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Numerical Results

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# Isogeometric Residual Distribution Scheme

### Algiane Froehly Rémi Abgrall Cécile Dobrzynski



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### Introduction

### Theorical Framework

Euler Equations Residual Distribution Schemes  $(\mathcal{RDS})$ 

### $\mathcal{RDS}$ with $\mathcal{NURBS}$

 $\mathcal{NURBS}$  Generalities Residuals Computation

 $\mathcal{NURBS}$  Mesh generation

Numerical Results

Introduction	Theorical Framework	$\mathcal{RDS}$ with $\mathcal{NURBS}$	NURBS Mesh generation	Numerical Results
	0	000		
	0	00		

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### Introduction

- Theorical Framework
- $\mathcal{RDS}$  with  $\mathcal{NURBS}$
- $\mathcal{NURBS}$  Mesh generation
- Numerical Results

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Theorical Framework

RDS with NURBS

 $\mathcal{NURBS}$  Mesh generation

Numerical Results

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### Introduction

Problematic



Subsonic flow around 2 cylindres: spurious entropy production

Solution Isogeometric analysis

Introduction	Theorical Framework	$\mathcal{RDS}$ with $\mathcal{NURBS}$	$\mathcal{NURBS}$ Mesh generation	Numerical Results
	0	000		

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### Introduction

### Theorical Framework

Euler Equations Residual Distribution Schemes  $(\mathcal{RDS})$ 

RDS with NURBS

NURBS Mesh generation

Numerical Results

Theorical Framework

*RDS* with *NURBS* 000 00 NURBS Mesh generation

Numerical Results

# **Euler Equations**

### Conservative form

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$$\mathbf{U} = \begin{pmatrix} \rho \\ \rho \vec{\mathbf{u}} \\ \rho E \end{pmatrix}, \begin{cases} \vec{\nabla} \cdot [\rho \vec{\mathbf{u}}] &= 0 \quad \mathbf{Cons} \\ \vec{\nabla} \cdot [\rho u_i \vec{\mathbf{u}} + p \delta_i] &= 0 \quad \mathbf{Cons} \\ \vec{\nabla} \cdot [(\rho E + \rho) \vec{\mathbf{u}}] &= 0 \quad \mathbf{Cons} \end{cases}$$

- = 0 **Conservation of mass**
- = 0 Conservation of momentum

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= 0 **Conservation of Energy** 

Theorical Framework

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Numerical Results

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# **Euler Equations**

### Conservative form

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### Problem to solve

$$\begin{cases} \overrightarrow{\nabla}.\overrightarrow{\mathcal{F}}(\mathbf{U}) = 0 & \forall x \in \Omega \\ \text{weak boundary conditions} & (\mathcal{BC}) \end{cases}$$

Theorical Framework

RDS with NURBS

 $\mathcal{NURBS}$  Mesh generation

Numerical Results

 $\mathcal{RDS}$ 

## Definitions

•  $\Phi^{\mathsf{T}}$ : total residual over  $\mathsf{T}$ 

$$\Phi^{\mathsf{T}} = \int_{\partial \mathsf{T}} \overrightarrow{\mathcal{F}} (\mathbf{U}^h) \cdot \vec{\mathbf{n}} d\partial \mathsf{T}$$

•  $\Phi_i^{\mathsf{T}}$ : local nodal residual  $\sum_{i \in T} \Phi_i^{\mathsf{T}} = \Phi^{\mathsf{T}}$   $\Phi_i^{\mathsf{T}} = \text{ High-order term } + \text{ Stabilization term}$ 



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Theorical Framework

RDS with NURBS 000  $\mathcal{NURBS}$  Mesh generation

Numerical Results

 $\mathcal{RDS}$ 

## Definitions

•  $\Phi^{\mathsf{T}}$ : total residual over  $\mathsf{T}$ 

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 $\mathcal{RDS}$  formulation

$$\forall M_i, \sum_{T \ni M_i} \Phi_i^{\mathsf{T}}(\mathbf{U}^h) = 0$$



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NURBS Mesh generation

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3

Numerical Results

#### Introduction

### Theorical Framework

### RDS with NURBS NURBS Generalities Residuals Computation

NURBS Mesh generation

Numerical Results

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Numerical Results

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- their polynomial order in the  $j^{th}$  dimension:  $p_j$
- d knot vectors  $(\xi_j)_{j=1,d}$



 $\mathcal{N}$  on  $\mathcal{U}$ niform  $\mathcal{R}$ ational  $\mathcal{B}$ - $\mathcal{S}$ pline Definitions B-Splines basis functions in  $\mathbb{R}^d$  defined by :

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# $\mathcal{NURBS}$ basis functions in $\mathbb{R}^d$ defined by :

- The  $n_j$  B-Splines basis functions in the  $j^{th}$  dimension
- A set of n<sub>j</sub> weighting factors: W<sub>l</sub>

B-Splines and  $\mathcal{NURBS}$  basis functions for d = 1 $i^{th} \mathcal{NURBS}$  function of order k:

$$M_{i,k}(\xi) = \frac{W_i N_{i,k}(\xi)}{\sum_{l=1,n} W_l N_{l,k}(\xi)}$$

 $\mathcal{N}$  on  $\mathcal{U}$ niform  $\mathcal{R}$ ational  $\mathcal{B}$ - $\mathcal{S}$ pline Definitions  $\mathcal{NURBS}$  curves and surfaces extrapolation

$$C(\xi) = \sum_{i=1,n} \tilde{P}_i M_{i,k}(\xi); \quad S(\xi,\eta) = \sum_{i=1,n \times m} \tilde{P}_i \Psi_{i,k}(\xi,\eta)$$



# $\mathcal{NURBS}$ adaptation for $\mathcal{RDS}$

Numerical simplifications and basis functions used

- Same nodal vector for all edges:  $\xi = [\underbrace{0, 0, \cdots, \underbrace{k \text{ times}}_{k \text{ times}}, \underbrace{k \text{$
- One unique weight per point
- Definition of B-Spline-like basis for triangular elements :  $\binom{k}{j1, j2, j3} \xi^{j1} \eta^{j2} \zeta^{j3}, \ j \in [|1, k|], \ j1 + j2 + j3 = k$

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Numerical Results

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## Residuals computation



Total residual computation

$$\Phi^{\mathsf{T}} = \int_{\partial \mathsf{T}} \overrightarrow{\mathcal{F}}_h \cdot \overrightarrow{\mathbf{n}} \, d\partial \mathsf{T} = \sum_{j=1, n_e} \int_{\Gamma_j} \Big( \sum_{l=1}^{n_{\mathsf{DoFs}}} \overrightarrow{\overrightarrow{\mathcal{F}}}_l \Psi_l(\xi, \eta) \Big) \cdot \overrightarrow{\mathbf{n}}_j(x, y) d\Gamma_j$$

3-points Gauss-Legendre quadrature at order 3 6-points Gauss-Legendre quadrature at order 4

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Numerical Results

## Residuals computation



Stabilization term computation

$$D_i^{\mathsf{T}} = \int_{\mathsf{T}} (\vec{\lambda} \cdot \overrightarrow{\nabla \Psi_i}) \bar{\boldsymbol{\tau}} (\vec{\lambda} \cdot \overrightarrow{\nabla \mathbf{U}_h}) d\mathsf{T}$$

7-points Gauss-Legendre quadrature at order 3 13-points Gauss-Legendre quadrature at order 4  $\mathcal{RDS}$  with  $\mathcal{NURBS}$ ••• Numerical Results

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# Slip wall Boundary Condition: $\vec{u}.\vec{n} = \vec{0}$



Weak formulation at the  $i^{th}$  boundary node

$$\Phi_{b,i,\Gamma} = \int_{\Gamma} \Psi_i^{\mathcal{T}} \overrightarrow{\mathcal{F}}_{slip}(\mathbf{U}_h^n, \vec{\mathbf{n}}) d\Gamma$$

$$\Phi_{b,i,\Gamma} = \sum_{q=1}^{n_{quad}} \left( \tilde{\omega}_q \Psi_i^{\mathcal{T}}(\xi_q) \overrightarrow{\mathcal{F}}_{slip}(\mathbf{U}_h^n(\xi_q), \vec{\mathbf{n}}(\xi_q)) \|\Gamma\| \right)$$

 $(k-1)^2$ -points Newton-Cotes quadrature at order k

Introduction	Theorical Framework	$\mathcal{RDS}$ with $\mathcal{NURBS}$	NURBS Mesh generation	Numerical Results
	0	000		
	0	00		

**Theorical Framework** 

 $\mathcal{RDS}$  with  $\mathcal{NURBS}$ 

 $\mathcal{NURBS}$  Mesh generation

Numerical Results

# $\mathcal{NURBS}$ Mesh generation

## Problematic

•  $\mathcal{NURBS}$  mesh needed to use isogeometric analysis

## Solution

- Classical meshing of the geometry (piecewise linear)
- Computing the control points and weights needed to extrapolate a  $\mathcal{NURBS}$  curve verifying that :
  - $\mathcal{G}_0$  continuity is preserved (the curve interpolate each P1 point)
  - Each boundary edge is approached by a portion of circle

Theorical Framework

RDS with NURBS

 $\mathcal{NURBS}$  Mesh generation

Numerical Results

# $\mathcal{NURBS}$ Mesh generation

### From classical mesh to $\mathcal{NURBS}$ mesh



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Theorical Framework

RDS with NURBS

 $\mathcal{NURBS}$  Mesh generation

Numerical Results

# $\mathcal{NURBS}$ Mesh generation

### From classical mesh to $\mathcal{NURBS}$ mesh



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Theorical Framework

RDS with NURBS

 $\mathcal{NURBS}$  Mesh generation

Numerical Results

# $\mathcal{NURBS}$ Mesh generation

### From classical mesh to $\mathcal{NURBS}$ mesh



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Introduction	Theorical Framework	$\mathcal{RDS}$ with $\mathcal{NURBS}$	NURBS Mesh generation	Numerical Results
	0	000 00		

**Theorical Framework** 

 $\mathcal{RDS}$  with  $\mathcal{NURBS}$ 

 $\mathcal{NURBS}$  Mesh generation

Numerical Results

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 $\mathcal{NURBS}$  Mesh generation

Numerical Results

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## Subsonic flow around a cylindre: Ma = 0.38

### Boundaries representation



P2 picewise-linear mesh, IsoP2 mesh and Nurbs mesh

## Subsonic flow around a cylindre: Ma = 0.38



L2-entropy error during iterations for  $8 \times 32$  nodes meshes

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Numerical Results

## Subsonic flow around a cylindre: Ma = 0.38



Pressure loss distribution along cylindre for  $8 \times 32$  nodes meshes

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NURBS Mesh generation

Numerical Results

## Subsonic flow around 2 cylindres: Ma = 0.38



 $\mathcal{NURBS}$  Mesh generation

Numerical Results

## Subsonic flow around 2 cylindres: Ma = 0.38



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# Subsonic flow over a bump: Ma = 0.63, slope=26%



Entropy error on the whole bump meshes

## Subsonic flow over a bump: Ma = 0.63, slope=26%



L2-Entropy error along the bump floor

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# Conclusion

### Done work

- $\mathcal{RDS}$  lsogeometric of order 3 and 4 for triangular, quadrangular and hybrid 2*D*-meshes
- $\mathcal{RDS}$  Isogeometric of order 3 for tetrahedral 3D-mesh
- Automatic generation of  $2D \ \mathcal{NURBS}$  meshes

### Perspectives

- To adapt Isogeometric analysis to Navier-Stokes equations
- Automatic generation of 3D NURBS meshes