Entropy-based artificial viscosity

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Outline Part 1

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4 COMPRESSIBLE EULER EQUATIONS

5 LAGRANGIAN HYDRODYNAMICS
INTRODUCTION
LINEAR TRANSPORT EQUATION
NONLINEAR SCALAR CONSERVATION

Why $L^1$ for PDEs?
A new idea based on $L^1$ minimization
Why L1 for PDEs?

- Solve 1D eikonal

\[ |u'(x)| = 1, \quad u(0) = 0, \quad u(1) = 0 \]

- Exists infinitely many weak solutions
Why L1 for PDEs?

- Exists a unique (positive) viscosity solution, $u$

\[ |u'_\epsilon| - \epsilon u''_\epsilon = 1, \quad u_\epsilon(0) = 0, \quad u_\epsilon(1) = 0. \]

- $\|u - u_\epsilon\|_{H^1} \leq c\epsilon^{1/2}$,

- Sloppy approximation.
Why L1 for PDEs?

One can do better with $L^1$ (of course 😊)

- Define mesh $T_h = \bigcup_{i=0}^{N} [x_i, x_{i+1}]$, $h = x_{i+1} - x_i$.
- Use continuous finite elements of degree 1.

$$V = \{ v \in C^0[0, 1]; \ v|_{[x_i, x_{i+1}]} \in P_1, \ v(0) = v(1) = 0 \}.$$

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High-Order Hydrodynamics
Why L1 for PDEs?

- Consider $p > 1$ and set

$$J(v) = \int_0^1 |v' - 1| \, dx + h^{2-p} \sum_{i=1}^N (v'(x_i^+) - v'(x_i^-))^p$$

- Define $u_h \in V$

$$u_h = \arg \min_{v \in V} J(v)$$
Why L1 for PDEs?

- Implementation: use mid-point quadrature
  \[ J_h(v) = \sum_{i=0}^{N} h \left| v'(x_{i+\frac{1}{2}}) - 1 \right| + \text{Entropy} \]

- Define
  \[ \tilde{u}_h = \arg\min_{v \in V} J_h(v) \]
Why L1 for PDEs?

Theorem (J.-L. G.&B. Popov (2008))

\[ u_h \rightarrow u \text{ and } \tilde{u}_h \rightarrow u \text{ strongly in } W^{1,1}(0, 1) \cap C^0[0, 1]. \]

- Fast solution in 1D (JLG&BP 2010) and in higher dimension (fast-marching/fast sweeping, Osher/Sethian) to compute \( \tilde{u}_h \).
- Similar results in 2D for convex Hamiltonians (JLG&BP 2008).
Some provable properties of minimizer $\tilde{u}_h$ (JLG&BP 2008, 2009, 2010). Minimizer $\tilde{u}_h$ is such that:

- **Residual is SPARSE:**

  $$|\tilde{u}'_h(x_{i+\frac{1}{2}})| - 1 = 0, \quad \forall i \text{ such that } \frac{1}{2} \not\in [x_i, x_{i+1}].$$

- **Entropy** makes it so that graph of $\tilde{u}'_h(x)$ is concave down in $[x_i, x_{i+1}] \ni \frac{1}{2}$. 
Conclusion:

- Residual is **SPARSE**: PDE solved almost everywhere. Entropy does not play role in those cells.
- Entropy plays a key role only in cell where PDE is not solved.
New idea:

- Go back to the notion of viscosity solution
- Add smart viscosity to the PDE:

\[ |u'_\epsilon| - \partial_x (\epsilon(u_\epsilon) \partial_x u_\epsilon) = 1 \]

- Make \( \epsilon \) depend on the entropy production
  1. Viscosity large (order h) where entropy production is large
  2. Viscosity vanish when no entropy production

- Entropy plays a key role in cell where PDE is not solved.
Transport, mixing

1 INTRODUCTION
2 LINEAR TRANSPORT EQUATION
3 NONLINEAR SCALAR CONSERVATION EQUATIONS
Solve the transport equation

\[ \partial_t u + \beta \cdot \nabla u = 0, \quad u|_{t=0} = u_0, \quad +\text{BCs} \]

Use standard discretizations (ex: continuous finite elements)
Deviate as little possible from Galerkin.
The idea

- Notion of renormalized solution (DiPerna/Lions (1989))
  Good framework for non-smooth transport.
- \( \forall E \in C^1(\mathbb{R}; \mathbb{R}) \) is an entropy
- If solution is smooth \( \Rightarrow E(u) \) solves PDE, \( \forall E \in C^1(\mathbb{R}; \mathbb{R}) \)
  (multiply PDE by \( E'(u) \) and apply chain rule)

\[
\partial_t E(u) + \beta \cdot \nabla E(u) = 0
\]

Entropy residual
Key idea 1:

Use entropy residual to construct viscosity
The idea

\[
\text{viscosity} \sim \text{entropy residual}
\]
The idea

- Viscosity $\sim$ residual (Hughes-Mallet (1986) Johnson-Szepessy (1990))
- Entropy Residual $\sim$ a posteriori estimator (Puppo (2003))
- Add entropy to formulation (For Hamilton-Jacobi equations Guermond-Popov (2007))
- Application to nonlinear conservation equations (Guermond-Pasquetti (2008))

viscosity $\sim$ entropy residual
The algorithm + time discretization

- **Numerical analysis 101:**
  
  \[
  \text{Up-winding} = \text{centered approx} + \frac{1}{2}|\beta|h \text{ viscosity}
  \]

- **Proof:**

  \[
  \beta_i \frac{u_i - u_{i-1}}{h_i} = \beta_i \frac{u_{i+1} - u_{i-1}}{2h_i} - \frac{1}{2\beta_i h_i} \frac{u_{i+1} - 2u_i + u_{i-1}}{h_i}
  \]
Key idea 2:
Entropy viscosity should not exceed $\frac{1}{2} |\beta|h$
The algorithm

- Choose one entropy functional.
  
  EX1: $E(u) = |u - \bar{u}_0|$,  
  EX2: $E(u) = (u - \bar{u}_0)^2$, etc.

- Define entropy residual $D_h := \partial_t E(u_h) + \beta \cdot \nabla E(u_h)$,

- Define local mesh size of cell $K$: $h_K = \text{diam}(K)/p^2$

- Construct a wave speed associated with this residual on each mesh cell $K$:
  
  $$v_K := h_K \| D_h \|_{\infty, K} / E(u_h)$$

- Define entropy viscosity on each mesh cell $K$:
  
  $$\nu_K := h_K \min\left(\frac{1}{2} \| \beta \|_{\infty, K}, v_K\right)$$
Summary

- Space approximation: Galerkin + entropy viscosity:

\[ \int_{\Omega} \left( \partial_t u_h + \beta \cdot \nabla u_h \right) v_h \, dx + \sum_K \int_K \nu_K \nabla u_h \cdot \nabla v_h \, dx = 0, \quad \forall v_h \]

  - Galerkin (centered approximation)
  - Entropy viscosity

- Time approximation: Use an explicit time stepping: BDF2, RK3, RK4, etc.

- Idea: make the viscosity explicit ⇒ Stability under CFL condition.
EX: 2nd-order centered finite differences 1D

Compute the entropy residual $D_i$ on each cell $(x_i, x_{i+1})$

$$D_i := \max \left( \left| \frac{E(u^n_i) - E(u^n_{i-1})}{\Delta t} + \beta \frac{E(u^n_{i+1}) - E(u^n_i)}{h_i} \right|, \left| \frac{E(u^n_{i+1}) - E(u^n_{i-1})}{\Delta t} + \beta \frac{E(u^n_{i+1}) - E(u^n_i)}{h_i} \right| \right)$$

Compute the entropy viscosity

$$\nu^n_i := h_i \min \left( \frac{1}{2} |\beta|, \frac{1}{2} \frac{D_i}{E(u^n) h_i} \right)$$
Use RK to solve on next time interval \([t^n, t^n + \Delta t]\)

\[
\begin{align*}
    u_i(t = t^n) &= u_i^n \\
    \partial_t u_i + \beta_{i + \frac{1}{2}} \frac{u_{i+1} - u_{i-1}}{2h_i} - \left( \nu_i^n \frac{u_{i+1} - u_i}{h_i} - \nu_{i-1}^n \frac{u_i - u_{i-1}}{h_{i-1}} \right) &= 0
\end{align*}
\]

- Centered approximation
- Centered viscous fluxes

The entropy viscosity can be computed on the fly for some RK techniques.
Space + time discretization: RK2 midpoint

- Advance half time step to get $w^n$

$$w^n_i = u^n_i - \frac{1}{2} \Delta t \beta_{i+\frac{1}{2}} \frac{u^n_{i+1} - u^n_{i-1}}{2h_i}$$

- Compute entropy viscosity on the fly

$$D_i := \max \left( \left| \frac{E(w^n_i) - E(u^n_i)}{\Delta t/2} + \beta_{i+\frac{1}{2}} \frac{E(w^n_{i+1}) - E(w^n_i)}{h_i} \right|, \left| \frac{E(w^n_{i+1}) - E(u^n_{i+1})}{\Delta t/2} + \beta_{i+\frac{1}{2}} \frac{E(w^n_{i+1}) - E(w^n_i)}{h_i} \right| \right)$$

- Compute $u^{n+1}$

$$u^{n+1}_i = u^n_i - \Delta t \beta_{i+\frac{1}{2}} \frac{w^n_{i+1} - w^n_{i-1}}{2h_i}$$

$$+ \left( \nu^n_i \frac{w^n_{i+1} - w^n_i}{h_i} - \nu^n_{i-1} \frac{w^n_i - w^n_{i-1}}{h_i} \right)$$
Theory for linear steady equations

Consider

$$\partial_t u + \beta \cdot \nabla u = f, \quad u|_{\Gamma^-} = 0.$$
Consider

\[ \partial_t u + \beta \cdot \nabla u = f, \quad u|_{\Gamma^-} = 0. \]

**Theorem**

Let \( u_h \) be the finite element approximation with Euler time approximation and \( u^2 \) entropy viscosity, then \( u_h \) converges to \( u \).
Theory for linear steady equations

- Consider

\[ \partial_t u + \beta \cdot \nabla u = f, \quad u|_{\Gamma^-} = 0. \]

**Theorem**

Let \( u_h \) be the finite element approximation with *Euler* time approximation and \( u^2 \) entropy viscosity, then \( u_h \) converges to \( u \).

**Theorem**

Let \( u_h \) be the \( \mathbb{P}_1 \) finite element approximation with *RK2* time approximation and \( u^2 \) entropy then \( u_h \) converges to \( u \).

**Conjecture**

The results should hold for nonlinear scalar conservation laws with convex, Lipschitz flux.
Why convergence is so difficult to prove?

- Key a priori estimate

\[ \int_0^T \nu(u)|\nabla u|^2 \, dx \leq c \]

- Ok in \( \{ \nu(u)(x, t) = \frac{1}{2} \| \beta \| h \} \) (non-smooth region)

- The estimate is useless in smooth region.

- Explicit time stepping makes the viscosity depend on the past.
1D Numerical tests, BV solution

- linear transport

\[ \partial_t u + \partial_x u = 0, \quad u_0(x) = \begin{cases} 
  e^{-300(2x-0.3)^2} & \text{if } |2x-0.3| \leq 0.25, \\
  1 & \text{if } |2x-0.9| \leq 0.2, \\
  \left(1 - \left(\frac{2x-1.6}{0.2}\right)^2\right)^{\frac{1}{2}} & \text{if } |2x-1.6| \leq 0.2, \\
  0 & \text{otherwise.} 
\end{cases} \]

- Periodic boundary conditions.
1D Numerical tests, BV solution, Spectral elements

- Spectral elements in 1D on random meshes.
- Long time integration, 100 periods.

Long time integration, \( t = 100 \), for polynomial degrees \( k = 2, \ldots, 8 \), \#d.o.f.=200. Galerkin (left); Constant viscosity (center); Entropy viscosity (right).
Second-order finite differences in 1D on uniform and random meshes.

Long time integration, 100 periods.

Long time integration, \( t = 100 \), for 2nd order finite differences
\( \#d.o.f. = 200 \). Uniform mesh (left); Random mesh (right).
Numerical tests, smooth solution

- \( \Omega = \{(x, y) \in \mathbb{R}^2, \sqrt{x^2 + y^2} \leq 1\} := B(0, 1) \)
- Speed: rotation about origin, angular speed \( 2\pi \)
- \( u(x, y) = \frac{1}{2} \left(1 - \tanh \left( \frac{(x-r_0 \cos(2\pi t))^2 + (y-r_0 \sin(2\pi t))^2}{a^2} - 1 \right) \right) + 1 \),
- \( a = 0.3, \ r_0 = 0.4 \)
2D numerical tests, smooth solution, $P_1$ FE

- $P_1$ finite elements

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**Table:** $P_1$ approximation.
2D numerical tests, smooth solution, spectral elements

Linear transport problem with smooth initial condition. Errors in $L^1$ (at left) and $L^2$ (at right) norms vs $h$ for $N = 2, 4, 6, 8, 12$. 
2D Numerical tests, BV solution

- \( \Omega = \{ (x, y) \in \mathbb{R}^2, \sqrt{x^2 + y^2} \leq 1 \} := B(0, 1) \),
- Speed: rotation about origin, angular speed \( 2\pi \)
- \( u(x, y) = \chi_{B(0,a)}\left( \sqrt{(x - r_0 \cos(2\pi t))^2 + (y - r_0 \sin(2\pi t))^2} \right) \),
- \( a = 0.3, r_0 = 0.4 \)
2D Numerical tests, BV solution, $P_2$ FE

- $P_2$ finite elements

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Table: $P_2$ approximation.
NONLINEAR SCALAR CONSERVATION EQUATIONS

Johannes Martinus Burgers
Solve

\[ \partial_t u + \partial_x f(u) + \partial_y g(u) = 0 \quad u|_{t=0} = u_0, \quad +\text{BCs.} \]

The unique entropy solution satisfies

\[ \partial_t E(u) + \partial_x F(u) + \partial_y G(u) \leq 0 \]

for all entropy pair \( E(u), F(u) = \int E'(u)f'(u)du, \)
\( G(u) = \int E'(u)g'(u)du \)
2D scalar nonlinear conservation laws

- Choose one entropy $E(u)$
- Define entropy residual, $D_h(u) := \partial_t E(u) + \partial_x F(u) + \partial_y G(u)$
- Define local mesh size of cell $K$: $h_K = \text{diam}(K)/p^2$
- Construct a speed associated with residual on each cell $K$:
  $$v_K := h_K \|D_h\|_{\infty,K}/E(u_h)$$

- Compute maximum local wave speed:
  $$\beta_K = \|\sqrt{f'(u)^2 + g'(u)^2}\|_{\infty,K}$$
- Define entropy viscosity on each mesh cell $K$:
  $$\nu_K := h_K \min\left(\frac{1}{2}\beta_K, v_K\right)$$
Summary

- Space approximation: Galerkin + entropy viscosity:
  \[
  \int_{\Omega} (\partial_t u_h + \partial_x f(u_h) + \partial_y g(u_h)) v_h dx + \sum_K \int_K \nu K \nabla u_h \nabla v_h dx = 0, \quad \forall v_h
  \]
  Galerkin (centered approximation)
  Entropy viscosity

- Time approximation: explicit RK
EX: 2nd-order centered finite differences 1D

- Compute local speed on each cell \((x_i, x_{i+1})\)

\[
\beta_{i+\frac{1}{2}} := \frac{1}{2} (f'(u_i) + f'(u_{i+1}))
\]

- Compute the entropy residual \(D_i\) on each cell \((x_i, x_{i+1})\)

\[
D_i := \max \left( \frac{E(u_i^n) - E(u_i^{n-1})}{\Delta t} + \beta_{i+\frac{1}{2}} \frac{E(u_{i+1}^n) - E(u_i^{n-1})}{h_i}, \right.
\]

\[
\frac{E(u_{i+1}^n) - E(u_{i+1}^{n-1})}{\Delta t} + \beta_{i+\frac{1}{2}} \frac{E(u_{i+1}^n) - E(u_i^{n-1})}{h_i} \left. \right)
\]
The algorithm + time discretization

- Compute the entropy viscosity

\[ \nu_i^n := h_i \min \left( \frac{1}{2} |\beta_{i+\frac{1}{2}}|, \frac{1}{2} \frac{D_i}{E(u^n)h_i} \right) \]

- Use RK to solve on next time interval \([t^n, t^n + \Delta t]\)

\[
\partial_t u_i + \frac{f(u_{i+1}) - f(u_{i-1})}{2h_i} - \left( \nu_i^n \frac{u_{i+1} - u_i}{h_i} - \nu_{i-1}^n \frac{u_i - u_{i-1}}{h_{i-1}} \right) = 0
\]
EX: 1D burgers + 2nd-order Finite Differences

- Second-order Finite Differences + RK2/RK3/RK4

Burgers, \( t = 0.25 \), \( N = 50, 100, \) and 200 grid points.

\[ u_h \]

\[ v_h(u_h)|\partial_x u_h| \]
EX: 1D burgers + Fourier

- Solution method: Fourier + RK4 + entropy viscosity

Burgers at $t = 0.25$ with $N = 50, 100, \text{ and } 200$. 

\[
\nu_N(t)
\]

\[
\nu_N(u_N)
\]
EX: 1D Nonconvex flux + Fourier (WENO5 + SuperBee (or minmod 2) fails)

Consider $\partial_t + \partial_x f(u) = 0$, $u(x, 0) = u_0(x)$

$$f(u) = \begin{cases} 
\frac{1}{4} u(1 - u) & \text{if } u < \frac{1}{2}, \\
\frac{1}{2} u(u - 1) + \frac{3}{16} & \text{if } u \geq \frac{1}{2},
\end{cases}$$

$$u_0(x) = \begin{cases} 
0, & x \in (0, 0.25], \\
1, & x \in (0.25, 1]
\end{cases}$$
EX: 1D Nonconvex flux + Fourier (WENO5 + SuperBee (or minmod 2) fails)

- Consider $\partial_t + \partial_x f(u) = 0$, $u(x, 0) = u_0(x)$

$$f(u) = \begin{cases} 
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\frac{1}{2}u(u - 1) + \frac{3}{16} & \text{if } u \geq \frac{1}{2}, 
\end{cases}$$

$$u_0(x) = \begin{cases} 
0, & x \in (0, 0.25], \\
1, & x \in (0.25, 1]
\end{cases}$$
Convergence tests, 2D Burgers

- Solve 2D Burgers

\[ \partial_t u + \partial_x \left( \frac{1}{2} u^2 \right) + \partial_y \left( \frac{1}{2} u^2 \right) = 0 \]

- Subject to the following initial condition

\[ u(x, y, 0) = u^0(x, y) = \begin{cases} 
-0.2 & \text{if } x < 0.5 \text{ and } y > 0.5 \\
-1 & \text{if } x > 0.5 \text{ and } y > 0.5 \\
0.5 & \text{if } x < 0.5 \text{ and } y < 0.5 \\
0.8 & \text{if } x > 0.5 \text{ and } y < 0.5 
\end{cases} \]

- Compute solution in \((0, 1)^2\) at \(t = \frac{1}{2}\).
Convergence tests, 2D Burgers

Initial data

$\mathbb{P}_1$ FE, $3 \times 10^4$ nodes
Convergence tests, 2D Burgers

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Table: Burgers, $P_1$ approximation.
Convergence tests, 2D Burgers

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<td>0.899</td>
<td>–</td>
<td>1.4588E-2</td>
<td>0.899</td>
</tr>
<tr>
<td>6.25E-3</td>
<td>6.8806E-2</td>
<td>0.473</td>
<td>7.6435E-3</td>
<td>0.932</td>
<td>–</td>
<td>7.6435E-3</td>
<td>0.932</td>
</tr>
</tbody>
</table>

Table: Burgers, $P_2$ approximation.
Solve $\partial_t u + \partial_x f(u) + \partial_y g(u) = 0$.

\[ f(u) = \frac{u^2}{u^2 + (1-u)^2}, \quad g(u) = f(u)(1 - 5(1-u)^2) \]

Non-convex fluxes (composite waves)

\[ u(x, y, 0) = \begin{cases} 
1, & \sqrt{x^2 + y^2} \leq 0.5 \\
0, & \text{else}
\end{cases} \]
Nonlinear scalar conservation laws
Convergence tests, 2D Burgers, $\mathbb{P}_1/\mathbb{P}_2$ FE
Buckley Leverett, FE
Kurganov, Petrova, Popov problem, FE

Buckley Leverett, $\mathbb{P}_2$ FE
Solve $\partial_t u + \partial_x f(u) + \partial_y g(u) = 0$.

$$f(u) = \sin(u), \quad g(u) = \cos(u)$$

Non-convex fluxes (composite waves)

$$u(x, y, 0) = \begin{cases} \frac{7}{2} \pi, & \sqrt{x^2 + y^2} \leq 1 \\ \frac{1}{4} \pi, & \text{else} \end{cases}$$
KPP (WENO + superbee limiter fails)

$P_2$ approx

$Q_4$ entrop visco.
COMPRESSIBLE EULER EQUATIONS
LAGRANGIAN HYDRODYNAMICS

NONLINEAR SCALAR CONSERVATION EQUATIONS

Leonhard Euler
Euler flows

- Solve compressible Euler equations

\[
\begin{align*}
\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) &= 0 \\
\partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u} + p \mathbb{I}) &= 0 \\
\partial_t (E) + \nabla \cdot (\mathbf{u}(E + p)) &= 0 \\
\rho e &= E - \frac{1}{2} \rho \mathbf{u}^2, \quad T = (\gamma - 1)e \quad T = \frac{p}{\rho}
\end{align*}
\]

Initial data + BCs

- Use continuous finite elements of degree \( p \).
- Deviate as little possible from Galerkin.
The algorithm

- Compute the entropy $S_h = \frac{\rho_h}{\gamma-1} \log(\rho_h/\rho_h^\gamma)$
- Define entropy residual, $D_h := \partial_t S_h + \nabla \cdot (u_h S_h)$
- Define local mesh size of cell $K$: $h_K = \text{diam}(K)/p^2$
- Construct a speed associated with residual on each cell $K$:
  \[ v_K := h_K \| D_h \|_{\infty,K} \]
- Compute maximum local wave speed:
  \[ \beta_K = \| u \| + (\gamma T)^{\frac{1}{2}} \| u \|_{\infty,K} \]
The algorithm

- Use Navier-Stokes regularization: define $\mu_K$ and $\kappa_K$.
- Entropy viscosity and thermal conductivity on each mesh cell $K$:
  \[
  \mu_K := h_K \min \left( \frac{1}{2} \beta_K \| \rho_h \|_{\infty, K}, \nu_K \right), \quad \kappa_K = \mathcal{P} \mu_K 
  \]
- In practice use $\mathcal{P} = \frac{1}{10}$.
- Solution method: Galerkin + entropy viscosity + thermal conductivity
1D Euler flows + Fourier

• Solution method: Fourier + RK4 + entropy viscosity
1D Euler flows + Fourier

- Solution method: Fourier + RK4 + entropy viscosity

**Figure**: Lax shock tube, $t = 1.3, 50, 100, 200$ points. Shu-Osher shock tube, $t = 1.8, 400, 800$ points. Right: Woodward-Collela blast wave, $t = 0.038, 200, 400, 800, 1600$ points.
2D Euler flows + Fourier

- Domain $\Omega = (-1, 1)^2$
- Riemann problem with the initial condition:
  
  $0 < x < 0.5$ and $0 < y < 0.5$, \quad $p = 1, \rho = 0.8, \mathbf{u} = (0, 0)$,

  $0 < x < 0.5$ and $0.5 < y < 1$, \quad $p = 1, \rho = 1, \mathbf{u} = (0.7276, 0)$,

  $0.5 < x < 1$ and $0 < y < 0.5$, \quad $p = 1, \rho = 1, \mathbf{u} = (0, 0.7276)$,

  $0 < x < 0.5$ and $0.5 < y < 1$, \quad $p = 0.4, \rho = 0.5313, \mathbf{u} = (0, 0)$.

- Solution at time $t = 0.2$. 
2D Euler flows + Fourier (Riemann test case 12)

Euler benchmark, Fourier approximation: Density (at left),
\[ 0.528 < \rho_N < 1.707 \] and viscosity (at right),
\[ 0 < \mu_N < 3.410^{-3} \], at 
\[ t = 0.2, \ N = 400. \]
Riemann problem test case no 12, $P_1$ FE

movie, Riemann no 12
Cylinder in a channel, Mach 2, $P_1$ FE (By M. Nazarov, KTH)

t = 0.2
Bubble, density ratio $10^{-1}$, Mach 1.65, $P_1$ FE (by M. Nazarov, KTH)
Mach 3 Wind Tunnel with a Step, $\mathbb{P}_1$ finite elements

- Mach 3 Wind Tunnel with a Step (Standard Benchmark since Woodward and Colella (1984))
- Inflow boundary, density 1.4, pressure 1, and $x$-velocity 3, (Mach =3)
Mach 3 Wind Tunnel with a Step, $P_1$ finite elements

- Mach 3 Wind Tunnel with a Step (Standard Benchmark since Woodward and Colella (1984))
- Inflow boundary, density 1.4, pressure 1, and x-velocity 3, (Mach = 3)

$P_1$ FE, $1.3 \times 10^5$ nodes

Flash Code, adaptive $PPM$, $\sim 4.9 \times 10^6$ nodes

Log(density)

Movie, density field (entropy visc.) Movie, density field (viscous)
Mach 3 Wind Tunnel with a Step, $P_1$ finite elements

Viscous flux of entropy Viscosity.
Mach 10 Double Mach reflection, $\mathbb{P}_1$ finite elements

- Right-moving Mach 10 shock makes $60^\circ$ angle with $x$-axis (Standard Benchmark, Woodward and Colella (1984))
- Shock interacts with flat plate $x \in \left(\frac{1}{6}, +\infty\right)$.
- The un-shocked fluid $\rho = 1.4$, $p = 1$, and $u = 0$
Mach 10 Double Mach reflection, $P_1$ finite elements

- Right-moving Mach 10 shock makes $60^\circ$ angle with $x$-axis (Standard Benchmark, Woodward and Colella (1984))
- Shock interacts with flat plate $x \in \left(\frac{1}{6}, +\infty\right)$.
- The un-shocked fluid $\rho = 1.4$, $p = 1$, and $u = 0$

$P_1$ FE, $4.5 \times 10^5$ nodes, $t = 0.2$

Movie, density field
Mach 10 Double Mach reflection

Entropy Viscosity
Leonhard Euler
EULER IN LAGRANGIAN COORDINATES

- Solve compressible Euler equations in Lagrangian form

\[
\begin{align*}
\rho \partial_t u + \nabla p &= 0 \\
\rho \partial_t e + p \nabla \cdot u &= 0 \\
J \rho &= \rho_0 \\
\partial_t x &= u(x, t) \\
T &= (\gamma - 1)e \\
T &= \frac{p}{\rho}
\end{align*}
\]

- Work with \( \rho \) and nonconservative variables \( u, e \).
Euler in Lagrangian Coordinates

- Weak forms

\[
\int_{\Omega_0} \rho_0 \partial_t u(\phi_t(x_0)) \psi(\phi_t(x_0)) \, dx_0 = - \int_{\Omega_t} \psi(x) \nabla p(x, t) \, dx \\
- \int_{\Omega_t} \nu(x, t) \nabla \psi(x) \nabla u(x, t) \, dx \\
\int_{\Omega_0} \rho_0 \partial_t e(\phi_t(x_0)) \psi(\phi_t(x_0)) \, dx_0 = - \int_{\Omega_t} \psi(x) p(x, t) \nabla \cdot u(x, t) \, dx \\
- \int_{\Omega_t} \frac{1}{2} \nu(x, t) \nabla \psi(x) \nabla |u(x, t)|^2 \, dx - \int_{\Omega_t} \kappa(x, t) \nabla \psi(x) \nabla T(x, t) \, dx
\]

\[
\int_{\Omega_t} \rho(x) \psi(x) \, dx = \int_{\Omega_0} \rho_0(x_0) \psi(\phi_t(x_0)) \, dx_0 \quad \partial_t x = u(x, t)
\]

\[
T = (\gamma - 1) e = \frac{p}{\rho}
\]
Specific entropy $s = \frac{1}{\gamma - 1} \log(p/\rho^\gamma)$

Entropy residual

$$D := \max(|\rho \partial_t s|, |s(\partial_t \rho + \rho \nabla \cdot \mathbf{u})|)$$

Algorithm similar to Eulerian formulation
1D NOH PROBLEM
WOODWARD/COLLELA BLAST WAVE
TWO WAVE PROBLEM