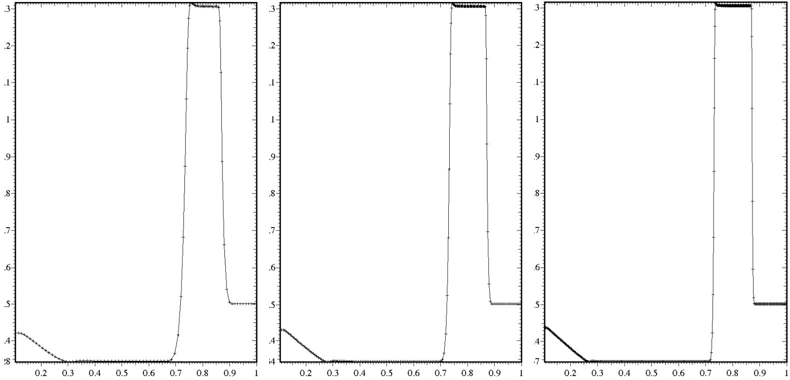
$$D := \max(|\rho\partial_t s|, |s(\partial_t\rho + \rho\nabla\cdot\mathbf{u})|)$$

- Algorithm similar to Eulerian formulation

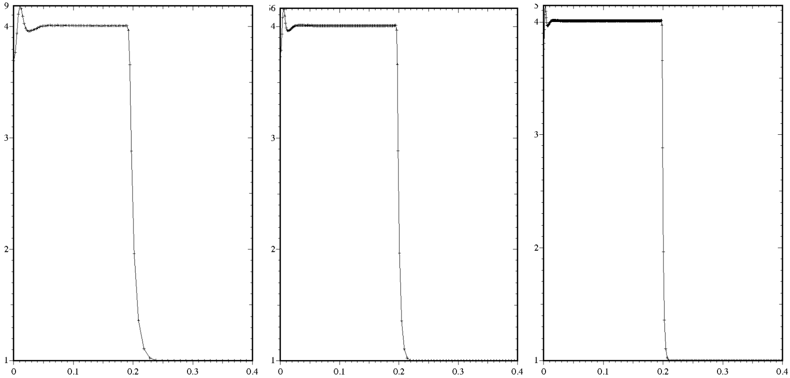
SOD TUBE



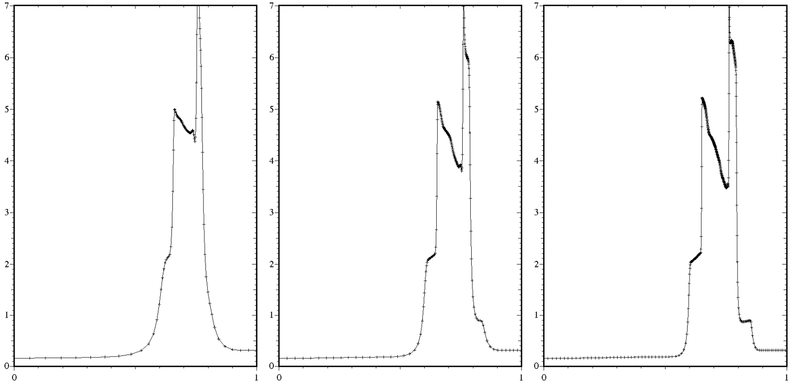
LAX TUBE


0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1

1D NOH PROBLEM



WOODWARD/COLLELA BLAST WAVE



TWO WAVE PROBLEM