Problèmes d’information dans les jeux en temps continu

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Description of the game

We investigate a stochastic differential game defined by

$$\left\{ \begin{array}{l} dX_t = f(X_t, u_t, v_t)dt + \sigma(X_t, u_t, v_t)dB_t, \ \ t \in [t_0, T], \\ X_{t_0} = x_0, \end{array} \right.$$

where

- B is a d-dimensional standard Brownian motion
- $f : \mathbb{R}^N \times U \times V \to \mathbb{R}^N$ and $\sigma : \mathbb{R}^n \times U \times V \to \mathbb{R}^{N \times d}$ are Lipschitz continuous and bounded,
- the processes u (controlled by Player I) and v (controlled by Player II) take their values in some compact sets U and V.

The solution to (*) is denoted by $t \to X_t^{t_0, x_0, u, v}$.

The payoffs

Let $g : \mathbb{R}^N \to \mathbb{R}$ be the terminal payoff,

- Player I minimises the payoff $\mathbf{E}[g(X_T)]$
- Player II maximises the payoff $\mathbf{E}[g(X_T)]$.

Problems

- Describe the fact that the players chose their controls
  - simultaneously
  - by observing each other
- Compute (or characterize) their best payoffs.
- Compute (or characterize) their optimal strategies.

Outline

Outline

1. Solving classical differential games
   - Formalisation
   - Existence of the value
2. Games with imperfect information
   - Description of the game
   - Existence and characterization of the value
   - Illustration through a simple game
3. Differential games with imperfect observation

Outline

Strategies

- A strategy for Player I is a Borel measurable map $\alpha : [t_0, T] \times L^0([t_0, T], V) \times \mathcal{C}^0([t_0, T], \mathbb{R}^N) \to U$ such that there is $\tau > 0$ with

$$\begin{array}{ll} v_1 = v_2 \text{ and } f_1 = f_2 \text{ on } [t_0, t] \\ \Rightarrow \quad \alpha(s, v_1, f_1) = \alpha(s, v_2, f_2) \text{ for } s \in [t_0, t_0 + \tau] \end{array}$$

  The set of strategies for Player I is denoted by $\mathcal{A}(t_0)$.

- The set of strategies for Player II is defined symmetrically and denoted by $\mathcal{B}(t_0)$.

Playing pure strategies together

Lemma

For all $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$, for all $(\alpha, \beta) \in \mathcal{A}(t_0) \times \mathcal{B}(t_0)$, there exists a unique couple of controls (u, v) that satisfies

$$(*) \qquad (u, v) = (\alpha(\cdot, v, B_\cdot - B_{t_0}), \beta(\cdot, u, B_\cdot - B_{t_0})) \text{ on } [t_0, T].$$

Notation :

$$X_T^{t_0, x_0, \alpha, \beta} = X_T^{t_0, x_0, u, v}$$

where (u, v) is given by $(*)$.

Upper and lower value functions

The upper value function is

$$V^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} \mathbf{E}\left[g(X_T^{t_0, x_0, \alpha, \beta}) \right]$$

while the lower value function is

$$V^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} \mathbf{E}\left[g(X_T^{t_0, x_0, \alpha, \beta}) \right]$$

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Isaacs' condition

We assume that Isaacs'condition holds : for all $(t,x)\in[0,T]\times\mathbb{R}^N$, $\xi\in\mathbb{R}^N$, and all $A\in\mathcal{S}_N$:

$$H(x,\xi,A) :=$$

$$\inf_u \sup_v \{\langle f(x,u,v),\xi\rangle + \frac{1}{2} Tr(A\sigma(x,u,v)\sigma^*(x,u,v))\}$$

$$= \sup_v \inf_u \{\langle f(x,u,v),\xi\rangle + \frac{1}{2} Tr(A\sigma(x,u,v)\sigma^*(x,u,v))\}$$

Existence of a value

Theorem (Fleming-Souganidis, 1989)

Under Isaacs' condition, the game has a value :

$$V^+(t,x) = V^-(t,x) \qquad \forall (t,x) \in [0,T] \times \mathbb{R}^N .$$

The value $V := V^+ = V^-$ is the unique viscosity solution to the (backward) Hamilton-Jacobi equation

$$\left\{ \begin{array}{l} \partial_t w + H(x, Dw, D^2 w) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^N \\ w = g \qquad \text{in } \{T\} \times \mathbb{R}^N \end{array} \right.$$

Idea of proof (1)

Assume for simplicity that V^+ and V^- are smooth.

Lemma (Dynamic programming)

For $(t_0, x_0) \in [0, T] \times \mathbb{R}^N$ and $h > 0$,

$$V^+(t_0, x_0) = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} \mathbf{E}\left[V^+ \left(t_0 + h, X_{t_0+h}^{t_0, x_0, \alpha, \beta} \right) \right]$$

and

$$V^-(t_0, x_0) = \sup_{\beta \in \mathcal{B}(t_0)} \inf_{\alpha \in \mathcal{A}(t_0)} \mathbf{E}\left[V^- \left(t_0 + h, X_{t_0+h}^{t_0, x_0, \alpha, \beta} \right) \right]$$

Idea of proof (2)

From dynamic programming :

$$0 = \inf_{\alpha \in \mathcal{A}(t_0)} \sup_{\beta \in \mathcal{B}(t_0)} \mathbf{E}\left[V^+ \left(t_0 + h, X_{t_0+h}^{t_0,x_0,\alpha,\beta} \right) - V^+(t_0, x_0) \right]$$

$$\approx \inf_{\alpha} \sup_{\beta} \mathbf{E} \left[h\, \partial_t V^+ + \int_{t_0}^{t_0+h} \langle DV^+, f(\alpha, \beta) \rangle + \frac{1}{2} \mathrm{Tr}(\sigma\sigma^*(\alpha, \beta) D^2 V^+) ds \right]$$

Divide by h and let $h \to 0$:

$$0 = \partial_t V^+ + \inf_{u \in U} \sup_{v \in V} \left\{ \langle DV^+, f(x_0, u, v) \rangle + \frac{1}{2} \mathrm{Tr}(\sigma\sigma^*(x_0, u, v) D^2 V^+) \right\}$$

$$= \partial_t V^+(t_0, x_0) + H(x_0, DV^+(t_0, x_0), D^2 V^+(t_0, x_0)) \ .$$

Sketch of proof (3)

So V^+ and V^- are both solutions to the Hamilton-Jacobi equation

$$\left\{ \begin{array}{l} \partial_t w + H(x, Dw, D^2 w) = 0 \qquad \text{in } (0,T) \times \mathbb{R}^N \\ w = g \qquad \text{in } \{T\} \times \mathbb{R}^N \end{array} \right.$$

Uniqueness of the solution $\Rightarrow V^+ = V^-$.

Sketch of proof (3)

So V^+ and V^- are both solutions to the Hamilton-Jacobi equation

$$\begin{cases} \partial_t w + H(x, Dw, D^2 w) = 0 & \text{in } (0,T) \times \mathbb{R}^N \\ w = g & \text{in } \{T\} \times \mathbb{R}^N \end{cases}$$

Uniqueness of the solution $\Rightarrow V^+ = V^-$.

Comments

- Differential games were first investigated by Pontryagin and Isaacs in the mid-50ies.
- First proof of existence of a value : Fleming, 1961
- The Hamilton-Jacobi equation has to be understood in the viscosity sense
  (introduced by Crandall-Lions, 1981)
- The above proof was made rigorous in
  - Evans-Souganidis, 1984 (deterministic D.G.)
  - Fleming-Souganidis, 1989 (stochastic D.G.)

Outline

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Dynamics and payoffs

As before the stochastic differential game is defined by

$$\left\{ \begin{array}{l} dX_t = f(X_t, u_t, v_t)dt + \sigma(X_t, u_t, v_t)dB_t, \;\; t \in [t_0, T], \\ X_{t_0} = x_0, \end{array} \right.$$

Let

- $g_i : \mathbb{R}^N \to \mathbb{R}$ a family of terminal payoffs, $i = 1, \ldots, I$,
- $p \in \Delta(I)$ be a probability on $\{1, \ldots, I\}$.

Organization of the game

The game is played in two steps :

- At initial time t_0 the index i is chosen at random according to probability p.
  Index i is communicated to Player I only.
- Then
  - Player I tries to minimise the terminal payoff $\mathbf{E}[g_i(X_T)]$
  - Player II tries to maximise the terminal payoff $\mathbf{E}[g_i(X_T)]$.
- Players observe each other.

This is a continuous-times version of a game introduced in the late 60s by Aumann and Maschler.

Upper- and lower value functions

The upper value function is

$$V^+(t_0, x_0, p) = \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sup_{\beta\in\mathcal{B}_r(t_0)} \sum_i p_i \mathbf{E}\left[g_i(X_T^{t_0,x_0,\alpha_i,\beta})\right]$$

while the lower value function is

$$V^-(t_0, x_0, p) = \sup_{\beta\in\mathcal{B}_r(t_0)} \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sum_i p_i \mathbf{E}\left[g_i(X_T^{t_0,x_0,\alpha_i,\beta})\right]$$

Outline

Existence of a value

Theorem (C.-Rainer, 2009)

Under Isaacs' condition, the game has a value :
$\forall (t, x, p) \in [0, T] \times \mathbb{R}^N \times \Delta(I)$

$$V^+(t, x, p) = V^-(t, x, p) \; .$$

Convexity of the value functions

Proposition

For all $(t,x)\in[0,T]\times\mathbb{R}^n$, the maps $(p,q)\to V^{\pm}(t,x,p)$ are convex in p.

Proof : Obvious for V^- :

$$\begin{array}{rcl} V^-(t_0,x_0,p) & = & \sup_{\beta\in\mathcal{B}_r(t_0)} \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sum_i p_i \mathbf{E}\left[g_i(X_T^{t_0,x_0,\alpha_i,\beta})\right] \\ & = & \sup_{\beta\in\mathcal{B}_r(t_0)} \sum_i p_i \inf_{\alpha\in\mathcal{A}_r(t_0))} \mathbf{E}\left[g_i(X_T^{t_0,x_0,\alpha_i,\beta})\right] \end{array}$$

For V^+ : “splitting method" (Aumann-Maschler).

Fenchel conjugate of V^-

We introduce the Fenchel conjugate of V^- :

$$V^{-*}(t,x,\hat{p}) = \sup_{p\in\Delta(I)} \left(p.\hat{p} - V^-(t,x,p,q)\right)$$

Then

$$\begin{array}{rcl} V^{-*}(t,x,\hat{p}) & = & \sup\limits_{p} \left(p.\hat{p} - \sup\limits_{\beta} \inf\limits_{(\alpha_i)} \sum\limits_i p_i \mathbf{E}\left[g_i\right] \right) \\ & = & \sup\limits_{p} \inf\limits_{\beta} \sup\limits_{(\alpha_i)} \sum\limits_i p_i \left(\hat{p}_i - \mathbf{E}\left[g_i\right]\right) \\ \text{“}=\text{”} & & \inf\limits_{\beta} \sup\limits_{p} \sup\limits_{(\alpha_i)} \sum\limits_i p_i \left(\hat{p}_i - \mathbf{E}\left[g_i\right]\right) \end{array}$$

Lemma

$$V^{-*}(t,x,\hat{p}) = \inf_{\beta\in\mathcal{B}_r(t))} \sup_{\alpha\in\mathcal{A}(t)} \max_{i\in\{1,\dots,I\}} \left\{ \hat{p}_i - \mathbf{E}\left[g_i(X_T^{t,x,\alpha,\beta}) \right] \right\}.$$

An inequation for V^-

As a consequence : for all $0 \leq t_0 \leq t_1 \leq T, x_0 \in \mathbb{R}^N, \hat{p} \in \mathbb{R}^I$,

$$V^{-*}(t_0, x_0, \hat{p}) \leq \inf_{\beta \in \mathcal{B}(t_0)} \sup_{\alpha \in \mathcal{A}(t_0)} \mathbf{E}[V^{-*}(t_1, X_{t_1}^{t_0, x_0, \alpha, \beta}, \hat{p})]$$

Corollary

For any $\hat{p} \in \mathbb{R}^I$, $(t, x) \to V^{-*}(t, x, \hat{p})$ is a subsolution in viscosity sense of

$$\partial_t w - H(x, -Dw, -D^2 w) \geq 0$$

Hence V^- is a supersolution to

$$(HJ) \qquad \min\left\{\partial_t w + H(x, Dw, D^2 w)\ ,\ \lambda_{\min}(D^2_{pp} w)\right\} \leq 0$$

in $(0, T) \times \mathbb{R}^N \times \Delta(I)$.

Analysis of V^+

V^+ satisfies the subdynamic programming :

$$\begin{aligned} V^+(t_0,x_0,p) &= \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sup_{\beta\in\mathcal{B}_r(t_0)} \sum_i p_i \mathbf{E}\left[g_i(X_T^{t_0,x_0,\alpha_i,\beta})\right] \\ &\leq \inf_{\alpha\in\mathcal{A}_r(t_0)} \sup_{\beta\in\mathcal{B}_r(t_0)} \mathbf{E}\left[V^+(t_1, X_{t_1}^{t_0,x_0,\alpha_i,\beta}, p)\right] \end{aligned}$$

Corollary

V^+ is a subsolution of

$$(HJ) \qquad \min\left\{\partial_t w + H(x, Dw, D^2 w)\ ,\ \lambda_{\min}(D^2_{pp} w)\right\} \geq 0$$

in $(0,T)\times\mathbb{R}^N\times\Delta(I)$.

Summary

- We have $V^- \leq V^+$ by construction.
- We have seen that
  (i) V^- is a supersolution of (HJ)
  (ii) V^+ is a subsolution of (HJ)
  (iii) $V^-(T,x,p,q) = V^+(T,x,p,q) = \sum_i p_i g_i(x)$
- Comparison principle for (HJ) $\Rightarrow V^+ \leq V^-$.

Hence the value $V^+ = V^-$ is the unique viscosity solution to

$$(HJ)\ \begin{cases} \min\left\{\partial_t w + H(x, Dw, D^2 w)\ ,\ \lambda_{\min}(D^2_{pp} w)\right\} = 0 \\ w = \sum_i p_i g_i \qquad \text{in } \{T\} \times \mathbb{R}^N \times \Delta(I) \end{cases}$$

Outline

Rules of the game

No dynamics

- At time t_0, i is chosen by nature in $\{1, \ldots, I\}$ according to probability p,
- the choice of i is communicated to Player 1 only,
- Player 1 minimizes the integral payoff

$$\int_{t_0}^{T} \ell_i(s, u(s), v(s))ds.$$

- Player 2 maximizes it.

Isaacs' condition takes the form

$$H(t,p) = \inf_{u \in U} \sup_{v \in V} \sum_{i=1}^{I} p_i \ell_i(t,u,v) = \sup_{v \in V} \inf_{u \in U} \sum_{i=1}^{I} p_i \ell_i(t,u,v)$$

Existence of a value

We already know that :

Under Isaacs' condition, the game has a value

$$\mathbf{V}(t_0,p) = \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sup_{\beta\in\mathcal{B}_r(t_0)} \sum_{i=1}^I p_i \mathbf{E}_{\alpha_i\beta}\left[\int_{t_0}^T \ell_i(s,\alpha_i(s),\beta(s))ds\right]$$

$$= \sup_{\beta\in\mathcal{B}_r(t_0)} \inf_{(\alpha_i)\in(\mathcal{A}_r(t_0))^I} \sum_{i=1}^I p_i \mathbf{E}_{\alpha_i\beta}\left[\int_{t_0}^T \ell_i(s,\alpha_i(s),\beta(s))ds\right]$$

Furthermore **V** is the unique viscosity solution of :

$$\begin{cases} \min\left\{\partial_t w + H(t,p)\,;\ \lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)\right\} = 0 & \text{in } [0,T]\times\Delta(I) \\ w(T,p) = 0 \quad \text{in } \Delta(I) \end{cases}$$

Optimal strategy : a representation theorem

Let $\mathcal{P}(t_0, p_0)$ be the set of càdlàg martingale processes $\mathbf{p} : [t_0^-, T] \to \Delta(I)$ such that

$$\mathbf{p}(t_0^-) = p_0 \qquad \text{and} \qquad \mathbf{p}(T) \in \{e_1, \ldots, e_I\} ,$$

where $\{e_1, \ldots, e_I\}$ is the canonical basis of $\mathbb{R}^I$.

Theorem

$\forall (t_0, p_0) \in [0, T] \times \Delta(I)$

$$\mathbf{V}(t_0, p_0) = \min_{\mathbf{p} \in \mathcal{P}(t_0, p_0)} \mathbf{E}\left[\int_{t_0}^{T} H(s, \mathbf{p}(s))ds\right]$$

Recall that $H(t, p) = \inf_{u \in U} \sup_{v \in V} \sum_{i=1}^{I} p_i \ell_i(t, u, v)$.

Optimal strategy for Player I

Let $u^* = u^*(t,p)$ be a Borel measurable selection of

$$\mathrm{argmin}_{u\in U}(\max_{v\in V}\sum_{i=1}^{I} p_i\ell_i(t,u,v)) \ .$$

For $(t_0,p_0)\in[0,T]\times\Delta(I)$ fixed, let $\overline{\mathbf{p}}$ be optimal for

$$\min_{\mathbf{p}\in\mathcal{P}(t_0,p_0)} \mathbf{E}\left[\int_{t_0}^{T} H(s,\overline{\mathbf{p}}(s))ds\right] .$$

Finally, $\forall i\in\{1,\dots,I\}$, let us define

$$\overline{u}_i(s) \stackrel{d}{=} u^*(s,\overline{\mathbf{p}}(s))|_{\{\overline{\mathbf{p}}(T)=e_i\}}.$$

Theorem

The random control $(\overline{u}_i) \in (\mathcal{U}_r(t_0))^I$ is optimal for $\mathbf{V}(t_0, p_0)$. Namely

$$\mathbf{V}(t_0, p_0) = \sup_{\beta \in \mathcal{B}(t_0)} \sum_{i=1}^{I} (p_0)_i E_{\overline{u}_i} \left[\int_{t_0}^{T} \ell_i(s, \overline{u}_i(s), \beta(\overline{u}_i)(s)) ds \right].$$

Example 1 : Stationary case

If the $\ell_i = \ell_i(u, v)$ do not depend on time, then

Proposition

$$\mathbf{V}(t,p) = (T-t)\mathrm{Vex}H(p) \qquad \forall p \in \Delta(I)\ .$$

Proof :

- Let $w(t,p) = (T-t)\mathrm{Vex}H(p)$. Then $w(T,p) = 0$ and
$$\partial_t w(t,p) = -\mathrm{Vex}H(p)\ .$$
If $\lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)(t,p) > 0$, then $\mathrm{Vex}H(p) = H(p)$.
- Hence
$$\min\left\{\partial_t w + H(t,p)\ ;\ \lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)\right\} = 0$$

Example 1 (continued)

For $p \in \Delta(I)$, let $(\lambda_k) \in \Delta(I)$, $p^k \in \Delta(I)$ $(k = 1, \ldots, I)$ such that

$$\sum_k \lambda_k p^k = p \qquad \text{and} \qquad \mathrm{Vex}\, H(p) = \sum_k \lambda_k H(p^k)\ .$$

Proposition

The martingale $\mathbf{p} \in \mathcal{P}(t_0, p)$ constant and equal to p^k with probability λ^k on $[t_0, T)$ is optimal.

Proof :

$$\begin{aligned} \mathbf{E}\left[\int_{t_0}^{T} H(\mathbf{p}_s)ds\right] &= (T - t_0)\sum_k \lambda_k H(p^k) \\ &= (T - t_0)\mathrm{Vex}\, H(p) = \mathbf{V}(t_0, p)\ . \end{aligned}$$

Example 2 : $I = 2$

We assume that $I = 2$. Then $\Delta(I) \approx [0, 1]$.

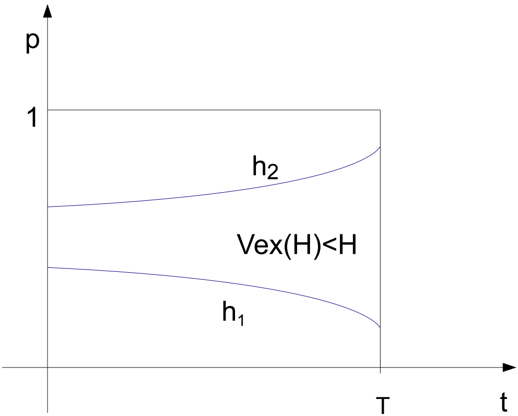
Assumption on H : There are $h_1 : [0, T] \to [0, 1]$ continuous non increasing and $h_2 : [0, T] \to [0, 1]$ continuous nondecreasing such that

$$Vex(H)(t, p) < H(t, p) \Leftrightarrow p \in (h_1(t), h_2(t))$$

Proposition

$$\mathbf{V}(t, p) = \int_t^T \mathrm{Vex} H(s, p) ds \qquad \forall (t, p) \in [0, T] \times \Delta(I) \ .$$

Example 2 continued



Example 2 continued

Proof :

- Let $w(t,p) = \int_t^T \mathrm{Vex} H(s,p)ds$. Then $w(t,\cdot)$ is convex and
$$\partial_t w(t,p) = -\mathrm{Vex} H(t,p)$$
Moreover if $\lambda_{\min}\left(\frac{\partial^2 w}{\partial p^2}\right)(t,p) > 0$ then $p \notin (p_1(t), p_2(t))$, i.e., $Vex(H)(t,p) = H(t,p)$.
- Hence
$$\min\left\{\partial_t w + H(t,p)\ ;\ \lambda_{\min}\left(p, \frac{\partial^2 w}{\partial p^2}\right)\right\} = 0$$

Example 2 continued

Proposition

If $p_0 \in (h_1(t_0), h_2(t_0))$, there is a unique optimal martingale $\mathbf{p}$. The process $\mathbf{p}$ is purely discontinuous and satisfies

$$\mathbf{p}(t) \in \{h_1(t), h_2(t)\} \qquad \forall t \in [t_0, T) .$$

In particular, if $s < t < T$

$$\mathbf{P}\left[\mathbf{p}(t) = h_1(t) \mid \mathbf{p}(s) = h_1(s)\right] = \frac{h_2(t) - h_1(s)}{h_2(t) - h_1(t)} .$$

Example 3 : $I = 2$

We suppose that $H(t,p) = \lambda(t)p(1-p)$ with λ Lipschitz and there exists $0 < a < b < T$ with

$$\lambda > 0 \text{ in } [0,b), \;\; \lambda < 0 \text{ on } (b,T] \qquad \text{and} \qquad \int_a^T \lambda(s)ds = 0$$

Proposition

$$\mathbf{V}(t,p) = \begin{cases} 0 & \text{if } t \in [0,a] \\ p(1-p)\int_t^T \lambda(s)ds & \text{if } t \in [a,T] \end{cases}$$

Hence

$$\mathbf{V}(t,p) \neq \int_t^T \mathrm{Vex} H(s,p)ds \text{ on } (a,b)$$

Extensions

- Characterization of the optimal martingale.
- Case where the unknown i is a continuous r.v.
- Representation formula for differential games with non-degenerate diffusion
  (via BSDE arguments). C. Grün
- Analysis of games in which the information is relieved to Player I progressively.

Outline

Deterministic differential game with finite horizon

We now consider a deterministic differential game

$$\left\{ \begin{array}{l} dX_t = f(X_t, u_t, v_t)dt \\ x_{t_0} = x_0 \end{array} \right.$$

The trajectory associated to (u, v) is denoted by $X_\cdot^{t_0,x_0,u,v}$.

Main assumption on the game : Player II does not observe anything.

Rules of the game

- At time t_0, the initial state x_0 is drawn at random according to a probability μ_0 on $\mathbb{R}^N$.
- Player I is informed on the initial state x_0, Player II just knows μ_0.
- Player I observes $x(t)$ and $v(t)$. He minimizes $g(X_T^{t_0,x_0,u,v})$.
- Player II observes nothing but has perfect recall about his own control v. He maximizes $g(X_T^{t_0,x_0,u,v})$.

The value functions

The lower value function is :

$$\mathbf{V}^-(t_0,\mu_0) = \sup_{v\in\mathcal{V}_r(t_0)} \inf_{(\alpha^x)\in(\mathcal{A}_r(t_0))^{\mathbb{R}^N}} \int_{\mathbb{R}^N} \mathbf{E}\left[g(X_T^{t_0,x,\alpha^x,v})\right] d\mu_0(x)$$

The upper value function is :

$$\mathbf{V}^+(t_0,\mu_0) = \inf_{(\alpha^x)\in(\mathcal{A}_r(t_0))^{\mathbb{R}^N}} \sup_{v\in\mathcal{V}_r(t_0)} \int_{\mathbb{R}^N} \mathbf{E}\left[g(X_T^{t_0,x,\alpha^x,v})\right] d\mu_0(x)$$

Framework

We work on the set of Borel probability measures

$$\mathcal{P}_2 := \{\mu / \int_{\mathbb{R}^N} |x|^2 d\mu(x) < \infty\}$$

endowed with the Wasserstein distance :

$$\mathbf{d}^2(\mu,\nu) = \min_{\pi \in \Pi(\mu,\nu)} \int_{\mathbb{R}^{2N}} |x-y|^2 d\pi(x,y)$$

We consider the Hamiltonian

$$\mathcal{H}(\mu,\mathbf{p}) = \sup_{\mathbf{v}\in\Delta(V)} \int_{\mathbb{R}^N} \inf_{\mathbf{u}\in\Delta(U)} \int_{U\times V} \langle f(x,u,v), \mathbf{p}(x)\rangle d\mathbf{u}(u) d\mathbf{v}(v) d\mu(x)$$

(for $\mathbf{p} \in L^2_\mu(\mathbb{R}^N, \mathbb{R}^N),\ \mu \in \mathcal{P}_2$)

Existence of the value

Theorem (C., Souquière)

For all (t, μ) *:*

$$\mathbf{V}^+(t,\mu) = \mathbf{V}^-(t,\mu)$$

Moreover $\mathbf{V}^+ = \mathbf{V}^-$ *is the unique viscosity solution of*

$$\begin{cases} \partial_t w + \mathcal{H}(\mu, D_\mu w) = 0 & \text{in } (0,T)\times\mathcal{P}_2 \\ w(T,\mu) = \displaystyle\int_{\mathbb{R}^N} g(x)d\mu(x) & \text{in } \mathcal{P}_2 \end{cases}$$

Idea of proof (1)

Proposition (Dynamic programming principle)

The upper value function satisfies :

$$\mathbf{V}^+(t_0,\mu_0) = \inf_{(\alpha^x)\in(\mathcal{A}_r(t_0))^{\mathbb{R}^N}} \sup_{v\in\mathcal{V}_r(t_0)} \mathbf{V}^+(t_1,\mu_{t_1}) .$$

where μ_{t_1} *is is the information of player II on the state of the system, knowing the strategy of his opponent :*

$$\int_{\mathbb{R}^N} \varphi(x)d\mu_{t_1}(x) = \int_{\mathbb{R}^N} \mathbf{E}\left[\varphi(X_{t_1}^{t_0,x,\alpha)^x,v})\right] d\mu_0(x)$$

for any $\varphi \in \mathcal{C}_b(\mathbb{R}^N,\mathbb{R})$.

Idea of proof (2)

The rest of the proof relies on

- P.D.E. characterization of $\mathbf{V}^+$.
- Comparison principle for (HJ) related to the “Euclidean structure" of $\mathcal{P}_2$.
  (See also Feng-Kurtz (2006), C.-Quincampoix (2007), Gangbo-Nguyen-Adrian (2008), Feng-Katsoulakis (2009), Lasry-Lions.)
- Sion's min-max Theorem for the equality $\mathbf{V}^+ = \mathbf{V}^-$.

Conclusion

- Differential games with imperfect information :
  - well understood for simple information structure.
  - a lot remains to be done in more general settings.
- Differential games with lack of observation : almost completely open.
- Nonzero sum differential games with lack of information : open.

Thank you for your attention !

Solution of the HJ Equation

Definition (Subsolution of the HJ Equation)

$\mathbf{V} : [t_0, T] \times \mathcal{P}_2 \to \mathbb{R}$, Lipschitz continuous, is a subsolution to (HJ) if, for any test function $\phi(t, \mu)$ of the form

$$\phi(t, \mu) = \frac{\alpha}{2}\mathbf{d}^2(\bar{\mu}, \mu) + \eta\mathbf{d}(\bar{\nu}, \mu) + \psi(t)$$

(where $\psi \in C^1(\mathbb{R}, \mathbb{R})$, $\alpha, \eta > 0$, $\bar{\nu}, \bar{\mu} \in \mathcal{P}_2$) such that $\mathbf{V} - \phi$ has a local maximum at $(\bar{\nu}, \bar{t})$, one has :

$$\psi'(\bar{t}) + \mathcal{H}(\bar{\nu}, -\alpha\mathbf{p}_y) \geq -\|f\|_\infty \eta$$

where, for a fixed $\bar{\pi} \in \Pi_{\rm opt}(\bar{\mu}, \bar{\nu})$, $\mathbf{p}_y \in L^2_{\bar{\nu}}(\mathbb{R}^N, \mathbb{R}^N)$ is defined by :
$\int_{\mathbb{R}^N} \langle \xi(y), x - y\rangle d\bar{\pi}(x, y) = \int_{\mathbb{R}^N} \langle \xi(y), \mathbf{p}_y(y)\rangle d\bar{\nu}(y) \qquad \forall \xi \in L^2_{\bar{\nu}}$

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Solution of the HJ Equation

Definition (Supersolution of the HJ Equation)

$\mathbf{V} : [t_0, T] \times \mathcal{P}_2 \to \mathbb{R}$, Lipschitz continuous, is a supersolution to (HJ) if, for any test function $\phi(t, \mu)$ of the form

$$\phi(t, \mu) = -\frac{\alpha}{2}\mathbf{d}^2(\bar{\mu}, \mu) - \eta\mathbf{d}(\bar{\nu}, \mu) + \psi(t)$$

(where $\psi \in C^1(\mathbb{R}, \mathbb{R})$, $\alpha, \eta > 0$ and $\bar{\mu}, \bar{\nu} \in \mathcal{P}_2$) such that $\mathbf{V} - \phi$ has a local minimum at $(\bar{\nu}, \bar{t}) \in (0, T) \times \mathcal{P}_2$, one has :

$$\psi'(\bar{t}) + \mathcal{H}(\bar{\nu}, \alpha\mathbf{p}_y) \leq \|f\|_\infty \eta \ .$$

A solution of (HJ) is a subsolution and a supersolution.

Solution of the HJ Equation

Lemma (Comparison principle)

Let w_1 be some subsolution of (HJ) and w_2 some supersolution such that $w_2(T,\mu) \geq w_1(T,\mu)$.
Then for all $(t,\mu) \in [t_0, T] \times \mu \in \mathcal{P}_2$:

$$w_2(t,\mu) \geq w_1(t,\mu)$$

- The definition comes from Cardaliaguet-Quincampoix (2007) (cf. also Gangbo-Nguyen-Adrian (2008), Feng-Katsoulakis (2009), Lasry-Lions).
- The proof of the comparison principle is an adaptation of Crandall, Lions (1986).

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