On Godunov type schemes accurate at any Mach number

Stéphane Dellacherie[1,3]

In collaboration with
P. Omnes[1,3] and P.A. Raviart[2,3]

[1]CEA-Saclay
[2]Université Paris 6
[3]LRC-Manon, LJLL, Paris 6

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Outline

Introduction

I - The low Mach number problem and the linear wave equation

II - The linear case at any Mach number

III - The non-linear case at any Mach number

Introduction

energie atomique • energies alternatives

When $M := \frac{u_*}{a_*} \ll 1$ and when the initial conditions are well-prepared in the sense

$$\begin{cases} \rho(t=0,x) = \rho_*(x), \\ p(t=0,x) = p_* + \mathcal{O}(M^2), \\ \mathbf{u}(t=0,x) = \widehat{\mathbf{u}}(x) + \mathcal{O}(M) \quad \text{with} \quad \nabla \cdot \widehat{\mathbf{u}}(x) = 0, \end{cases}$$

the solution $(\rho, \mathbf{u}, p)$ of the (dimensionless) **compressible Euler system**

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \dfrac{\nabla p}{M^2} = 0, \\ \partial_t (\rho E) + \nabla \cdot [(\rho E + p)\mathbf{u}] = 0 \end{cases}$$

is close to $(\rho, \mathbf{u}, p)$ which satisfies the **incompressible Euler system**

$$\begin{cases} \partial_t \rho + \mathbf{u} \cdot \nabla \rho = 0, \quad \rho(t=0,x) = \rho_*(x), \\ \nabla \cdot \mathbf{u} = 0 \quad \text{and} \quad \mathbf{u}(t=0,x) = \widehat{\mathbf{u}}(x), \\ \rho(t,x)(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla \Pi \end{cases}$$

(with variable density when $\rho_*'(x) \neq 0$) and $p = p_*$.

Idem for the Navier-Stokes syst. when the thermal fluxes are not high.

Introduction

Nevertheless, when we apply a (2D or 3D) Godunov type scheme

on a mesh that is not triangular, the discrete compressible Euler solution:

- converges with high difficulties to an incompressible solution when $\Delta x \to 0$ ($M \ll 1$ is given);
- does not converge to an incompressible solution when $M \to 0$ (Δx is given).

Introduction

For example, we find in [Guillard *et al.*, 1999] when the mesh is not triangular:

On the behavior of upwind schemes in the low Mach number limit : II. 7

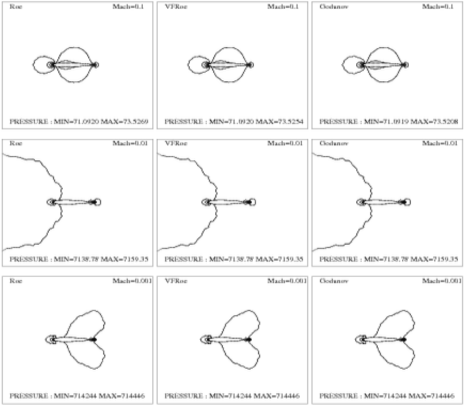
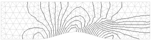
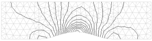


Figure 1: **Isovalues of the pressure, on a 3114 node mesh for** $M_\infty = 0.1$ **(top),** $M_\infty = 0.01$ **(middle),** $M_\infty = 0.001$ **(bottom) and for Roe scheme (left), VFRoe scheme (middle), Godunov scheme (right).**

Nevertheless, when the mesh is TRIANGULAR, the results seem to remain accurate:



Iso-Mach, VFRoe scheme, $M = 10^{-2}$



Iso-press., VFRoe scheme, $M = 10^{-2}$

WHAT HAPPENS !?!?

I.1 - From the non-linear case to the linear case

With $\rho(t,x) := \rho_*[1 + \frac{M}{a_*} r(t,x)]$ ($\rho_* = \mathcal{O}(1)$, $a_* = \sqrt{p'(\rho_*)}$), the (dimensionless) **barotropic Euler system**

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla(\rho \mathbf{u} \otimes \mathbf{u}) + \dfrac{\nabla p(\rho)}{M^2} = 0. \end{cases}$$

is **equivalent** to

$$\partial_t q + \mathcal{H}(q) + \frac{\mathcal{L}}{M}(q) = 0$$

with

$$\begin{cases} q = \begin{pmatrix} r \\ \mathbf{u} \end{pmatrix}, \\ \mathcal{H}(q) = \begin{pmatrix} \mathbf{u} \cdot \nabla r \\ (\mathbf{u} \cdot \nabla) \mathbf{u} \end{pmatrix} := (\mathbf{u} \cdot \nabla) q, \\ \mathcal{L}(q) = \begin{pmatrix} (a_* + Mr) \nabla \cdot \mathbf{u} \\ \dfrac{p'[\rho_*(1 + \frac{M}{a_*} r)]}{a_*(1 + \frac{M}{a_*} r)} \nabla r \end{pmatrix}. \end{cases}$$

- $\mathcal{H}$ = non-linear transport operator (time scale = 1);
- $\mathcal{L}/M$ = non-linear acoustic operator (time scale = M).

- **Linearization without convection:** Let us define the linearization of $\mathcal{L}(q)$ with

$$\begin{cases} q = \begin{pmatrix} r \\ \mathbf{u} \end{pmatrix}, \\ \\ Lq = a_* \begin{pmatrix} \nabla \cdot \mathbf{u} \\ \\ \nabla r \end{pmatrix} \end{cases}$$

where $a_* = C_2^{st}$ such that $\mathcal{O}(a_*) = 1$.

  - ▶ $L/M =$ linear acoustic operator (time scale $= M$).

So, we replace the (dimensionless) barotropic Euler system

$$\partial_t q + \mathcal{H}(q) + \frac{\mathcal{L}}{M}(q) = 0$$

with the linear wave equation

$$\partial_t q + \frac{L}{M} q = 0. \tag{1}$$

Let us note that (1) may be seen as a linearization of the comp. Euler system (without convection) **with**

$$r(t,x) \quad \text{such that} \quad p(t,x) := p_* \left[1 + \frac{M}{a_*} r(t,x) \right].$$

In the sequel, r will be considered as a *pressure perturbation*.

Let us now introduce the sets

$$(L^2(\mathbb{T}^d))^{1+d} := \left\{ q := \begin{pmatrix} r \\ \mathbf{u} \end{pmatrix} : \int_{\mathbb{T}^d} r^2 dx + \int_{\mathbb{T}^d} |\mathbf{u}|^2 dx < +\infty \right\}$$

equipped with the inner product $\langle q_1, q_2 \rangle = \int_{\mathbb{T}^d} q_1 q_2 dx$ and

$$\begin{cases} \mathcal{E} = \left\{ q \in (L^2(\mathbb{T}^d))^{1+d} : \nabla r = 0 \text{ and } \nabla \cdot \mathbf{u} = 0 \right\}, \\ \mathcal{E}^\perp = \left\{ q \in (L^2(\mathbb{T}^d))^{1+d} : \int_{\mathbb{T}^d} r dx = 0, \exists \phi \in H^1(\mathbb{T}^d), \mathbf{u} = \nabla \phi \right\} \end{cases}$$

($\mathbb{T}^d$ is the torus in $\mathbb{R}^d$, $d \in \{1, 2, 3\}$). Let us recall that:

Lemma 2.1 (Hodge decomposition)

$$\mathcal{E} \oplus \mathcal{E}^\perp = (L^2(\mathbb{T}^d))^{1+d} \qquad \textit{and} \qquad \mathcal{E} \perp \mathcal{E}^\perp.$$

In other words, any $q \in (L^2(\mathbb{T}^d))^{1+d}$ *can be decomposed into*

$$q = \widehat{q} + q^\perp \qquad \textit{where} \qquad (\widehat{q} := \mathbb{P}q, q^\perp) \in \mathcal{E} \times \mathcal{E}^\perp.$$

- **The low Mach asymptotics and the linear wave equation:**

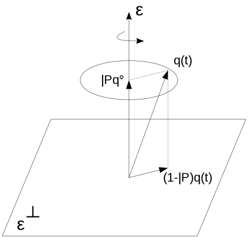
Lemma 2.2

Let $q(t,x)$ be solution of the linear wave equation

$$\begin{cases} \partial_t q + \dfrac{L}{M} q = 0, \\ q(t=0,x) = q^0(x). \end{cases} \tag{2}$$

Thus, we have $q = q_1 + q_2$ with $q_1 = \mathbb{P}q^0$ and $q_2 = (\mathbf{1} - \mathbb{P})q =: q^\perp$ where q_2 is solution of (2) with the initial condition $q_2(t=0,x) = (\mathbf{1} - \mathbb{P})q^0(x)$.
Moreover, we have

$$||q^0 - \mathbb{P}q^0|| = \mathcal{O}(M) \quad \Longrightarrow \quad ||q - \mathbb{P}q^0||(t \geq 0) = \mathcal{O}(M). \tag{3}$$



The previous results are obtained by using the properties:

- Conservation of the energy $E(t) := \langle q, q \rangle = C^{st}$.
- $\mathcal{E} = KerL$.

- **We can relax these two properties with:**

Theorem 2.3
Let $q(t,x)$ be solution of the linear PDE

$$\begin{cases} \partial_t q + \dfrac{\mathcal{L}}{M} q = 0, \\ q(t=0) = q^0 \end{cases} \tag{4}$$

supposed to be well-posed in such a way $||q||(t \geq 0) \leq C||q^0||$ where C does not depend on M. Then, when $\mathcal{L}$ is such that

$$\mathcal{E} \subseteq Ker\mathcal{L},$$

the solution $q(t,x)$ of (4) verifies

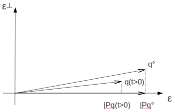
$$||q^0 - \mathbb{P}q^0|| = \mathcal{O}(M) \quad \Longrightarrow \quad ||q - \mathbb{P}q^0||(t \geq 0) = \mathcal{O}(M).$$

Definition 2.4

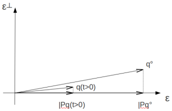
The solution $q(t,x)$ of $\left\{\begin{array}{l} \partial_t q + \dfrac{\mathcal{L}}{M} q = 0, \\ q(t=0) = q^0 \end{array}\right.$ *is said to be accurate at low Mach number in the incompressible regime if and only if the estimate*

$$||q^0 - \mathbb{P}q^0|| = \mathcal{O}(M) \quad \Longrightarrow \quad ||q - \mathbb{P}q^0||(t \geq 0) = \mathcal{O}(M) \tag{5}$$

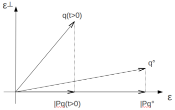
is satisfied.



(5) is verified



(5) is NOT verified



(5) is NOT verified

- **We deduce from Theorem 2.3 that:**

$\mathcal{E} \subseteq Ker\mathcal{L}$ is a **sufficient** condition to be accurate in the sense of Definition 2.4.

- The low Mach problem can be explained by replacing L with

$$\mathcal{L} = L + \delta L$$

where $\delta L =$ *perturbation due to the spatial discretization*.

The Godunov scheme applied to the linear wave equation is given by

$$\begin{cases} \dfrac{d}{dt} r_i + \dfrac{a_*}{M} \cdot \dfrac{1}{2|\Omega_i|} \displaystyle\sum_{\Gamma_{ij} \subset \partial\Omega_i} |\Gamma_{ij}| [(\mathbf{u}_i + \mathbf{u}_j) \cdot \mathbf{n}_{ij} + r_i - r_j] = 0, \\ \dfrac{d}{dt} \mathbf{u}_i + \dfrac{a_*}{M} \cdot \dfrac{1}{2|\Omega_i|} \displaystyle\sum_{\Gamma_{ij} \subset \partial\Omega_i} |\Gamma_{ij}| [r_i + r_j + \kappa(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij}] \mathbf{n}_{ij} = 0 \end{cases}$$

with $\kappa := 1$. This scheme can be written in the compact form

$$\begin{cases} \dfrac{d}{dt} q_h + \dfrac{\mathbb{L}_{\kappa,h}}{M} q_h = 0, \\ q_h(t=0) = q_h^0 \end{cases} \quad \text{with} \quad q_h := \begin{pmatrix} r_i \\ \mathbf{u}_i \end{pmatrix} \tag{6}$$

Lemma 2.5

$$Ker \mathbb{L}_{\kappa=1,h} = \left\{ q_h := \begin{pmatrix} r_h \\ \mathbf{u}_h \end{pmatrix} \in \mathbb{R}^{3N} \quad s.t. \quad \exists c, \ \forall i : \ r_i = c \ and \ (\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij} = 0 \right\}$$

$$Ker \mathbb{L}_{\kappa=0,h} = \left\{ q_h := \begin{pmatrix} r_h \\ \mathbf{u}_h \end{pmatrix} \in \mathbb{R}^{3N} \quad s.t. \quad \exists c, \ \forall i : \ r_i = c \ and \ \sum_{\Gamma_{ij} \subset \partial\Omega_i} |\Gamma_{ij}| \frac{\mathbf{u}_i + \mathbf{u}_j}{2} \cdot \mathbf{n}_{ij} = 0 \right\}.$$

Do we have $\mathcal{E}_h \subseteq Ker \mathbb{L}_{\kappa,h}$??? Let us note that $\sum_{\Gamma_{ij} \subset \partial\Omega_i} |\Gamma_{ij}| \frac{\mathbf{u}_i + \mathbf{u}_j}{2} \cdot \mathbf{n}_{ij} \simeq \int_{\Omega_i} \nabla \cdot \mathbf{u} dx$.

In the cartesian and triangular cases: Let us define

$$\mathcal{E}_h^{\square} = \left\{ \begin{array}{ll} q := \begin{pmatrix} r \\ \mathbf{u} \end{pmatrix} \in \mathbb{R}^{3N_xN_y} \quad \text{such that} & \exists (a, b, c, (\psi_{i,j})) \in \mathbb{R}^3 \times \mathbb{R}^{N_xN_y} : \\ & r_{i,j} = c, \ \mathbf{u}_{i,j} = \begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y} \\ -\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x} \end{pmatrix} \end{array} \right.$$

and in the triangular case

$$\mathcal{E}_h^{\Delta} = \left\{ \begin{array}{ll} q := \begin{pmatrix} r \\ \mathbf{u} \end{pmatrix} \in \mathbb{R}^{3N} \quad \text{such that} & \exists (a, b, c, \psi_h) \in \mathbb{R}^3 \times V_h : \\ & r_i = c, \ \mathbf{u}_i = \begin{pmatrix} a \\ b \end{pmatrix} + (\nabla \times \psi_h)_{|T_i} \end{array} \right\}$$

where $V_h := \left\{ \psi_h \in C_0(\overline{\mathbb{T}^d}),\ \psi_h \text{ periodic on } \overline{\mathbb{T}^d} \text{ such that } \forall T_i :\ (\psi_h)_{|T_i} \in P^1(T_i) \right\}$.

We can prove that:

$$\left\{ \begin{array}{lll} \textit{On a triangular mesh}: & Ker\mathbb{L}_{\kappa=1,h} = \mathcal{E}_h^{\Delta} \subset Ker\mathbb{L}_{\kappa=0,h}, \\ \textit{On a 1D cartesian mesh:} & Ker\mathbb{L}_{\kappa=1,h} = \mathcal{E}_h^{\square} \subseteq Ker\mathbb{L}_{\kappa=0,h}, \\ \textit{On a 2D cartesian mesh:} & Ker\mathbb{L}_{\kappa=1,h} \subsetneq \mathcal{E}_h^{\square} \subseteq Ker\mathbb{L}_{\kappa=0,h}. \end{array} \right.$$

These results show that at low Mach number[1]:

- The Godunov scheme is accurate on a triangular mesh.
- The Godunov scheme modified in such a way the pressure gradient is centered is accurate on cartesian and triangular meshes.
- The Godunov scheme should not be accurate at low Mach number:

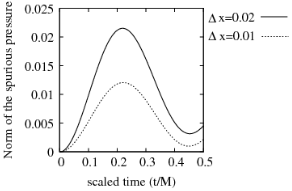
  $\longrightarrow$ It should transfer energy from $\mathcal{E}$ to $\mathcal{E}^{\perp}$ in a time $t = \mathcal{O}(M)$!

[1] With periodic boundary conditions.

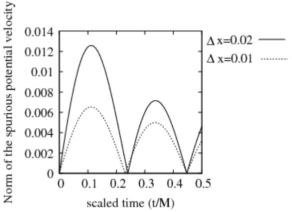
- **Linear Godunov scheme on a 2D CARTESIAN mesh:**
  - $a_* = 1$ and $M = 10^{-4}$.
  - Explicit Godunov scheme.
  - Cartesian mesh with $\Delta x = \Delta y = \mathcal{O}(10^{-2}) \gg M$.
  - Continuous initial condition: $r^0 = 1$ and $\mathbf{u}^0 = \nabla \times \psi$ with $\psi(x,y) = \frac{1}{\pi}\left[\sin^2(\pi x)\sin^2(2\pi y) - \frac{1}{4}\right]$. Thus: $q^0 \in \mathcal{E}$.
  - Discrete initial condition: $q_h^0 \in \mathcal{E}_h^{\square}$ that is to say

$$q_h^0 = \begin{pmatrix} r_{i,j} \\ \mathbf{u}_{i,j} \end{pmatrix} \quad \text{where} \quad \begin{cases} r_{i,j} = 1, \\ \mathbf{u}_{i,j} = \begin{pmatrix} \dfrac{\psi_{i,j+1} - \psi_{i,j-1}}{2\Delta y} \\ -\dfrac{\psi_{i+1,j} - \psi_{i-1,j}}{2\Delta x} \end{pmatrix} \end{cases}.$$

  - **Results:** $q(t^n)_h \neq q_h^0$ since $Ker\mathbb{L}_{\kappa=1,h} \subset \mathcal{E}_h^{\square}$. Moreover, **we numerically verify that:** $\exists \tau = \mathcal{O}(M) : \ ||q_h - \mathbb{P}_{\mathcal{E}_h^{\square}} q_h||(\tau) = \mathcal{O}(\Delta x)$ $\rightarrow$ spurious acoustic waves.



$||r_h^{\perp}||(t^n)$



$||\nabla\phi_h||(t^n)$

Introduction

I - The low Mach number problem and the linear wave equation

II - The linear case at any Mach number

III - The non-linear case at any Mach number

- **We introduce the new definition:**

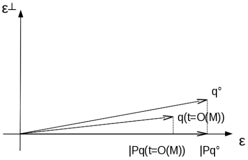
Definition 3.1
The solution $q(t,x)$ of

$$\begin{cases} \partial_t q + \dfrac{\mathcal{L}}{M} q = 0, \\ q(t=0) = q^0 \end{cases}$$

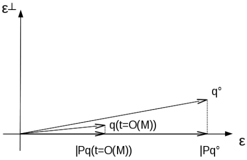
is said to be accurate at low Mach number in the incompressible regime iff the estimate

$$||q^0 - \mathbb{P}q^0|| = \mathcal{O}(M) \implies ||q - \mathbb{P}q^0||(t = \mathcal{O}(M)) = \mathcal{O}(M) \qquad (7)$$

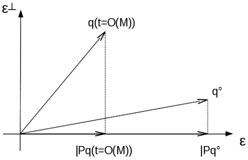
is satisfied.



(7) is verified



(7) is NOT verified



(7) is NOT verified

- **1^{st} order modified equation associated to the Godunov scheme:**

$$\begin{cases} \mathcal{L}_\nu := L - MB_\nu, \\ B_\nu q = \begin{pmatrix} \nu_r \Delta r \\ \nu_u \dfrac{\partial^2 u}{\partial x^2} \\ \nu_v \dfrac{\partial^2 v}{\partial y^2} \end{pmatrix}. \end{cases} \tag{8}$$

We define $\nu := (\nu_r, \nu_{\mathbf{u}})$ with $\nu_{\mathbf{u}} := (\nu_u, \nu_v)$.

Godunov scheme: $\nu_r = \nu_u = \nu_v = a_* \frac{\Delta x}{2M}$ (for the sake of simplicity: $\Delta x = \Delta y$).

$\nu_{\mathbf{u}}^{Godunov} := a_* \frac{\Delta x}{2M}(1, 1)$.

- **QUESTION:** What can we say about the equation

$$\begin{cases} \partial_t q + \dfrac{\mathcal{L}_\nu}{M} q = 0, \\ q(t=0) = q^0 \end{cases}$$

when $\nu_{\mathbf{u}} = \nu_{\mathbf{u}}^{Godunov}$ or when $\nu_{\mathbf{u}} \neq \nu_{\mathbf{u}}^{Godunov}$???

- **We have the following results:**

Lemma 3.2

1) In 1D: $Ker\mathcal{L}_\nu = \mathcal{E}$.

2) In 2D/3D with $\nu_{\mathbf{u}} = 0$: $Ker\mathcal{L}_\nu = \mathcal{E}$.

3) In 2D/3D with $\nu_{\mathbf{u}} := (\nu_{u_1}, \dots, \nu_{u_d}) \neq 0$ *such that* $\nu_{u_k} > 0$:

$$Ker\mathcal{L}_\nu = \left\{ q := \begin{pmatrix} r \\ \mathbf{u} \end{pmatrix} \in (L^2(\mathbb{T}))^{1+d} \quad \text{such that} \quad \nabla r = 0 \quad \text{and} \quad \partial_{x_k} u_k = 0 \right\} \subsetneq \mathcal{E}.$$

Remember that $\mathcal{E} \subseteq Ker\mathcal{L}_\nu$ is only a **sufficient** condition to be accurate !

→ **We have to be more precise ! What we have to proove:**

Let $q(t,x)$ *be solution of* $\begin{cases} \partial_t q + \dfrac{\mathcal{L}_\nu}{M} q = 0, \\ q(t=0) = q^0 \end{cases}$. *In 2D/3D:*

i) When $|\nu_{\mathbf{u}}| = \mathcal{O}(\frac{1}{M})$ *(e.g.* $\nu_{\mathbf{u}} = \nu_{\mathbf{u}}^{Godunov}$*),* $q(t,x)$ *is not accurate at low Mach numb.*

ii) When $|\nu_{\mathbf{u}}| = \mathcal{O}(1)$, $q(t,x)$ *is accurate at low Mach number that is to say*

$$||q^0 - \mathbb{P}q^0|| = \mathcal{O}(M) \quad \Longrightarrow \quad ||q - \mathbb{P}q^0||(t = \mathcal{O}(M)) = \mathcal{O}(M).$$

[Work in progress: OK for $\partial_t q = B_\nu q$ knowing that $\mathcal{L}_\nu := L - MB_\nu$]

II.2 - The discrete case

- The previous results incite us to propose the **All Mach linear scheme**

$$\frac{d}{dt}\begin{pmatrix} r \\ \mathbf{u} \end{pmatrix}_i + \frac{1}{|\Omega_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}|\Phi_{ij}^{AM} = 0 \tag{9}$$

with the **two following expressions for** Φ_{ij}^{AM} which are **equivalent** in this **linear** case:

1 - All Mach Godunov scheme = Godunov scheme + pressure correction:

$$\Phi_{ij}^{AM} = \Phi_{ij}^{Godunov} + (1-\theta)\frac{a_*}{2M}\begin{pmatrix} 0 \\ [(\mathbf{u}_j - \mathbf{u}_i)\cdot\mathbf{n}_{ij}]\mathbf{n}_{ij} \end{pmatrix} \quad \text{where} \quad \theta = \min(M, 1). \tag{10}$$

2 - All Mach Godunov scheme = Godunov scheme + corrected Riemann pressure:

$$\Phi_{ij}^{AM} = \frac{a_*}{M}\begin{pmatrix} (\mathbf{u}\cdot\mathbf{n})^* \\ r^{**}\mathbf{n} \end{pmatrix}_{ij} \tag{11}$$

with

$$r_{ij}^{**} = \theta r_{ij}^* + (1-\theta)\frac{r_i + r_j}{2} = \text{corrected Riemann pressure} \tag{12}$$

where $(\mathbf{u}\cdot\mathbf{n})^*$ is solution of the 1D linear Riemann problem that is to say

$$(\mathbf{u}\cdot\mathbf{n})_{ij}^* = \frac{(\mathbf{u}_i + \mathbf{u}_j)\cdot\mathbf{n}_{ij}}{2} + \frac{r_i - r_j}{2}.$$

Let us note that r^{**} replaces $r^* = \dfrac{r_i + r_j}{2} + \dfrac{(\mathbf{u}_i - \mathbf{u}_j)\cdot\mathbf{n}_{ij}}{2}$.

Introduction

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III.1 - Accurate Godunov type scheme at any Mach number

We define the **All Mach Godunov type scheme** in the **NON-linear** case

$$\frac{d}{dt}\left(\begin{array}{c}\rho\\ \rho\mathbf{u}\end{array}\right)_i + \frac{1}{|\Omega_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\Phi_{ij}^{AM\ X} = 0 \tag{13}$$

(X = Godunov type scheme) with the **two following expressions**:

1 - All Mach Godunov type scheme = Godunov type scheme + pressure correction:

$$\Phi_{ij}^{AM\ X} = \Phi_{ij}^{X} + (1-\theta_{ij})\frac{\rho_{ij}a_{ij}}{2}\left(\begin{array}{c}0\\ [(\mathbf{u}_j-\mathbf{u}_i)\cdot\mathbf{n}_{ij}]\mathbf{n}_{ij}\end{array}\right) \quad \text{where} \quad \theta_{ij} = \min(M_{ij},1). \tag{14}$$

Example: X = Roe scheme $\rightarrow$ (13)(14) defines an **All Mach Roe scheme**.

2 - All Mach Godunov type sch = Godunov type sch + corrected Riemann pressure:

$$\Phi_{ij}^{AM\ X} = \left(\begin{array}{c}\rho^*(\mathbf{u}\cdot\mathbf{n})^*\\ \rho^*(\mathbf{u}^*\cdot\mathbf{n})\mathbf{u}^* + p^{**}\mathbf{n}\end{array}\right)_{ij} \quad \text{with} \quad p_{ij}^{**} = \theta_{ij}p_{ij}^* + (1-\theta_{ij})\frac{p_i+p_j}{2} \tag{15}$$

where $(\rho^*,\mathbf{u}^*)$ is solution of a 1D **linearized** or **non-linearized** Riemann problem.
Let us note that p^{**} replaces $p(\rho^*)$.

Example: X = VFRoe scheme $\rightarrow$ (13)(15) defines an **All Mach VFRoe scheme**.

(13)(14) and (13)(15) are **NOT equivalent** in the non-linear case.

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Conjecture 4.1

These two **All Mach Godunov type schemes** *are stable and accurate at low Mach.*

These two schemes are already known. **Indeed:**

- These two **All Mach Godunov type schemes** have been proposed *(without any justification)* in:

  Fillion F., Chanoine A., D.S. and Kumbaro A. (2011). *FLICA-OVAP : a new platform for core thermal-hydraulic studies.*
  To appear in Nucl. Eng. and Design., 2011.

- The **All Mach Roe scheme** has been proposed *(and justified with a formal asymptotic expansion for a perfect gas EOS)* in:

  Rieper F. (2011). *A low Mach number fix for Roe's approximate Riemann solver.* J. Comp. Phys., **230**, pp. 5263-5287, 2011.

  Moreover, numerical results justify this scheme in this paper.

We formally justify the **All Mach Roe scheme** in the non-linear case

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t(\rho \mathbf{u}) + \nabla(\mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = 0 \end{cases} \tag{16}$$

on any mesh type with an asymptotic expansion (we recall that $a^2 = p'(\rho)$).

- *Roe scheme* applied to (16):

$$\begin{cases} \dfrac{d}{dt}\rho_i + \dfrac{1}{2|\Omega_i|} \displaystyle\sum_{\Gamma_{ij} \subset \partial\Omega_i} |\Gamma_{ij}| \left\{ (\rho_i \mathbf{u}_i + \rho_j \mathbf{u}_j) \cdot \mathbf{n}_{ij} + \dfrac{\rho_{ij}}{a_{ij}} (\mathbf{u}_{ij} \cdot \mathbf{n}_{ij})(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij} + a_{ij}(\rho_i - \rho_j) \right\} = 0, \\ \dfrac{d}{dt}(\rho_i \mathbf{u}_i) + \dfrac{1}{2|\Omega_i|} \displaystyle\sum_{\Gamma_{ij} \subset \partial\Omega_i} |\Gamma_{ij}| \left\{ \rho_i (\mathbf{u}_i \cdot \mathbf{n}_{ij}) \mathbf{u}_i + \rho_j (\mathbf{u}_j \cdot \mathbf{n}_{ij}) \mathbf{u}_j + a_{ij}(\rho_i - \rho_j)(\mathbf{u}_{ij} + (\mathbf{u}_{ij} \cdot \mathbf{n}_{ij}) \mathbf{n}_{ij}) \right. \\ \qquad + \rho_{ij} |\mathbf{u}_{ij} \cdot \mathbf{n}_{ij}| [(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{t}_{ij}] \mathbf{t}_{ij} + \dfrac{\rho_{ij}(\mathbf{u}_{ij} \cdot \mathbf{n}_{ij})}{a_{ij}} [(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij}] \mathbf{u}_{ij} \\ \qquad \left. + [p_i + p_j + \rho_{ij} a_{ij} (\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij}] \mathbf{n}_{ij} \right\} = 0. \end{cases}$$

- **All Mach Roe scheme**:

We add the pressure correction

$$(1-\theta_{ij})\frac{\rho_{ij}a_{ij}}{2}\begin{pmatrix} 0 \\ [(\mathbf{u}_j-\mathbf{u}_i)\cdot\mathbf{n}_{ij}]\mathbf{n}_{ij}\end{pmatrix} \quad \text{where} \quad \theta_{ij}=\min(M_{ij},1)$$

to the Roe flux which allows to obtain the **All Mach Roe scheme**

$$\left\{\begin{aligned}
&\frac{d}{dt}\rho_i+\frac{1}{2|\Omega_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\left\{(\rho_i\mathbf{u}_i+\rho_j\mathbf{u}_j)\cdot\mathbf{n}_{ij}+\frac{\rho_{ij}}{a_{ij}}(\mathbf{u}_{ij}\cdot\mathbf{n}_{ij})(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{n}_{ij}+a_{ij}(\rho_i-\rho_j)\right\}=0,\\
&\frac{d}{dt}(\rho_i\mathbf{u}_i)+\frac{1}{2|\Omega_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\,\{\rho_i(\mathbf{u}_i\cdot\mathbf{n}_{ij})\mathbf{u}_i+\rho_j(\mathbf{u}_j\cdot\mathbf{n}_{ij})\mathbf{u}_j+a_{ij}(\rho_i-\rho_j)(\mathbf{u}_{ij}+(\mathbf{u}_{ij}\cdot\mathbf{n}_{ij})\mathbf{n}_{ij})\\
&\qquad\qquad +\rho_{ij}|\mathbf{u}_{ij}\cdot\mathbf{n}_{ij}|[(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{t}_{ij}]\mathbf{t}_{ij}+\frac{\rho_{ij}(\mathbf{u}_{ij}\cdot\mathbf{n}_{ij})}{a_{ij}}[(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{n}_{ij}]\mathbf{u}_{ij}\\
&\qquad\qquad [p_i+p_j+\theta_{ij}\rho_{ij}a_{ij}(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{n}_{ij}]\mathbf{n}_{ij}\}=0.
\end{aligned}\right.$$

Let us note that if we choose $M_{ij}:=\dfrac{|u_{ij}|}{a_{ij}}$, we have

$$M_{ij}\leq 1: \qquad \theta_{ij}\rho_{ij}a_{ij}[(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{n}_{ij}]\mathbf{n}_{ij}=\rho_{ij}|u_{ij}|[(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{n}_{ij}]\mathbf{n}_{ij}.$$

- **Asymptotic expansion:** The **dimensionless All Mach Roe scheme** is given by

$$\left\{\begin{aligned}
&\frac{d}{dt}\rho_i + \frac{1}{2|\Omega_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\left\{(\rho_i\mathbf{u}_i+\rho_j\mathbf{u}_j)\cdot\mathbf{n}_{ij} + M\frac{\rho_{ij}}{a_{ij}}(\mathbf{u}_{ij}\cdot\mathbf{n}_{ij})(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{n}_{ij} + \frac{a_{ij}}{M}(\rho_i-\rho_j)\right\} = 0\\
&\frac{d}{dt}(\rho_i\mathbf{u}_i) + \frac{1}{2|\Omega_i|}\sum_{\Gamma_{ij}\subset\partial\Omega_i}|\Gamma_{ij}|\left\{\rho_i(\mathbf{u}_i\cdot\mathbf{n}_{ij})\mathbf{u}_i + \rho_j(\mathbf{u}_j\cdot\mathbf{n}_{ij})\mathbf{u}_j + \frac{a_{ij}}{M}(\rho_i-\rho_j)(\mathbf{u}_{ij}+(\mathbf{u}_{ij}\cdot\mathbf{n}_{ij})\mathbf{n}_{ij})\right.\\
&\qquad +\rho_{ij}|\mathbf{u}_{ij}\cdot\mathbf{n}_{ij}|[(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{t}_{ij}]\mathbf{t}_{ij} + M\frac{\rho_{ij}(\mathbf{u}_{ij}\cdot\mathbf{n}_{ij})}{a_{ij}}[(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{n}_{ij}]\mathbf{u}_{ij}\\
&\qquad \left. + \left[\frac{1}{M^2}(p_i+p_j) + \frac{\theta_{ij}}{M}\rho_{ij}a_{ij}(\mathbf{u}_i-\mathbf{u}_j)\cdot\mathbf{n}_{ij}\right]\mathbf{n}_{ij}\right\} = 0.
\end{aligned}\right. \tag{17}$$

Let us note that $\mathcal{O}\left(\frac{\theta_{ij}}{M}\right) = 1$ (since $\theta_{ij} = M_{ij}$ when $M_{ij} \leq 1$ and $M_{ij} = \mathcal{O}(M)$).

Let us suppose that $\phi \in \{\rho, p, u\}$ are such that

$$\phi = \phi^{(0)} + M\phi^{(1)} + M\phi^{(2)} + \dots \tag{18}$$

Then:

- We inject (18) in (17).
- We separate the orders M^{-2}, M^{-1} and M^0.

This gives ...

$$\rho^{(0)} = \rho_* \qquad \text{and} \qquad p^{(0)} = p(\rho_*) = p_*$$

$$\text{and} \qquad \left\{ \begin{array}{l} \displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}| \left[\frac{1}{\rho_* a_*}(p_i^{(1)} - p_j^{(1)}) + (\mathbf{u}_i^{(0)} + \mathbf{u}_j^{(0)}) \cdot \mathbf{n}_{ij} \right] = 0, \\ \\ \displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}| \frac{1}{\rho_* a_*}(p_i^{(1)} + p_j^{(1)})\mathbf{n}_{ij} = 0 \end{array} \right.$$

with $a_*^2 = p'(\rho_*)$. By defining $r^{(1)} := \frac{p^{(1)}}{\rho_* a_*}$, we obtain $q := \begin{pmatrix} r^{(1)} \\ \mathbf{u}^{(0)} \end{pmatrix} \in Ker\mathbb{L}_{\kappa=0,h}$.

As a consequence, the asymptotic expansion will be valid if the NECESSARY condition

$$q(t=0) \in Ker\mathbb{L}_{\kappa=0,h} \quad \text{is VALID !} \tag{19}$$

But, let us recall that we have proven that

$$Ker\mathbb{L}_{\kappa=0,h} = \left\{ q := \begin{pmatrix} r \\ \mathbf{u} \end{pmatrix} \in \mathbb{R}^{3N} \; s.t. \; \exists c : \; r_i = c \; and \sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}| \frac{\mathbf{u}_i + \mathbf{u}_j}{2} \cdot \mathbf{n}_{ij} = 0 \right\}$$

which seems to be a good approximation of $\mathcal{E}$.

More precisely, in the **cartesian** and **triangular** cases, we have proven that

$$\mathcal{E}_h^{\square} \subseteq Ker\mathbb{L}_{\kappa=0,h} \quad \text{and} \quad \mathcal{E}_h^{\Delta} \subset Ker\mathbb{L}_{\kappa=0,h}.$$

As a consequence, when the initial condition belongs to $\mathcal{E}_h^{\square}$ OR $\mathcal{E}_h^{\Delta}$, necessary condition (19) is verified at least in the **cartesian** OR **triangular** cases !

The **All Mach Roe scheme** applied to the linear system

$$\begin{cases} \partial_t q + Hq + \dfrac{L}{M} q = 0, & \text{(a)} \\ q(t=0,x) = q^0(x). & \text{(b)} \end{cases}$$

is given by

$$\begin{cases} \dfrac{d}{dt} r_i + \dfrac{1}{2|\Omega_i|} \displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}| & \Big\{ (\mathbf{u}_* \cdot \mathbf{n}_{ij})[r_i + r_j + (\mathbf{u}_i - \mathbf{u}_j)\cdot \mathbf{n}_{ij}] \\ & + \dfrac{a_*}{M}[(\mathbf{u}_i + \mathbf{u}_j)\cdot \mathbf{n}_{ij} + r_i - r_j] \Big\} = 0, \\ \dfrac{d}{dt} \mathbf{u}_i + \dfrac{1}{2|\Omega_i|} \displaystyle\sum_{\Gamma_{ij}\subset\partial\Omega_i} |\Gamma_{ij}| & \Big\{ (\mathbf{u}_* \cdot \mathbf{n}_{ij})[(\mathbf{u}_i + \mathbf{u}_j) + (r_i - r_j)\mathbf{n}_{ij}] + |\mathbf{u}_* \cdot \mathbf{n}_{ij}|[(\mathbf{u}_i - \mathbf{u}_j)\cdot \mathbf{t}_{ij}]\mathbf{t}_{ij} \\ & + \dfrac{a_*}{M}[r_i + r_j + \theta(\mathbf{u}_i - \mathbf{u}_j)\cdot \mathbf{n}_{ij}]\mathbf{n}_{ij} \Big\} = 0. \end{cases} \tag{20}$$

Remark: r in (20) and ρ in the dimensionless non-linear Roe scheme (17) are linked through $d\rho = \dfrac{\rho_*}{a_*} M dr$.

Let us define the energy

$$E_h = \sum_i |\Omega_i|(r_i^2 + |\mathbf{u}_i|^2).$$

We have the following L^2-stability result:

Proposition 4.1

i) When $\theta := \dfrac{|\mathbf{u}_*|}{a_*} M$*:*

$$\frac{d}{dt} E_h \leq 0. \tag{21}$$

ii) When $\theta := 0$*:*

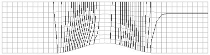
$$\frac{d}{dt} E_h \leq -\sum_{\Gamma_{ij}} |\Gamma_{ij}|(\mathbf{u}_* \cdot \mathbf{n}_{ij})(r_i - r_j)[(\mathbf{u}_i - \mathbf{u}_j) \cdot \mathbf{n}_{ij}]. \tag{22}$$

Remarks:

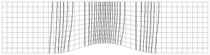
- $\theta := \dfrac{|\mathbf{u}_*|}{a_*} M$ = Mach number since a_*/M = sound velocity.
- **Inequality (21) justifies the choice** $\theta_{ij} = \min(M_{ij}, 1)$ **from a stability pt of view.**
- When $\theta := 0$ (in that case, the pressure gradient is centered), inequality (22) shows that we may observe instabilities (except when $\mathbf{u}_* := 0$).

III.4 - Numerical results

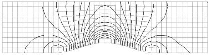
- **When the mesh is quadrangular:**



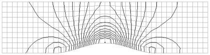
Iso-Mach, Roe scheme, $M = 10^{-2}$



Iso-pressure, Roe scheme, $M = 10^{-2}$



Iso-Mach, *low Mach Roe scheme*, $M = 10^{-2}$



Iso-pressure, *low Mach Roe scheme*, $M = 10^{-2}$

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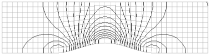
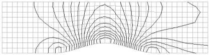
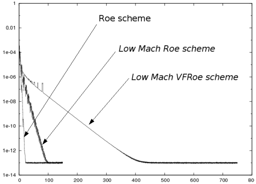
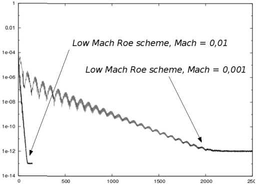


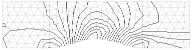


Iso-Mach, *low Mach Roe scheme*, $M = 10^{-3}$ Iso-Mach, *low Mach VFRoe scheme*, $M = 10^{-2}$

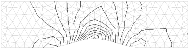




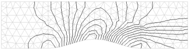
- **When the mesh is triangular:**



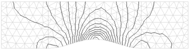
Iso-Mach, *low Mach VFRoe sch.*, $M = 10^{-2}$



Iso-pr., *low Mach VFRoe sch.*, $M = 10^{-2}$



Iso-Mach, VFRoe scheme, $M = 10^{-2}$



Iso-press., VFRoe scheme, $M = 10^{-2}$

Thank you for your attention !