Problèmes de variance en homogénéisation stochastique

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Classical periodic and stochastic homogenization

Periodic setting : Introduction to homogenization (I)

- Consider the family $(u_\varepsilon)_{\varepsilon>0}$ of solutions to the following PDEs :

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \quad \text{in} \quad \mathcal{D}, \\ u_\varepsilon = 0 \quad \text{on} \quad \partial\mathcal{D}, \end{cases}$$

with $A(y) = [a_{ij}(y)] \in \left(L^\infty(\mathbb{R}^d)\right)^{d\times d}$ is a Q-periodic function ($Q = [0,1]^d$) such that :

$$\exists \gamma > 0,\ \forall x_i \in \xi \in \mathbb{R}^d, \forall y \in \mathbb{R}^d \ \ \xi^T A(y)\xi \geq \gamma|\xi|^2,$$

and $f \in L^2(\mathcal{D})$.

- Up to extraction of a subsequence, the limit u_* in the weak sense of $(u_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$ can be shown to satisfy the following equation :

$$\begin{cases} -\operatorname{div}\left(A^\star\nabla u_\star\right) = f \quad \text{in} \quad \mathcal{D}, \\ u_\star = 0 \quad \text{on} \quad \partial\mathcal{D}. \end{cases}$$

- A^* is the homogenized matrix, which reflects the properties of some limit material. Its definition involves solutions w_p of the so-called corrector problems :

$$\begin{cases} \operatorname{div}\left[A\left(y\right)\left(p+\nabla w_p\right)\right]=0, \\ w_p \text{ is periodic,} \end{cases}$$

- Definition of the homogenized matrix :
  - general matrix :

$$\left[A^\star\right]_{ij}=\int_Q\left(e_i+\nabla w_{e_i}\right)^T A\left(e_j+\nabla w_{e_i}\right)dy,$$

  - symmetric matrix :

$$\left[A^\star\right]_{ij}=\int_Q\left(e_i+\nabla w_{e_i}\right)^T Ae_jdy.$$

- Consider the family $(u_\varepsilon)_{\varepsilon>0}$ of solutions to the following SPDEs :

$$\begin{cases} -\operatorname{div}\left(A\left(\frac{x}{\varepsilon},\omega\right)\nabla u_\varepsilon\left(\cdot,\omega\right)\right) = f \quad \text{in} \quad \mathcal{D}, \\ u_\varepsilon\left(\cdot,\omega\right) = 0 \quad \text{on} \quad \partial\mathcal{D}, \end{cases}$$

  with $A(y,\omega)$ stationnary such that :

$$\exists\gamma > 0,\ \forall x_i \in \xi \in \mathbb{R}^d, \forall \in \mathbb{R}^d\ \ \xi^T A(y)\xi \geq \gamma|\xi|^2, \text{almost surely}$$

  and $f \in L^2\left(\mathcal{D}\right)$.

- The deterministic limit in the weak sense $u_\star$ of $(u_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$ can be shown to satisfy a deterministic elliptic equation.

- The computation of A^* involves solutions w_p of the stochastic corrector problems :

$$\begin{cases} \operatorname{div}\left[A\left(y,\omega\right)\left(p+\nabla w_p\right)\right]=0, \\ \nabla w_p \text{ is stationary,} \\ \mathbb{E}\left(\int_Q \nabla w_p(y,\cdot)dy\right)=0. \end{cases}$$

- Definition of the deterministic homogenized matrix :

$$\boxed{\left[A^\star\right]_{ij}=\mathbb{E}\left(\int_Q \left(e_i+\nabla w_{e_i}\right)^T A\left(e_j+\nabla w_{e_i}\right)dy\right),}$$

- Numerical computation of $A^\star$ is thus quite a difficult task.
- Discretization of a SPDE posed on the whole space $\mathbb{R}^d$.

Variance reduction for general stochastic homogenization

- Our Goal: approximating $A^\star$
- First step: the truncated corrector problem

$$\begin{cases} -\mathrm{div}\left(A(\cdot,\omega)\left(p+\nabla w_p^N(\cdot,\omega)\right)\right)=0 \quad \text{on} \quad \mathbb{R}^d, \\ w_p^N(\cdot,\omega) \text{ is } Q_N\text{-periodic.} \end{cases}$$

where $Q_N=[-N,N]$.

- Second step : defining the approximated homogenized matrix $A_N^\star$

$$[A_N^\star]_{ij}(\omega)=\frac{1}{|Q_N|}\int_{Q_N}\left(e_i+\nabla w_{e_i}^N(y,\omega)\right)^T A(y,\omega)\left(e_j+\nabla w_{e_j}^N(y,\omega)\right)\,dy$$

- The randomness originates from truncation

Variance issues: Monte Carlo methods (I)

- Underlying assumption: $\mathbb{E}\left(A_N^\star\right) \approx A^\star$
- Monte Carlo method for computations of $\mathbb{E}\left(A_N^\star\right)$
  - Definition of estimators associated to $M = 2\mathcal{M}$ realizations

$$\boxed{\begin{aligned} \mu_M\left([A_N^\star]_{ij}\right) &= \frac{1}{M}\sum_{\mathbf{m}=1}^{M}\left[A_N^{\star,\mathbf{m}}\right]_{ij}, \\ \sigma_M\left([A_N^\star]_{ij}\right) &= \frac{1}{M-1}\sum_{\mathbf{m}=1}^{M}\left(\left[A_N^{\star,\mathbf{m}}\right]_{ij} - \mu_M\left([A_N^\star]_{ij}\right)\right)^2. \end{aligned}}$$

  - Properties of $\mu_M\left([A_N^\star]_{ij}\right)$:
    - Strong law of large numbers
    - Central Limit Theorem

$$\sqrt{M}\left(\mu_M\left([A_N^\star]_{ij}\right) - \mathbb{E}\left([A_N^\star]_{ij}\right)\right) \xrightarrow[M\to+\infty]{\mathcal{L}} \sqrt{\mathbb{V}\mathrm{ar}\left([A_N^\star]_{ij}\right)}\,\mathcal{N}(0,1)$$

Variance issues: Monte Carlo methods (II)

- The context of numerical practice
- $\mathbb{E}\left(A_N^\star\right)$ is in the confidence interval

$$\mu_M\left(\left[A_N^\star\right]_{ij}\right) \pm 1.96 \frac{\sqrt{\sigma_M\left(\left[A_N^\star\right]_{ij}\right)}}{\sqrt{M}}.$$

- Precision of the estimation depends on $\mathbb{V}\text{ar}\left(\left[A_N^\star\right]_{ij}\right)$

Our approach:

Defining new estimators of $A_N^\star$ with smaller variance than $\mu_M\left(\left[A_N^\star\right]_{ij}\right)$

Variance issues: antithetic variables (I)

- We give ourselves $\mathcal{M}$ i.i.d. copies $(A^{\mathbf{m}}(x,\omega))_{1\leq \mathbf{m}\leq \mathcal{M}}$ of $A(x,\omega)$
- Defining
  - an antithetic initial field

$$B^{\mathbf{m}}(x,\omega) = T\left(A^{\mathbf{m}}(x,\omega)\right), \quad 1 \leq \mathbf{m} \leq \mathcal{M},$$

  - an antithetic corrector problem whose solution is denoted v_p
- We finally build an antithetic homogenized matrix

$$\left[B_N^{\star,\mathbf{m}}\right]_{ij}(\omega) = \frac{1}{|Q_N|}\int_{Q_N} \left(e_i + \nabla v_{e_i}^{N,\mathbf{m}}(\cdot,\omega)\right)^T B^{\mathbf{m}}(\cdot,\omega)\,\left(e_j + \nabla v_{e_j}^{N,\mathbf{m}}(\cdot,\omega)\right).$$

Variance issues: antithetic variables (II)

- We introduce a new random variable

$$\widetilde{A}_N^{\star,\mathbf{m}}(\omega) := \frac{1}{2}\left(A_N^{\star,\mathbf{m}}(\omega) + B_N^{\star,\mathbf{m}}(\omega)\right).$$

- It is
  - unbiaised

$$\mathbb{E}\left(\widetilde{A}_N^{\star,\mathbf{m}}\right) = \mathbb{E}\left(A_N^{\star,\mathbf{m}}\right)$$

  - convergent

$$\widetilde{A}_N^{\star,\mathbf{m}} \underset{N\to+\infty}{\longrightarrow} A^\star \text{ almost surely,}$$

    because B is ergodic.

$\widetilde{A}_N^\star$ requires the solution of 2 correctors problems instead of 1 for the classical estimator !

Variance issues: antithetic variables (III)

- We define new estimators with identical properties

$$\boxed{\begin{aligned}\mu_{\mathcal{M}}\left(\left[\widetilde{A}_N^{\star}\right]_{ij}\right) &= \frac{1}{\mathcal{M}}\sum_{\mathbf{m}=1}^{\mathcal{M}}\left[\widetilde{A}_N^{\star,\mathbf{m}}\right]_{ij},\\ \sigma_{\mathcal{M}}\left(\left[\widetilde{A}_N^{\star}\right]_{ij}\right) &= \frac{1}{\mathcal{M}-1}\sum_{\mathbf{m}=1}^{\mathcal{M}}\left(\left[\widetilde{A}_N^{\star,\mathbf{m}}\right]_{ij} - \mu_{\mathcal{M}}\left(\left[\widetilde{A}_N^{\star}\right]_{ij}\right)\right)^2.\end{aligned}}$$

- Efficiency criteria
  - Equal computational costs

    $\boxed{M = 2\mathcal{M} \text{ realizations of } A_N^{\star} \text{ vs } \mathcal{M} \text{ of } \widetilde{A}_N^{\star}}$

  - Theoretic condition

$$\mathbb{C}\mathrm{ov}\left([A_N^{\star}]_{ij}, [B_N^{\star}]_{ij}\right) \leq 0.$$

Results

Theoretic validation (I)

- Hypotheses on the initial field A
  - A deterministic function of uniform r.v on finite-size cells

    There exists an integer n and a function $\mathcal{A}$, defined on $Q_N \times \mathbb{R}^n$, such that

    $$\forall x \in Q_N, \quad A(x,\omega) = \mathcal{A}(x, X_1(\omega), \dots, X_n(\omega)) \quad \text{a.s.},$$

    where $\{X_k(\omega)\}_{1\le k\le n}$ are independent scalar random variables, which are all distributed according to the uniform law $\mathcal{U}\,[0,1]$

  - Monotonicity in the sense of symmetric matrices

    For all $x \in Q_N$, and any vector $\xi \in \mathbb{R}^d$, the map

    $$(x_1, \dots, x_n) \in \mathbb{R}^n \mapsto \xi^T \mathcal{A}(x, x_1, \dots, x_n)\xi$$

    is non-decreasing with respect to each of its arguments.

Proposition 0.1 *We assume that $A(x,\omega)$ satisfies the above hypotheses. We define on Q_N the field*

$$B(x,\omega) := \mathcal{A}(x, 1-X_1(\omega), \ldots, 1-X_n(\omega)),$$

antithetic to $A(\cdot,\omega)$. We associate to this field the antithetic corrector problem (replacing A by B), the solution of which is denoted by v_p^N, and the matrix $B_N^\star(\omega)$, defined from v_p^N. Then,

$$\forall \xi \in \mathbb{R}^d, \quad \mathbb{C}\mathit{ov}\left(\xi^T A_N^\star \xi, \xi^T B_N^\star \xi\right) \leq 0$$

Otherwise stated, $\widetilde{A}_N^\star$ is an unbiased estimator of $\mathbb{E}\left(A_N^\star\right)$, and

$$\mathbb{V}\mathrm{ar}\left(\xi^T \widetilde{A}_N^\star \xi\right) \leq \frac{1}{2}\mathbb{V}\mathrm{ar}\left(\xi^T A_N^\star \xi\right).$$

Theoretic validation (III)

- This result guarantees the efficiency of the technique for
  - a wide range of initial fields characterized by
    1. different local behaviours
    2. different correlation structures
    3. different isotropy properties
  - a wide range of outputs including
    1. diagonal terms $[A_N^\star]_{ii}$
    2. eigenvalues of $A_N^\star$
    3. eigenvalues of $L_{A_N^\star} = -\mathrm{div}\left(A_N^\star \nabla \cdot\right)$

- The case of an isotropic initial field with no correlation

$$A(x,\omega) = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{Q+k}(x) a_k(\omega)\mathrm{Id} = \sum_{k\in\mathbb{Z}^d} \mathbf{1}_{Q+k}(x) f\left(X_k(\omega)\right)\mathrm{Id}.$$

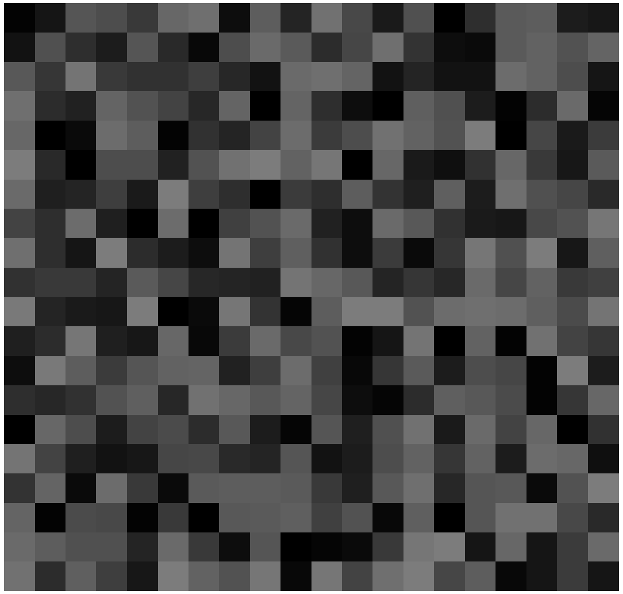
  - Values a_k and $a_{k'}$ of A on two distinct cells $Q+k$ and $Q+k'$ are independent.
  - A is invariant under rotations of angle $\pi/2$
  - The law of A is determined by that of a_0
    - case (i): $a_0 \sim \mathcal{U}([\alpha,\beta])$; $f(x) = \alpha + (\beta-\alpha)x$
    - case (ii): $a_0 \sim \mathcal{B}(1/2)$, $\mathbb{P}\left(a_0 = \alpha\right) = 1/2$ and $\mathbb{P}\left(a_0 = \beta\right) = 1/2$;

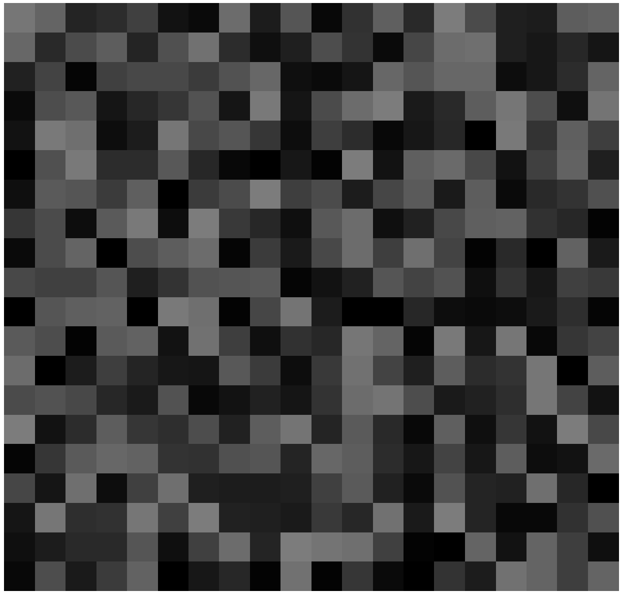
$$f(x) = \alpha + (\beta-\alpha)\mathbf{1}_{\left\{\frac{1}{2}\le x\le 1\right\}}.$$

- In the sequel, $\alpha = 3, \beta = 20$.

Numerical experiment: case (i) (I)

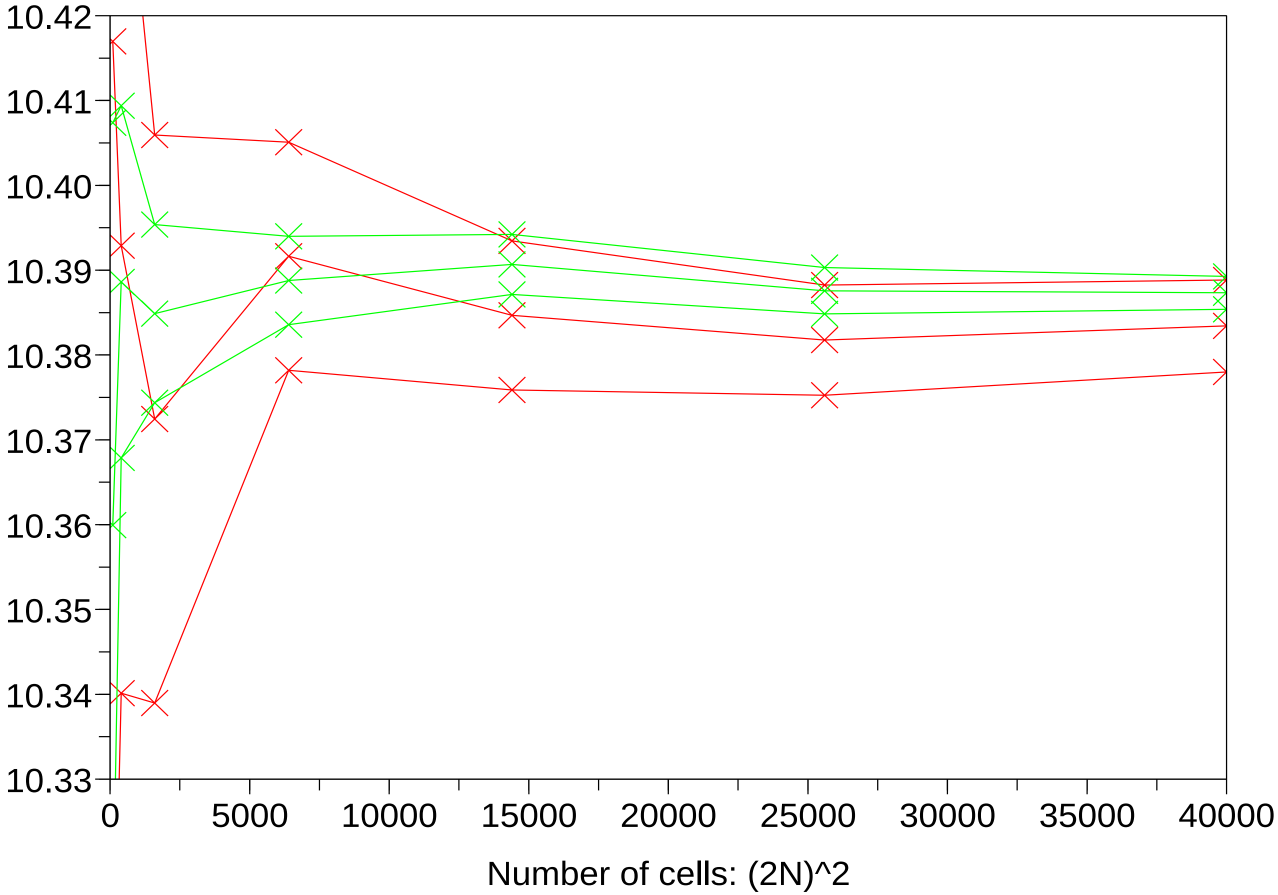
- One realization of A and its couterpart B when $a_0 \sim \mathcal{U}([\alpha, \beta])$





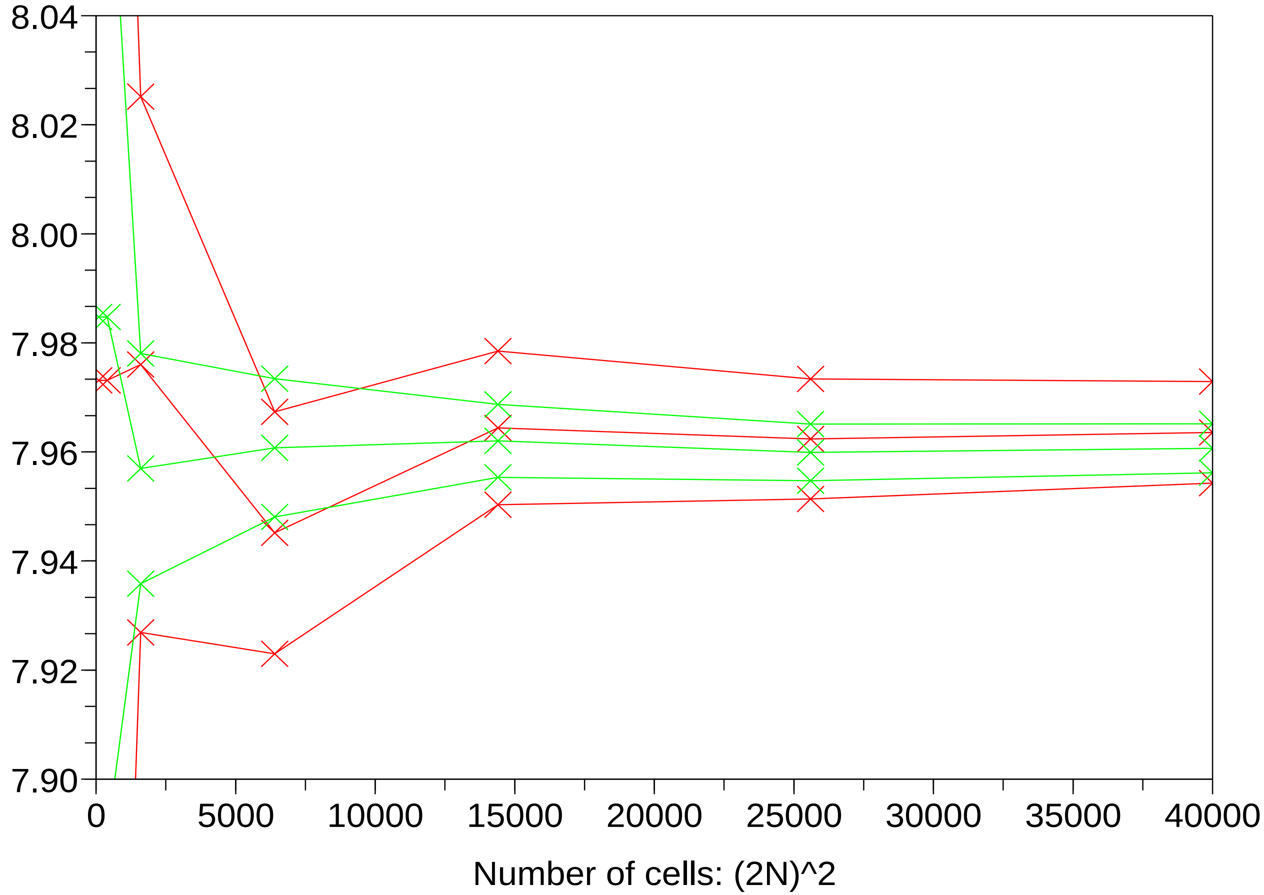
Numerical experiment: case (i) (II)

- Estimated means $\mu_{100}\left(A_N^\star\right)$ and $\mu_{50}\left(\widetilde{A}_N^\star\right)$ and respective confidence intervals.



- One realization of A and its couterpart B when $a_0 \sim \mathcal{B}(\frac{1}{2})$

- Estimated means $\mu_{100}\left(A_N^\star\right)$ and $\mu_{50}\left(\widetilde{A}_N^\star\right)$ and respective confidence intervals.



- Ratio of the widths of confidence intervals

$$R\left([A_N^\star]_{11}\right) = \frac{\sigma_{100}\left([A_N^\star]_{11}\right)}{2\sigma_{50}\left(\left[\widetilde{A}_N^\star\right]_{11}\right)}.$$

- Numerical values in the isotropic case

N	$a_0 \sim \mathcal{B}(1/2)$	$a_0 \sim \mathcal{B}(1/3)$	$a_0 \sim \mathcal{U}([\alpha,\beta])$
5	5.34	2.06	6.31
10	3.91	1.56	6.46
20	5.41	2.92	10.2
40	3.07	2.31	6.67
60	4.41	2.47	6.16
80	4.49	1.95	5.68
100	4.28	2.99	7.89

Weakly stochastic cases

- Linear perturbations
  - First model:

$$A_\eta(x,\omega) = A_{\mathrm{per}}(x) + \eta A_1(x,\omega) + O(\eta^2), \text{ in } \left(L^\infty\left(\Omega \times \mathbb{R}^d\right)\right)^{d\times d}$$

    where $A(y)$ is a periodic function satisfying the same conditions of boundedness and coercivity as above.
  - Second model:

$$A_\eta(x,\omega) = A_{\mathrm{per}}(x) + B_\eta(x,\omega) + O(\eta^2), \text{ in } \left(L^\infty\left(\mathbb{R}^d; L^p(\Omega)\right)\right)^{d\times d}$$

- Perturbations of alternative models

$$A_\eta(x,\omega) = A_{\mathrm{per}}\left(\Phi_\eta^{-1}(x,\omega)\right)$$

The Blanc Le Bris Lions framework

- Consider the family $(u_\varepsilon)_{\varepsilon>0}$ of solutions to the following SPDEs :

$$\begin{cases} -\operatorname{div}\left(A\left(\Phi^{-1}\left(\frac{x}{\varepsilon},\omega\right)\right)\nabla u_\varepsilon\left(\cdot,\omega\right)\right) = f \quad \text{in} \quad \mathcal{D}, \\ u_\varepsilon\left(\cdot,\omega\right) = 0 \quad \text{on} \quad \partial\mathcal{D}, \end{cases}$$

where $A(y)$ is a periodic function satisfying the same conditions of boundedness and coercivity as above.

- $\Phi(x,\omega)$ is a stationnary stochastic diffeomorphism :

$$\begin{cases} \underset{\omega\in\Omega,\, x\in\mathbb{R}^d}{\operatorname{EssInf}}\left[\det(\nabla\Phi(x,\omega))\right] = \nu > 0, \\ \underset{\omega\in\Omega,\, x\in\mathbb{R}^d}{\operatorname{EssSup}}\left(|\nabla\Phi(x,\omega)|\right) = M < \infty, \\ \nabla\Phi(x,\omega) \text{is stationnary.} \end{cases}$$

- Main results proved within the seminal paper :
  - The deterministic limit in the weak sense $u_\star$ of $(u_\varepsilon)_{\varepsilon>0}$ as $\varepsilon \longrightarrow 0$ satisfies a deterministic elliptic equation.
  - A new corrector problem :

$$\begin{cases} \operatorname{div}\left[A\left(\Phi^{-1}(y,\omega)\right)(p+\nabla w_p)\right]=0, \\ w_p(y,\omega)=\widetilde{w}_p\left(\Phi^{-1}(y,\omega),\omega\right), \quad \nabla\widetilde{w}_p \text{ is stationnary,} \\ \mathbb{E}\left(\int_{\Phi(Q,\cdot)}\nabla w_p(y,\cdot)dy\right)=0, \end{cases}$$

  - A new definition of the homogenized matrix :

$$[A^\star]_{ij}=\det\left(\mathbb{E}\left(\int_Q \nabla\Phi(z,\cdot)dz\right)\right)^{-1}$$
$$\mathbb{E}\left(\int_{\Phi(Q,\cdot)}(e_i+\nabla w_{e_i}(y,\cdot))^T A\left(\Phi^{-1}(y,\cdot)\right)e_j\,dy\right).$$

The perturbative result (I)

- Consider now the family $(\Phi_\eta)_{\eta \geq 0}$ of stochastic stationnary diffeomorphism, satisfying :

$$\Phi(x,\omega) = x + \eta \Psi(x,\omega) + O(\eta^2),$$

  when $\eta \to 0$ in $C^1(\mathbb{R}^d, L^\infty(\Omega))$.

- The gradient of the corrector $w_p = \widetilde{w}_p \left(\Phi^{-1}(x,\omega)\right)$ possesses an analogous expansion :

$$\nabla \widetilde{w}_p(x,\omega) = \nabla w_p^0(x) + \eta \nabla w_p^1(x,\omega) + O(\eta^2)$$

  in $L^2(Q \times \Omega)$.

- Introducing such an expansion into the formula of $A^\star$ leads to :

$$A^\star = A^0 + \eta A^1 + O(\eta^2),$$

- The equations satisfied by w_p^0 and w_p^1 are known.

The perturbative result (II)

- Deterministic cell problem satisfied by w^0 :

$$\begin{cases} -\operatorname{div}\left[A\,(p+\nabla w_p^0)\right] = 0, \\ w_p^0 \text{ is } Q\text{-periodic.} \end{cases}$$

- Stochastic cell problem satisfied by w^1 :

$$\begin{cases} -\operatorname{div}\left[A\,\nabla w_p^1\right] = \operatorname{div}\left[-A\,\nabla\Psi\,\nabla w_p^0 - (\nabla\Psi^T - (\operatorname{div}\Psi)\mathrm{Id})\,A\,(p+\nabla w_p^0)\right], \\ \nabla w_p^1 \text{ is stationnary and } \mathbb{E}\left(\int_Q \nabla w_p^1\right) = 0. \end{cases}$$

- First terms of $A^\star$'s expansion :

$$\begin{aligned} A_{ij}^0 &= \int_Q \left(e_i + \nabla w_{e_i}^0\right)^T A e_j \\ A_{ij}^1 &= -\int_Q \mathbb{E}(\operatorname{div}\Psi)\,A_{ij}^0 + \int_Q (e_i + \nabla w_{e_i}^0)^T A\,e_j\,\mathbb{E}(\operatorname{div}\Psi) \\ &+ \int_Q \left(\mathbb{E}(\nabla w_{e_i}^1) - \mathbb{E}(\nabla\Psi)\nabla w_{e_i}^0\right)^T A\,e_j. \end{aligned}$$

The perturbative result (III)

- In order to compute A^1 the knowledge of the deterministic $\overline{w}^1_{e_i} = \mathbb{E}(\nabla w^1_{e_i})$ is only required.
- $\overline{w}^1_{e_i} = \mathbb{E}(\nabla w^1_{e_i})$ is the unique periodic solution to the deterministic cell problem :

$$-\operatorname{div}\left[A\,\nabla\overline{w}^1_p\right] = \operatorname{div}\left[-A\,\mathbb{E}(\nabla\Psi)\,\nabla w^0_p - \left(\mathbb{E}(\nabla\Psi^T) - \mathbb{E}(\operatorname{div}\Psi)\mathrm{Id}\right)A\,(p+\nabla w^0_p)\right]$$

- Numerical strategy : instead of tackling the very general stochastic problem, one deals with two independent deterministic problems.

Do these results still hold when dealing with approximation spaces ?

A discretized version of BLL results

Discretization of the corrector problem (I)

- After domain truncation and introduction of finite element spaces, one obtains the following discrete formulation

$$\left\{ \begin{array}{l} \text{Find } \widetilde{w}_p^{h,N}(\cdot,\omega) \in V_h^{\mathrm{per}}(Q_N) \text{ such that, for all } \widetilde{v}_h \in V_h^{\mathrm{per}}(Q_N), \\ \int_{Q_N} \det(\nabla\Phi)\,(\nabla\widetilde{v}_h)^T\,(\nabla\Phi)^{-T}\,A\left(p + (\nabla\Phi)^{-1}\,\nabla\widetilde{w}_p^{h,N}(\cdot,\omega)\right) = 0, \text{ a.s.} \end{array} \right.$$

- The chosen method is thus a Monte-Carlo Method Finite Elements Method.
  - At each realization ω corresponds a divergence-form deterministic elliptic equation on the bounded domain Q_N, whose solution existence is guaranteed by Lax-Milgram Lemma.
  - No discretization of the random space Ω (Galerkin Method)

Discretization of the corrector problem (II)

- Thanks to the approximated corrector $\widetilde{w}_p^{h,N}(\cdot,\omega)$, one can compute an approximated random tensor

$$\left(A_\star^{h,N}\right)_{ij}(\omega) = \frac{\displaystyle\frac{1}{|Q_N|}\int_{Q_N} \det(\nabla\Phi)\left(e_i + (\nabla\Phi)^{-1}\,\nabla\widetilde{w}_{e_i}^{h,N}\right)^T A\, e_j.}{\det\left(\displaystyle\frac{1}{|Q_N|}\int_{Q_N}\nabla\Phi\right)}$$

- $A_\star^{h,N}(\omega)$ is random due to the loss of ergodicity resulting from the discretization procedure.
- What do the former results become within this discretized framework ?
- One introduces the formal expansion : $\widetilde{w}_p^{h,N} = w_p^{0,h,N} + \eta w_p^{1,h,N} + O(\eta^2)$

Discretization of the corrector problem (III)

- Introducing such a developement within the discrete corrector problem
  - $w_p^{0,h,N} = w_p^{0,h}$ is deterministic. It does not depend on N.

$$\left\{\begin{array}{l}\text{Find } w_p^{0,h} \in V_h^{\rm per}(Q) \text{ such that, for all } v_h \in V_h^{\rm per}(Q),\\ \int_Q (\nabla v_h)^T A \,(p + \nabla w_p^{0,h}) = 0,\end{array}\right.$$

  - $w_p^{1,h,N}$ is stochastic. It does depend on N :

$$\left\{\begin{array}{l}\text{Find } w_p^{1,h,N}(\cdot,\omega) \in V_h^{\rm per}(Q_N) \text{ such that, for all } v_h \in V_h^{\rm per}(Q_N),\\ \int_{Q_N} (\nabla v_h)^T A\, \nabla w_p^{1,h,N} = \int_{Q_N} (\nabla v_h)^T A\, \nabla\Psi\, \nabla w_p^{0,h}\\ + \int_{Q_N} \left(\nabla\Psi^T - (\operatorname{div}\Psi)\,\mathrm{Id}\right) A \left(p + \nabla w_p^{0,h}\right).\end{array}\right.$$

- First terms of the expansion

$$
\begin{aligned}
\left[A^{0,h}\right]_{ij} &= \int_Q \left(e_i + \nabla w^{0,h}_{e_i}\right)^T A \\
\left[A^{1,h,N}\right]_{ij} &= -(A^{0,h})_{ij} \, \frac{1}{|Q_N|} \int_{Q_N} \operatorname{div} \Psi \\
&+ \frac{1}{|Q_N|} \int_{Q_N} (e_i + \nabla w^{0,h}_{e_i})^T A \, e_j \, \operatorname{div} \Psi \\
&+ \frac{1}{|Q_N|} \int_{Q_N} \left(\nabla w^{1,h,N}_{e_i} - \nabla \Psi \, \nabla w^{0,h}_{e_i}\right)^T A \, e_j .
\end{aligned}
$$

- **Does the expansion $\left(A^{h,N}_{\star}\right)_{ij}(\omega) = (A^{0,h})_{ij} + \eta (A^{1,h,N})_{ij}$ hold ?**

Main result

- The discrete expansion

Proposition 0.2 *Suppose* $\Phi(x,\omega) = x + \eta\Psi(x,\omega) + O(\eta^2)$ *when* $\eta \to 0$, *in* $C^1(\mathbb{R}^d, L^\infty(\Omega))$, *with* $\nabla\Psi$ *stationary. A constant* C *exists such that, for small values of parameter* η *,*

$$\eta^{-2}\left|\nabla\widetilde{w}_p^{h,N}(x,\omega) - \nabla w_p^{0,h}(x) - \eta\nabla w_p^{1,h,N}(x,\omega)\right| \leq C,$$

where $\widetilde{w}_p^{h,N}$, $w_p^{0,h}$ *et* $w_p^{1,h,N}$ *are respectively solutions of the discrete, order 0 and order 1 corrector problems. In addition we have*

$$\eta^{-2}\left|A_\star^{h,N}(\omega) - A^{0,h} - \eta A^{1,h,N}(\omega)\right| \leq C,$$

- Sketch of the proof
  - uniform bounds in η thanks to the domain boundedness.
  - passing to the limit int the variatonal equation which corresponds to the discrete corrector problem.

Numerical experiments

- Reference periodic structure

$$\forall x \in Q,\ A(x) = a_{\text{per}}(x)\,\mathsf{Id}_2, \quad a_{\text{per}}(x_1, x_2) = \beta + (\alpha - \beta)\sin^2(\pi x_1)\sin^2(\pi x_2).$$

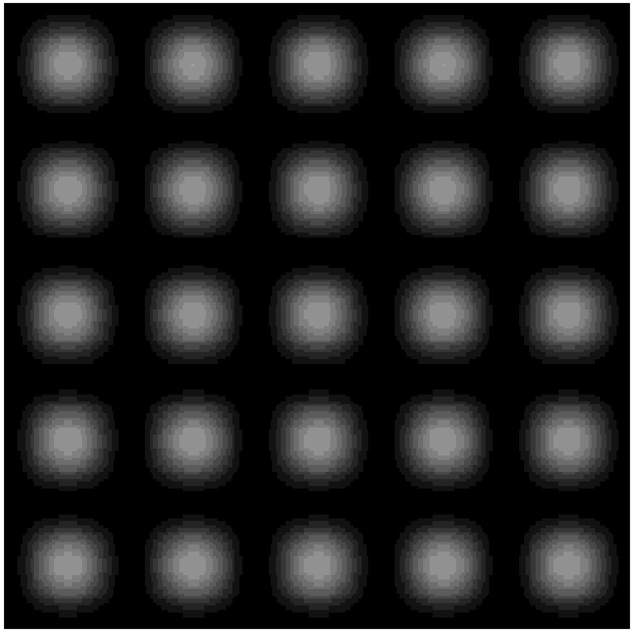
- Stochastic diffeomorphism :
  - Two independant family of i.i.d random variables $(X_k)_{k\in\mathbb{Z}}$ et $(Y_k)_{k\in\mathbb{Z}}$, who follow a uniform law $\mathcal{U}([a,b])$
  - $\Phi_\eta(x) = x + \eta\Psi(x,\omega)$, with $\Psi(x,\omega) = (\psi_X(x_1,\omega), \psi_Y(x_2,\omega))$ and

$$\psi_X(x_1,\omega) = \sum_{k\in\mathbb{Z}} 1_{[k,k+1[}(x_1)\left(\sum_{q=0}^{k-1} X_q(\omega) + 2X_q(\omega)\int_k^{x_1} \sin^2(2\pi t)\,dt\right),$$

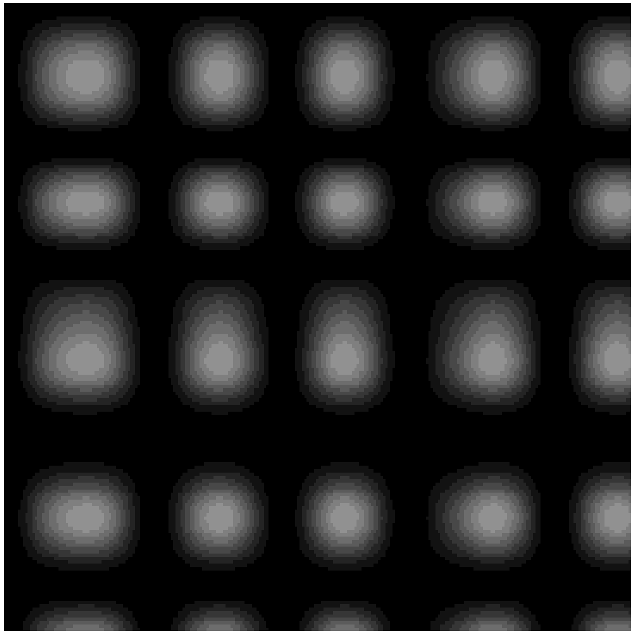
- Parameters

$$\alpha = 10 \quad \beta = 1 \quad a = 5.75 \quad b = -2.25$$

Reference and deformed structure

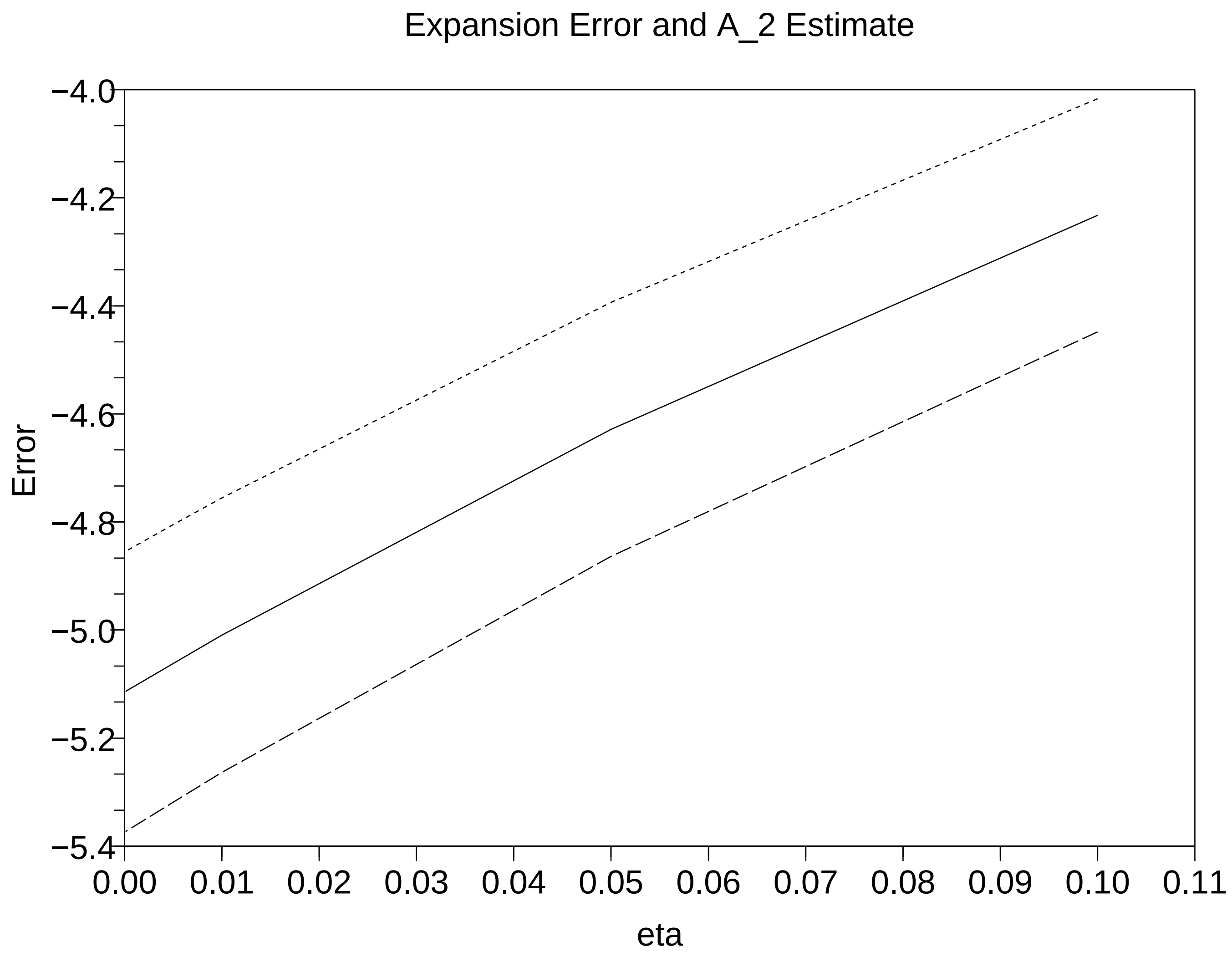


(a) Periodic structure



(b) Periodic structure through diffeomorphism $\Phi_\eta(x,\omega)$, $\eta = 0.05$

Several realizations



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