Applications Radar de la Géométrie de l'information associée aux matrices de covariances : traitements spatio-temporels

Frédéric BARBARESCO
Thales Air Operations
Domaine Surface Radar, Direction Technique
Département Développements Avancés

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Radar Applications of Information Geometry based on covariance matrices : Space-Time Processing

Frédéric BARBARESCO
Thales Air Operations
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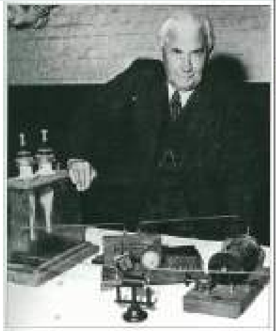
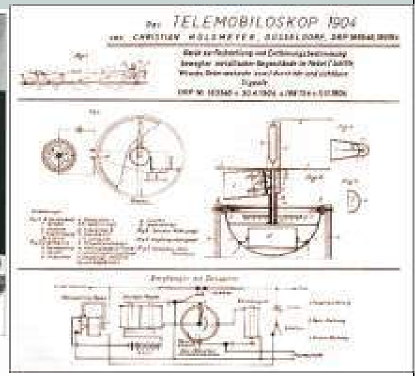
Radar Signal Processing : Doppler Processing

Doppler Sensors : Radar, Sonar & Lidar

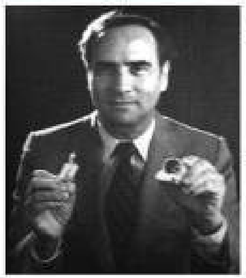
- Radar (RAdio Detection And Ranging) Technology
  - 107 Years old (C. Hülsmeyer, 1904)

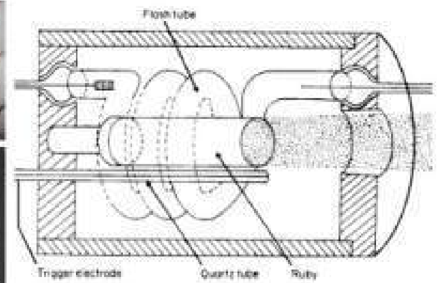

C. Hülsmeyer présentant des éléments de sa maquette historique (1956)



- Sonar (SOund Navigation And Ranging)
  - (piezoelectric effect) 95 years old (Paul Langevin & Constantin Chilowski, 1916)

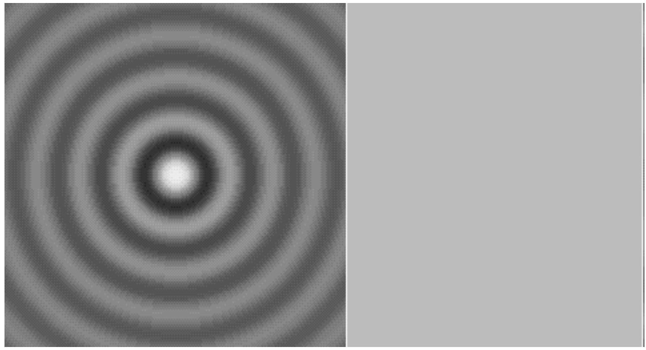
- Lidar (LIght Detection And Ranging) Technology
  - (laser) 51 years old (T. Maiman, 1960)

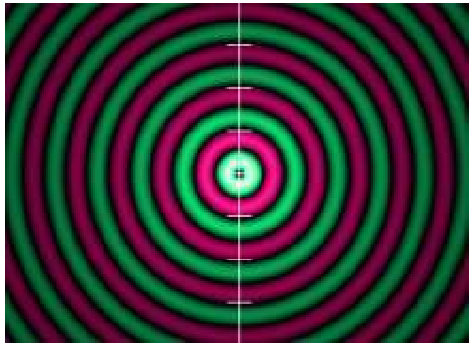


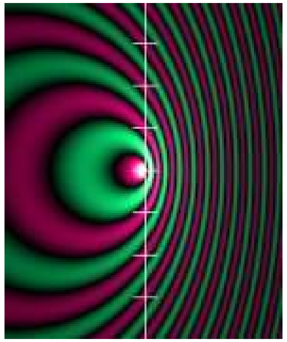


Thomas Maiman, 1960 ....
This man ignored the ridicule of his peers and easiliy succeeded in producing history's first visible light laser from this simple photographic coiled flashlamp and his ruby crystalline rod.

All these sensors use Doppler-Fizeau Effects

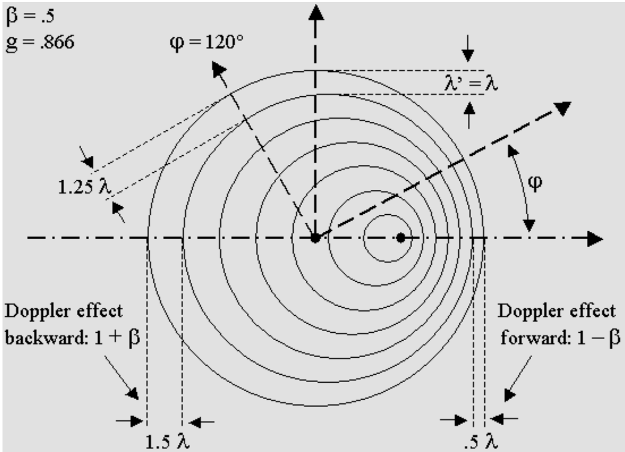


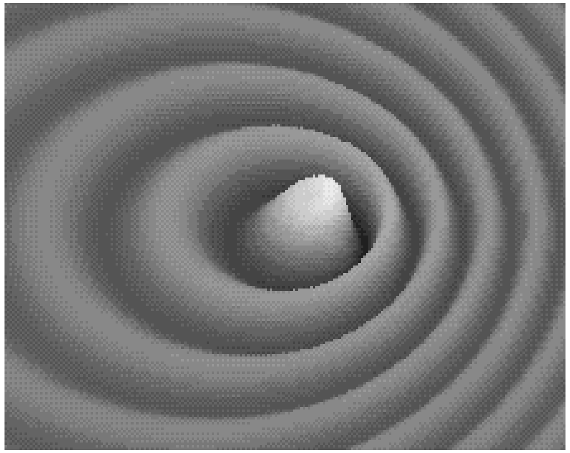


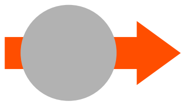


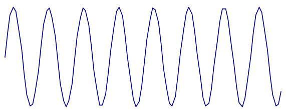
Armand Hippolyte Louis Fizeau (1819 – 1896)

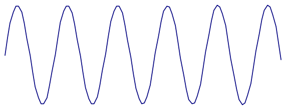
Christian Andreas Doppler (1803- 1853)





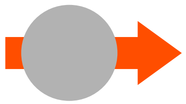


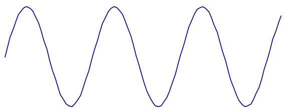




$\Delta freq \Rightarrow$ Radial Velocity (Doppler Spectrum Mean)

$Var(\Delta freq) \Rightarrow$ Turbulence (Doppler Spectrum Width)





Woldemar Voigt (1850 - 1919)

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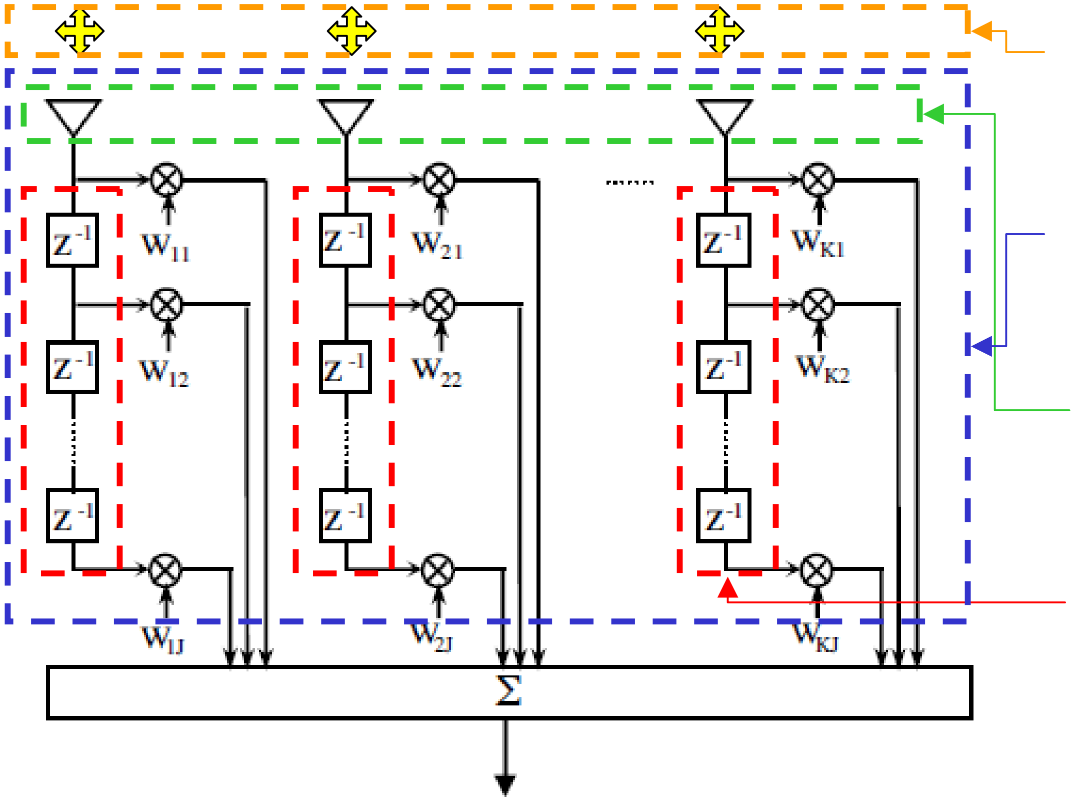
Radar Processing based on Covariance Matrix

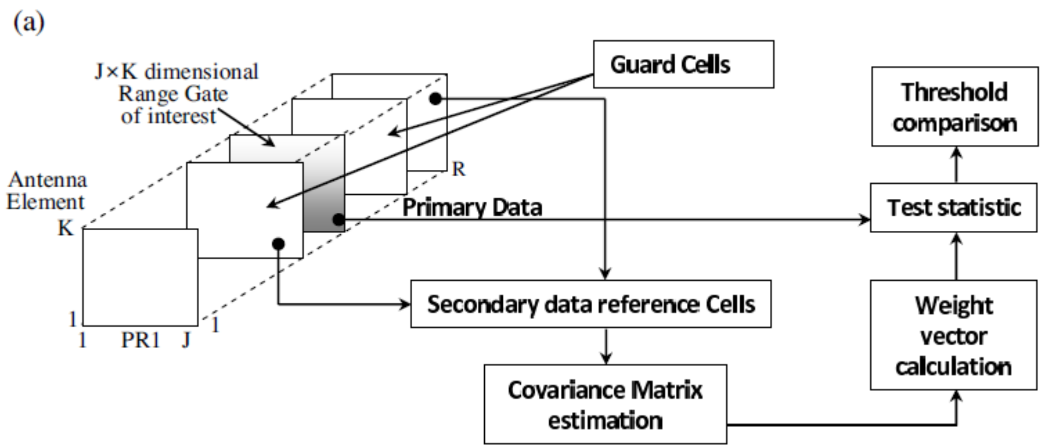


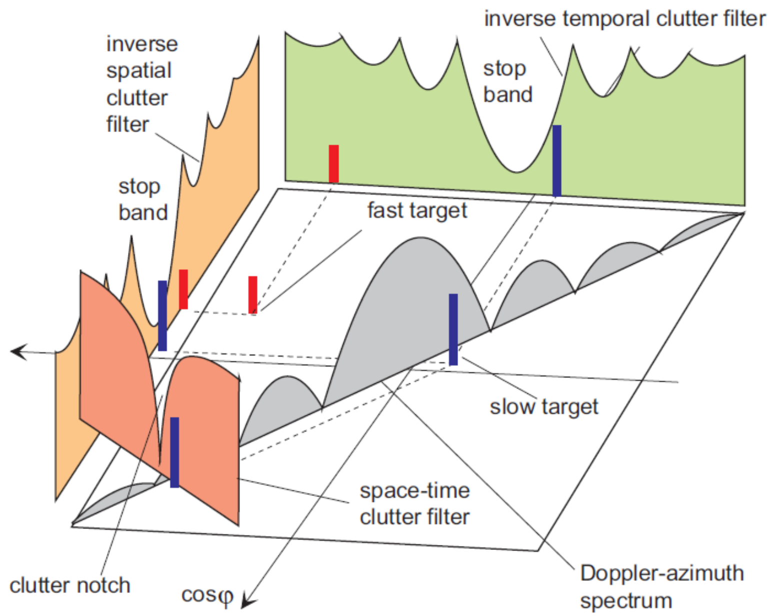
Polarimetric Radar Processing (Polarimetry Covariance Matrix)

STAP Processing (Space-Time Covariance Matrix)

Antenna Processing (Space Covariance Matrix)

Doppler Processing (Time Covariance Matrix)





Military Air Defense Application

Detection of slow/stealth targets in inhomogeneous Clutter

- **New requirements in Air Defense to detect low altitude or surface targets at low elevation. Target Doppler is very close to fluctuating Clutter Doppler (Ground & Sea Clutters)**
- **Detection of asymmetric & stealth targets in Ground Clutter**
  - Microlight Airplane
  - General Aviation
  - UAV & Micro UAV
  - Micro Helicopter

- **Detection of small targets in Sea Clutter**
  - Wooden & inflatable canoe
  - Jetski
  - Unmanned Boat
  - Naval Micro Helicopter
  - Periscope

Detection of low RCS targets
Detection of teneous Doppler Signal
Increase Range & Reactivity

Monitoring of turbulences: New Requirement in ATC (SESAR)

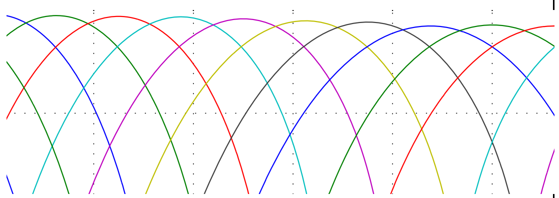
- **In Turbulences, signal is no longer characterized by Doppler velocity Mean but by Doppler Spectrum Width & “shape”**
- **Atmospheric Air turbulences**
  - Eddy Dissipation Rate
  - Turbulent Kinetic Energy

- **Airplane Wake-Vortex turbulences (A380, B747-8)**
  - Circulation
  - Decay Rate
- **Windshear in Final Approach**
  - Headwind
  - Crosswind

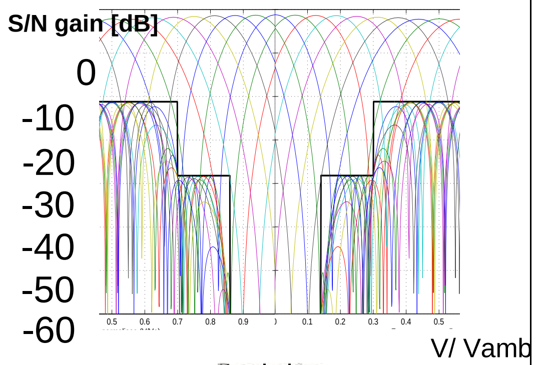
Improve Safety by mitigating weather hazards
Increase Capacity by reducing safe separations

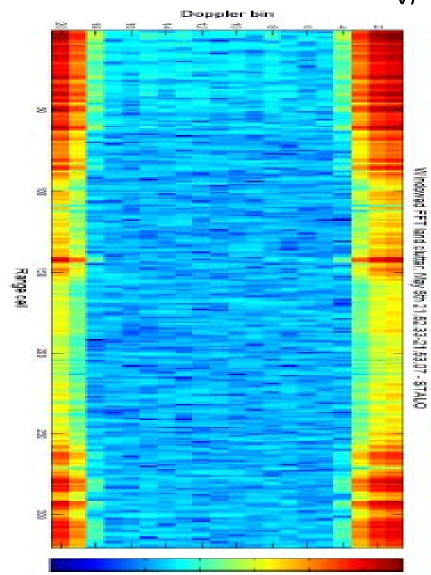
Challenges of Doppler Radar Processing: Detection on Ground

Detection of asymmetric & stealth targets in Inhomogenesous Ground Clutter



- **Classical Doppler Filter Banks (or FFT) are not efficient with very short bursts (<16 pulses) :**
  - Low Resolution of Doppler Filters with short Bursts (Low sidelobes / high loss, wide filter)
  - If Target Doppler is between two Doppler filters, energy is spread on adjacent filters. Gain between 2 filters is lower than gain at filter center ("Straddling loss")
  - Ground Clutter Energy is not limited to zero-Doppler filter but pollution is spread over all filters due to poor Filter-Banks Resolution & Doppler side lobes in case of very short Bursts.



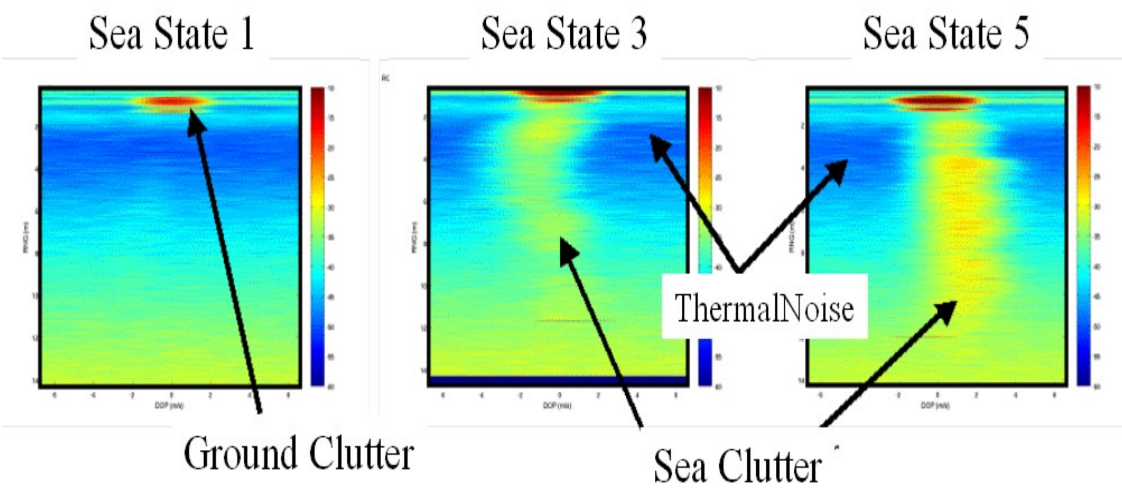


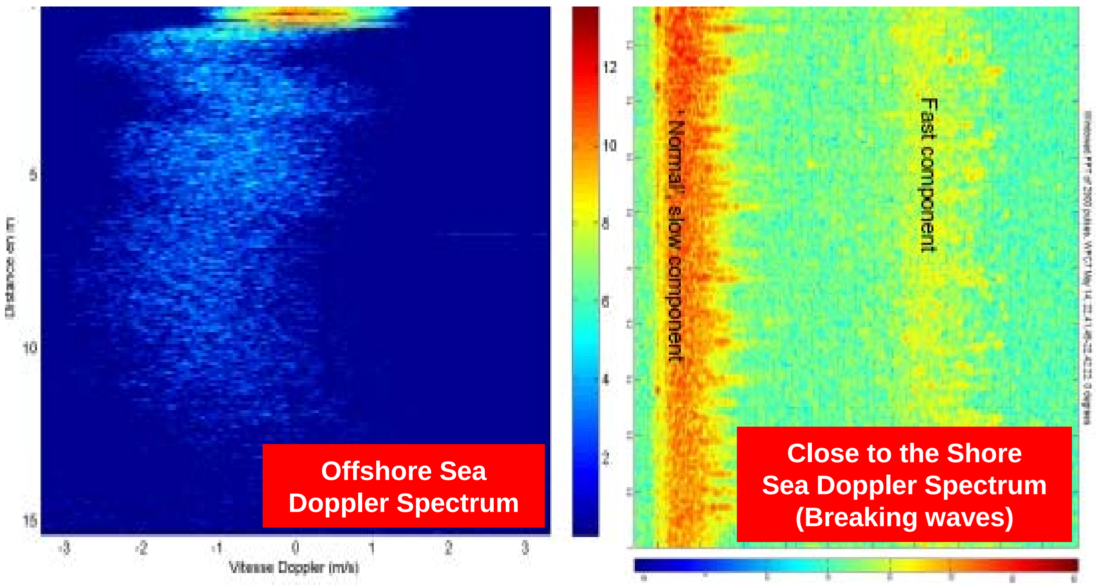
Filter 0 Filter 1 Filter 2 Filter 3 Filter 4 Filter 5 Filtre 6 Filter 7

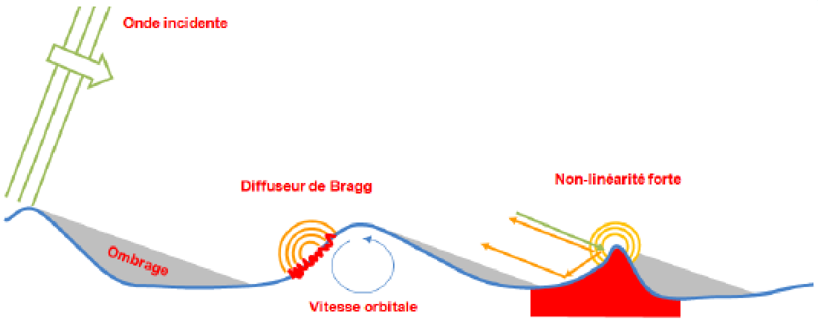
Challenges of Doppler Radar Processing:Detection in Sea

Detection of slow targets in Sea Clutter

- **Sea Clutter is highly inhomogeneous**
  - Doppler fluctuation
  - Time/space Fluctuation
- **Sea Clutter is dependant of**
  - Sea current
  - Surface wind
  - fetch
  - Bathymetry
- **Sea Clutter is corrupted by**
  - Spikes due to breaking waves
  - “Moutonement”







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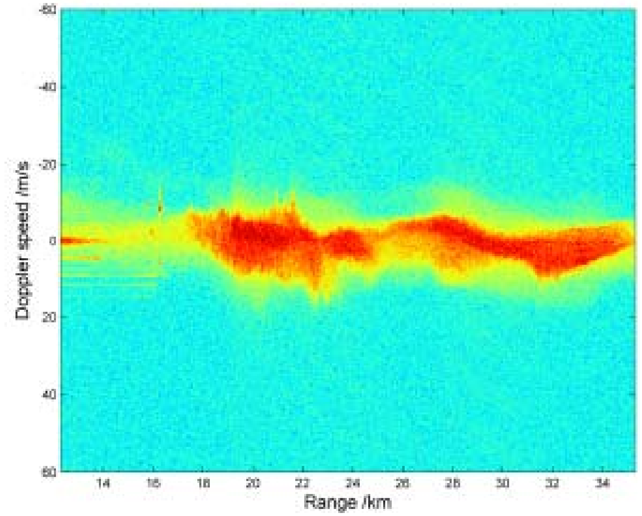
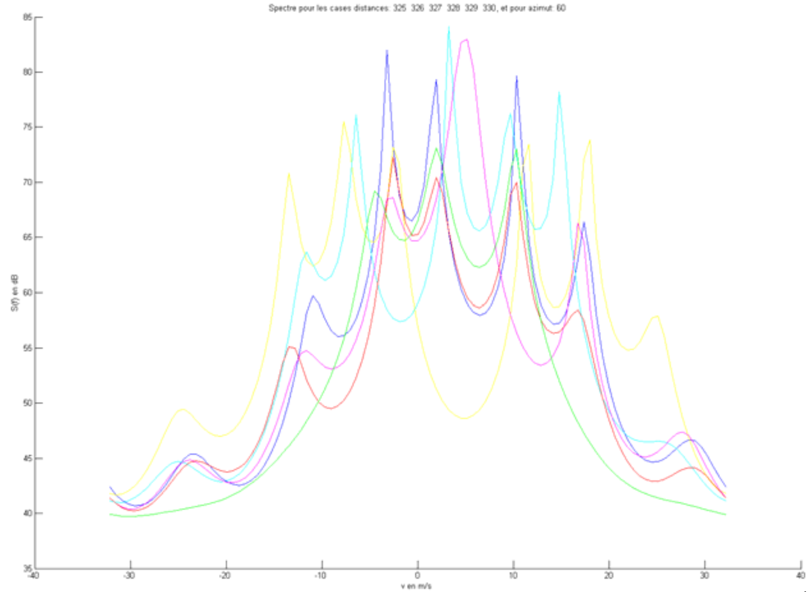
Challenges of Doppler Radar Processing: Atm. turbulences

Monitoring of atmospheric turbulences

- Air turbulence is characterized by Spread of fluctuating speeds
- This composition of different Doppler speeds in Radar cells generates a Widen Doppler Spectrum
- Speed variance of Doppler Spectrum Width are related to 2 measures of turbulence :
  - EDR: Eddy Dissipation Rate
  - TBE: Turbulent Kinetic Energy





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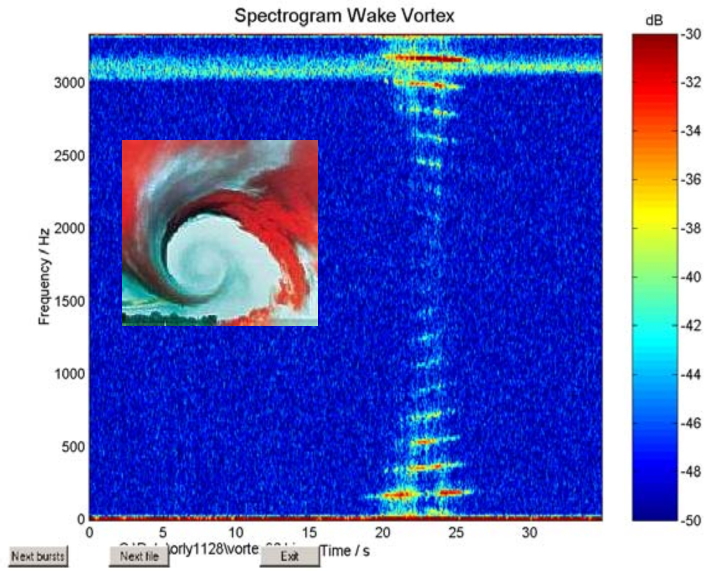
Challenges of Doppler Radar Processing: Wake Turbulences

Monitoring of Wake-Vortex Turbulences

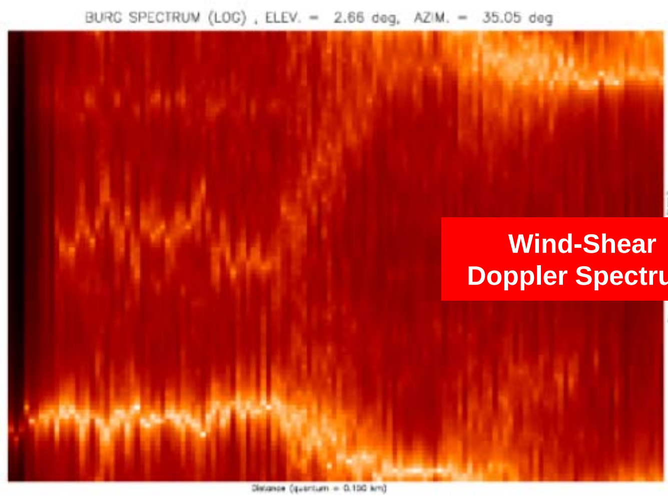
- Wake vortex generate to contra-rotative roll-up spirals
- Mean speed depends on cross-wind
- Wake-Vortex has spiral geometry with increasing speed in the core and decreasing speed outside the core
- Wake-Vortex Speed and structure depend on Wake-Vortex age/decay phase
- Wake-Vortex Strength is characterized by Circulation in m^2/s

Wake-Vortex Doppler Spectrum



Monitoring of Windshear

- Inversion of speed in range or in altitude
- Microburst in the same radar cell



Wind-Shear Doppler Spectrum

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Probability Metrics: Information Geometry & Optimal Transport

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Probability Metrics: Distance between 2 random variables

Question in Probability and Statistic: Could we define distance between 2 random variables ?

$$Z^{(1)} = \begin{bmatrix} z_1^{(1)} \\ z_2^{(1)} \\ \vdots \\ z_{n-1}^{(1)} \\ z_n^{(1)} \end{bmatrix} \longleftrightarrow Z^{(2)} = \begin{bmatrix} z_1^{(2)} \\ z_2^{(2)} \\ \vdots \\ z_{n-1}^{(2)} \\ z_n^{(2)} \end{bmatrix}$$

$$d\left(Z^{(1)}, Z^{(2)}\right) = ???$$

Or equivalently, could we define distance between 2 probability densities of these 2 random variables ?

$$p\left(Z^{(1)} / \theta^{(1)}\right) \longleftrightarrow p\left(Z^{(2)} / \theta^{(2)}\right)$$

$$d\left(\theta^{(1)}, \theta^{(2)}\right) = ???$$

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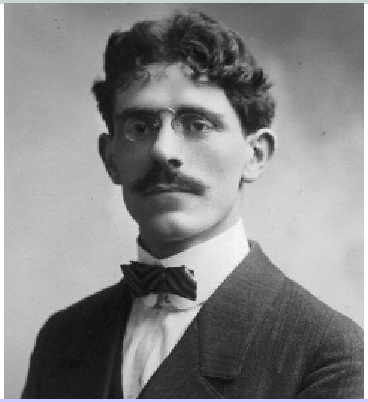
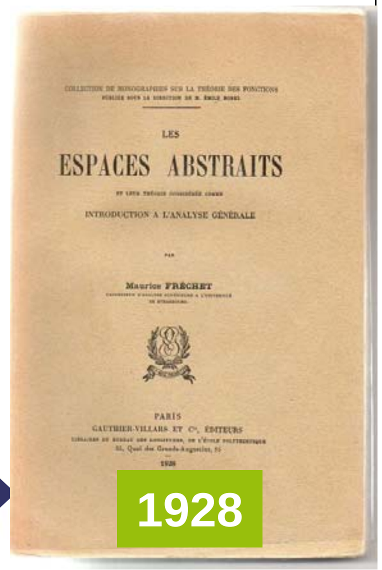
2 Definitions of Probability Metric



Maurice René Fréchet

1939 (IHP Lecture)	1957 (CRAS)
Fréchet Bound (Cramer-Rao)	Fréchet-Levy Distance
Information Geometry	Optimal Transport Theory

1948 (Annales de l'IHP)
Les éléments aléatoires de nature quelconque dans un espace distancié
Extension of Probability/Statistic in abstract space

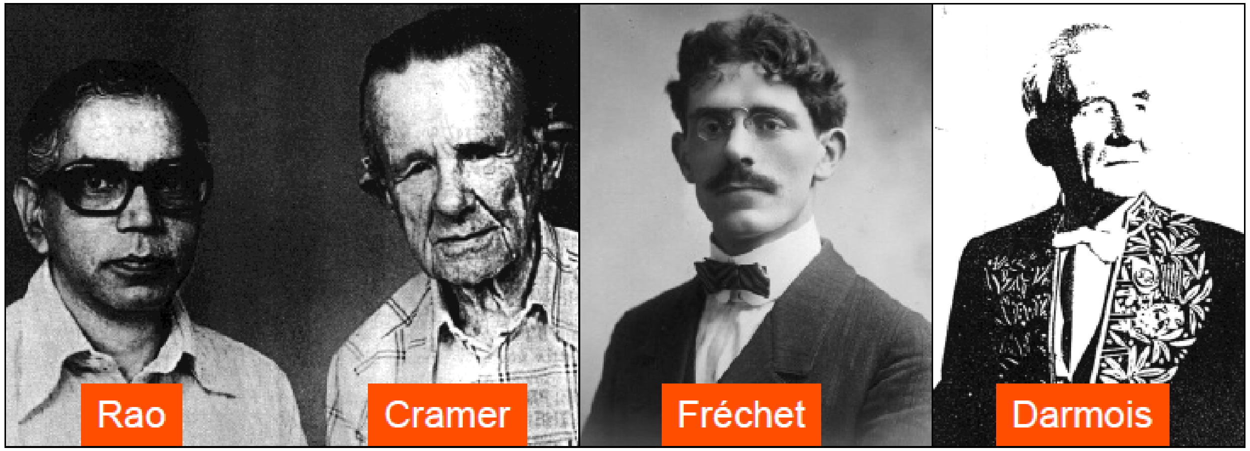


What is the good geometry : geometry with best properties for our applications in Radar Processing

Information Geometry: Fréchet-Darmois-Cramer-Rao Bound

Information Geometry :

- **Fisher Information Matrix and Fréchet-Darmois-Cramer-Rao Bound**



1945

Information and the Accuracy Attainable in the Estimation of Statistical Parameters

C. Radhakrishna Rao

SUR L'EXTENSION DE CERTAINES EVALUATIONS STATISTIQUES AU CAS DE PETITS ECHANTILLONS

par Maurice Fréchet.

1943 (1939 IHP Lecture)

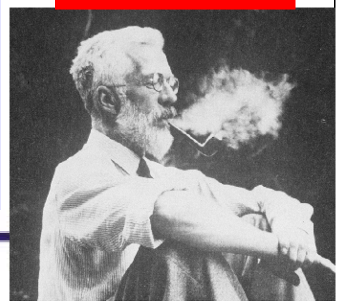
FDCR Bound

$$E\left[\left(\theta-\hat{\theta}\right)\left(\theta-\hat{\theta}\right)^{+}\right] \geq I(\theta)^{-1}$$

Fisher Information Matrix

$$[I(\theta)]_{i,j} = -E\left[\frac{\partial^2 \ln p\left(X/\theta\right)}{\partial\theta_i \partial\theta_j^*}\right]$$

R.A. Fisher



Information Geometry : Rao-Chentsov metric

Information Geometry :

- **Rao-Chentsov Metric defined with Kullback divergence**

Calyampudi Radhakrishna Rao — **1945**

Nikolai Nikolaevich Chentsov — **1964 (axiomatisation)**

- **Kulback-Leibler Divergence :**

$$K(p,q)=\underset{\phi}{Sup}\left[E_p(\phi)-\ln E_q\left(e^{\phi}\right)\right]=\int p(x/\theta)\ln\left(\frac{p(x/\theta)}{q(x/\theta)}\right)dx$$

- **Rao-Chentsov Metric (invariance by parameter changes)**

$$\boxed{ds^2=K\left[p(X/\theta),p(X/\theta+d\theta)\right]=d\theta^{+}I(\theta)d\theta=\sum_{i,j}g_{i,j}d\theta_i d\theta_j^{*}}$$

$$g_{i,j}=\left[I(\theta)\right]_{i,j}$$

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·Dual Coordinates systems & Potential functions

- Potential Functions are Dual and related by **Legendre transformation** :

$$\text{Dual coordinates}\quad \begin{cases} \tilde{\Theta} = (\theta, \Theta) = \left(\Sigma^{-1}m, (2\Sigma)^{-1}\right) \\ \tilde{H} = (\eta, H) = \left(m, -\Sigma + mm^T\right) \end{cases}$$

$$\Rightarrow \begin{cases} \tilde{\Psi}\left(\tilde{\Theta}\right) = 2^{-2}Tr\left(\Theta^{-1}\theta\theta^T\right) - 2^{-1}\log\left(\det\Theta\right) + 2^{-1}n\log(\pi) \\ \tilde{\Phi}\left(\tilde{H}\right) = -2^{-1}\log\left(1 + \eta^T H^{-1}\eta\right) - 2^{-1}\log\left(\det(-H)\right) - 2^{-1}n\log(2\pi e) \end{cases}$$

$$\begin{cases} \dfrac{\partial\tilde{\Psi}}{\partial\theta} = \eta \\ \dfrac{\partial\tilde{\Psi}}{\partial\Theta} = H \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{\partial\tilde{\Phi}}{\partial\eta} = \theta \\ \dfrac{\partial\tilde{\Phi}}{\partial H} = \Theta \end{cases}$$

$$\tilde{\Phi} \equiv \left\langle \tilde{\Theta}, \tilde{H} \right\rangle - \tilde{\Psi}$$

$$\text{with}\quad \left\langle \tilde{\Theta}, \tilde{H} \right\rangle = Tr\left(\theta\eta^T + \Theta H^T\right)$$

$$\tilde{\Phi}\left(\tilde{H}\right) = E\left[\log p\right]$$

Entropy

- Hessians are convexe and define Riemannian metrics :

$$g_{ij} = \frac{\partial^2\tilde{\Psi}}{\partial\Theta_i\partial\Theta_j} \quad \text{and} \quad g_{ij}^* \equiv \frac{\partial^2\tilde{\Phi}}{\partial H_i \partial H_j}$$

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Combinatorial/Variational Foundation of Kullback Divergence

Combinatorial Fundation of Kullback Divergence

- Kullback Divergence can be naturally introduced by combinatorial elements and stirling formula :

Let multinomial Law of N elements spread on M levels $\{n_i\}$

$$P_M(n_1, n_2, \ldots, n_M / q_1, \ldots, q_M) = N! \prod_{i=1}^{M} \frac{q_i^{n_i}}{n_i!}$$

with q_i **priors ,** $\sum_{i=1}^{M} n_i = N$ **and** $p_i = \frac{n_i}{N}$

Sirling formula gives : $n! \approx n^n . e^{-n} . \sqrt{2.\pi.n}$ when $n \to +\infty$

$$\boxed{\underset{N \to +\infty}{Lim} \frac{1}{N} \log[P_M] = \sum_{i=1}^{M} p_i . \log\left[\frac{p_i}{q_i}\right] = K(p,q)}$$

Variational Foundation of Kullback Divergence

- Donsker and Varadhan have proposed a variational definition of Kullback divergence :

$$\boxed{K(p,q) = \underset{\phi}{Sup}\left[E_p(\phi) - \log E_q(e^{\phi})\right]}$$

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Kullback Divergence & VARADHAN's Variational Approach

- Donsker and Varadhan have proposed a variational definition of Kullback divergence :

$$\boxed{K(p,q) = \underset{\phi}{Sup}\left[E_p(\phi) - \log E_q(e^{\phi})\right]}$$

$$\text{Consider} : \phi(\omega) = \ln\left(\frac{p(\omega)}{q(\omega)}\right)$$

$$\Rightarrow E_p(\phi) - \ln\left(E_q(e^{\phi})\right) = \sum_{\omega} p(\omega)\ln\left(\frac{p(\omega)}{q(\omega)}\right) - \ln\left[\sum_{\omega} q(\omega)\frac{p(\omega)}{q(\omega)}\right] = K(p,q) - \ln(1) = K(p,q)$$

This proves that the supremum over all ϕ is no smaller than the divergence

$$E_p(\phi) - \ln\left(E_q(e^{\phi})\right) = E_p\left[\ln\left(\frac{e^{\phi}}{E_q(e^{\phi})}\right)\right] = \sum_{\omega} p(\omega)\left(\ln\left[\frac{q^{\phi}(\omega)}{q(\omega)}\right]\right)$$

$$\text{with} \quad q^{\phi}(\omega) = \frac{q(\omega)e^{\phi(\omega)}}{\sum_{\theta} q(\theta)e^{\phi(\theta)}} \Rightarrow K(p,q) - \left[E_p(\phi) - \ln\left(E_q(e^{\phi})\right)\right] = \sum_{\omega} p(\omega)\left[\ln\left(\frac{p(\omega)}{q^{\phi}(\omega)}\right)\right] \geq 0$$

Using the divergence inequality,

- Link with « Large Deviation Theory » & Fenchel-Legendre Transform which gives that logarithm of generating function are dual to Kullback Divergence :

$$\log\left[\int e^{V(x)}q(x)dx\right] = \underset{p}{Sup}\left[\int V(x)p(x)dx - K(p,q)\right]$$

$$\Leftrightarrow K(p,q) = \underset{V(.)}{Sup}\left[\int V(x)p(x)dx - \log\left[\int e^{V(x)}q(x)dx\right]\right]$$

$$\Leftrightarrow K(p,q) = \underset{V(.)}{Sup}\left[E_p(V) - \log E_q\left[e^{V(x)}\right]\right]$$

Optimal Transport Theory: Wasserstein distance

Wasserstein distance

Cédric Villani, Field Medal 2010, Book on « Optimal Transport Theory : Old & New »

- **Framework of optimal transport theory**

$$W_n(\mu,\nu)=\left(\underset{\pi\in\Pi(\mu,\nu)}{Inf}\int_{\wp} d(x,y)^n\,d\pi(x,y)\right)^{1/n}$$

$$W_n(\mu,\nu)=Inf\left\{\left(E\left[d(X,Y)^n\right]\right)^{1/n},law(X)=\mu,law(Y)=\nu\right\}$$

Particular cases of Wasserstein distance

- **Case *n = 1* : Monge-Kantorovich-Rubinstein distance**

$$W_1(\mu,\nu)=d(P,Q)=\underset{X,Y}{Inf}\;E\left[\left|X-Y\right|\right]$$

- **Case *n=2* : Fréchet distance**

$$W_2(\mu,\nu)=d(P,Q)=\underset{X,Y}{Inf}\;E\left[\left|X-Y\right|^2\right]$$

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Optimal Transport Theory : Fréchet-Wasserstein distance

Fréchet distance

- **In 1957, Maurice René Fréchet has introduced a distance, based on Paul Levy's paper (particular case of Wasserstein distance)**

ACADÉMIE DES SCIENCES

SÉANCE DU LUNDI 4 FÉVRIER 1957.

CALCUL DES PROBABILITÉS. — *Sur la distance de deux lois de probabilité.*
Note de M. MAURICE FRÉCHET.

Une formule explicite et simple est donnée pour représenter la distance de deux lois de probabilités quand on utilise la première des trois définitions de cette distance proposées par Paul Lévy. Une quatrième définition est proposée.

Nous prendrons ici, pour cette distance, l'écart quadratique moyen de X et de Y. En appelant $F(x)$, $G(y)$, $H(x, y)$ les fonctions de répartition respectives de X, de Y et du couple (X, Y), cet écart quadratique moyen D_H a pour carré

$$D_H^2 = \mathfrak{M}_H(X-Y)^2 = \iint_P (x-y)^2 \, d_x d_y H(x, y),$$

$$d^2(F,G) = \underset{X,Y}{Inf}\ E\left[\left|X-Y\right|^2\right] = \iint (x-y)^2 H(x,y)dxdy$$

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Optimal Transport Theory : Fréchet-Wasserstein distance

Fréchet Distance

- **Fréchet's paper from 1957 in CRAS :**

Paul Lévy a proposé ([1]) trois définitions de la distance de deux lois de probabilité L, L′.

Nous examinerons ici la première, qui est la plus intuitive et qui, contrairement à ce que l'on aurait pu attendre, conduit à des formules très simples.

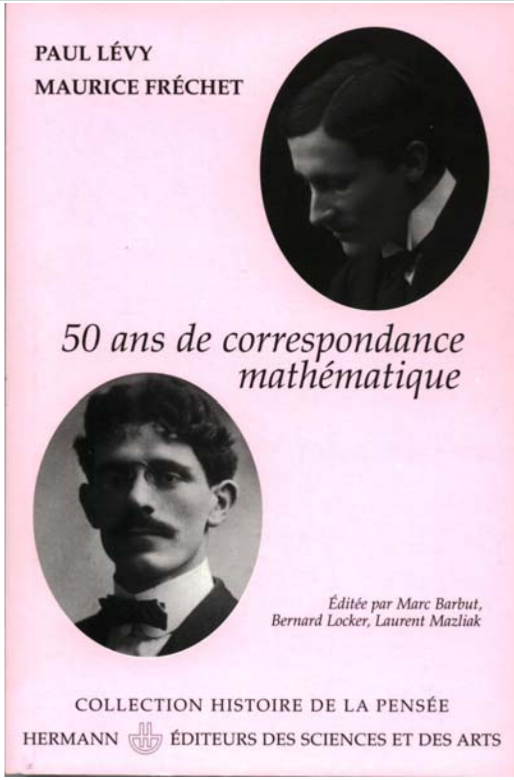
Selon cette première définition, la distance (L, L′) de ces deux lois est la borne inférieure de la « distance globale »

$$([X], [Y])$$

de deux nombres aléatoires X, Y qui ont respectivement L et L′ comme lois de probabilités individuelles, quand la corrélation entre X et Y varie.

Il est clair que la distance (L, L′) va dépendre de la définition adoptée pour la distance globale de X et de Y.

C.R., 1957, 1[er] *Semestre*. (T. 244, N° 6.) 44



- **Paul Levy's letter to Maurice Fréchet (2nd of April 1958)**
  - … J'ai ainsi pu apprécier ce que vous aviez fait, en prenant comme point de départ de votre mémoire ce que vous appelez ma première définition de la distance de deux lois de probabilité (en fait ce n'était pas la première). Vous l'avez d'ailleurs généralisée, en ce sens que je ne l'avais associée qu'à une de vos définitions de deux variables aléatoires. Et j'ai beaucoup admiré comment avec votre quatrième définition, vous arrivez à faire quelque chose de maniable d'une idée qui pour moi était surtout théorique, vu la difficulté de déterminer le minimum de la distance de deux variables aléatoires ayant les répartitions marginales données.

4th definition of Fréchet distance

4th Fréchet Distance

$$d^2(F,G) = \iint (x-y)^2 H_1(x,y)\,dxdy$$

with $H_1(x,y) = Min[F(x),G(y)]$

- Extreme Fréchet Copulas

Nous pouvons, en effet, considérer $H(x, y)$ comme définissant un « tableau de corrélation » dont les « marges » sont définies par les fonctions $F(x)$, $G(y)$.
Or nous avons montré ([2]) que l'ensemble des fonctions $H(x, y)$ est identique à l'ensemble des fonctions de répartition dont les valeurs sont comprises entre deux d'entres elles, à savoir

$$(4) \quad \boxed{\begin{cases} H_0(x, y) = \mathrm{Max}[F(x) + G(y) - 1, 0], \\ H_1(x, y) = \mathrm{Min}[F(x), G(y)]. \end{cases}}$$

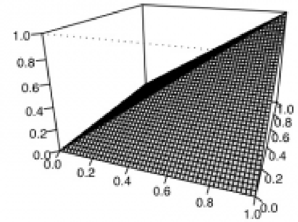
Poursuivant cette étude (d'ailleurs dans un autre but), Salvemini avait conjecturé que $\mathfrak{M}_H(X - Y)^2$ atteignait sa borne inférieure pour $H \equiv H_1$. Bass a énoncé ([3]) le résultat correspondant pour r_H dans le cas où X et Y sont bornés (et m'en a communiqué la démonstration). Un peu plus tard, Dall'Aglio ([4]) a validé la conjecture de Salvemini dans un cas plus général encore.

$H_1(x,y) \le H(x,y) \le H_0(x,y)$

with

$H_0(x,y) = Max[F(x)+G(y)-1,0]$

$H_1(x,y) = Min[F(x),G(y)]$

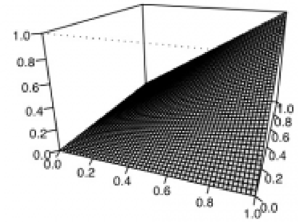


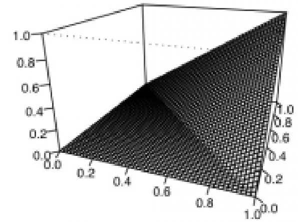
- 4th Fréchet's Distance

Pour esquiver ces deux difficultés, nous allons proposer une quatrième définition. Si celle-ci les supprime, en effet, il faut reconnaître qu'elle est moins intuitive que celle de Lévy.
Nous poserons, *a priori*, sans explication

$$(L, L') = ([X], [Y])_{H_1}.$$

On peut en donner deux justifications. D'une part, elle coïncide avec celle de Lévy, au moins dans le cas, examiné plus haut, où la distance globale de X et Y est égale à leur écart quadratique moyen. D'autre part, on peut prouver que cette valeur de (L, L') vérifie bien même dans le cas général les trois conditions imposées à la notion de distance.





ALES

Application for Multivariate Circular Gaussian Law of 0 mean

Model : Multivariate Circular Gaussian Law of zero mean

$$p(Z/m,R)=\frac{1}{\pi^n \det(R)}e^{-Tr\left(\hat{R}R^{-1}\right)} \text{ with } \begin{cases}\hat{R}=(Z-m)(Z-m)^+\\ E\left[\hat{R}\right]=R\end{cases}$$

Case with $m=0$

- **Rao-Chentsov(-Siegel) Metric & distance**

$$ds^2=Tr\left(\left(R^{-1}dR\right)^2\right)=\left\|R^{-1/2}dRR^{-1/2}\right\|_F^2$$

$$\boxed{\begin{array}{l} d^2(R_X,R_Y)=\left\|\log\left(R_X^{-1/2}.R_Y.R_X^{-1/2}\right)\right\|_F^2=\sum_{k=1}^{n}\ln^2(\lambda_k)\\ \text{with } \det\left(R_X^{-1/2}.R_Y.R_X^{-1/2}-\lambda.I\right)=\det(R_Y-\lambda R_X)=0\end{array}}$$

- **Fréchet-Levy(-Wasserstein) distance**

$$\boxed{d^2(R_X,R_Y)=Tr[R_X]+Tr[R_Y]-2.Tr\left[\left(R_X^{1/2}R_YR_X^{1/2}\right)^{1/2}\right]}$$

THALES

Properties of each geometry

Information Geometry

- **Geodesic**

$$\gamma(t) = R_X^{1/2} e^{t \log\left(R_X^{-1/2} R_Y R_X^{-1/2}\right)} R_X^{1/2} = R_X^{1/2} \left(R_X^{-1/2} R_Y R_X^{-1/2}\right)^t R_X^{1/2}$$

$$\gamma(0) = R_X \quad , \quad \gamma(1) = R_Y \;\text{ and }\; \gamma(1/2) = R_X \circ R_Y$$

- Space with sectional **Negative Curvature**
- Invariance by parameters changes

Existence and Unicity of Barycenter only proved in case of negative curvature by Elie Cartan during 20's

Wasserstein Geometry

- **Geodesic**

$$R_{(t)} = \left((1-t)I_k + t.D_{X,Y}\right) R_Y \left((1-t)I_k + t.D_{X,Y}\right)$$

$$\text{with } \; D_{X,Y} = R_X^{1/2}\left(R_X^{1/2} R_Y R_X^{1/2}\right)^{-1/2} R_X^{1/2} = R_X \circ R_Y^{-1}$$

- Space with sectional **Positive Curvature**

Information Geometry : Multivariate Gaussian Case, m=0

Information Geometry for Multicariate Gaussian Laws (cas m=0)

- **Rao-Chentsov distance**

$$d^2(R_X,R_Y)=\left\|\log\left(R_X^{-1/2}.R_Y.R_X^{-1/2}\right)\right\|^2=\sum_{k=1}^{n}\ln^2(\lambda_k)$$

$$\text{avec } \det\left(R_X^{-1/2}.R_Y.R_X^{-1/2}-\lambda.I\right)=\det\left(R_Y-\lambda R_X\right)=0$$

Particular case of Carl-Ludwig Siegel case (in the framework of symplectic geometry)

C.L. Siegel

- **Siegel Metric**

$$SH_n=\{Z=X+iY\in Sym(n,C)/\operatorname{Im}(Z)=Y>0\}$$

$$ds^2_{Siegel}=Tr\left(Y^{-1}.dZ.Y^{-1}.d\bar{Z}\right)\quad \text{avec}\quad Z=X+iY$$

  - Invariant by all automorphisms of SH_n

$$M(Z)=(AZ+B)(CZ+D)^{-1}\quad \text{avec}\quad A^TD-C^TB=I_n$$

  - Particular Case : $\begin{cases}X=0\\Y=R\end{cases}\Rightarrow ds^2=Tr\left[\left(R^{-1}dR\right)^2\right]$

THALES

Example : Monovariate Gauss-Laplace Law

Gauss-Laplace Law

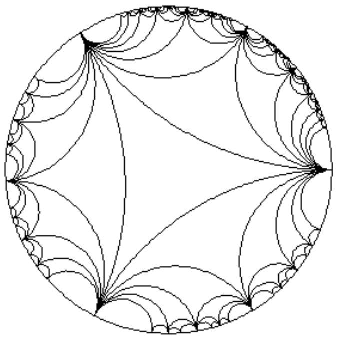
- Fisher Information Matrix for Gaussian Law :

$$I(\theta) = \sigma^{-2}.\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{with} \quad E\left[\left(\theta - \hat{\theta}\right)\left(\theta - \hat{\theta}\right)^T\right] \geq I(\theta)^{-1} \quad \text{and} \quad \theta = \begin{pmatrix} m \\ \sigma \end{pmatrix}$$

- Rao-Chentsov Metric from Information Geometry

$$\boxed{ds^2 = d\theta^T.I(\theta).d\theta = \frac{dm^2}{\sigma^2} + 2.\frac{d\sigma^2}{\sigma^2} = 2.\sigma^{-2}\left[\left(\frac{dm}{\sqrt{2}}\right)^2 + (d\sigma)^2\right]}$$

H. Poincaré

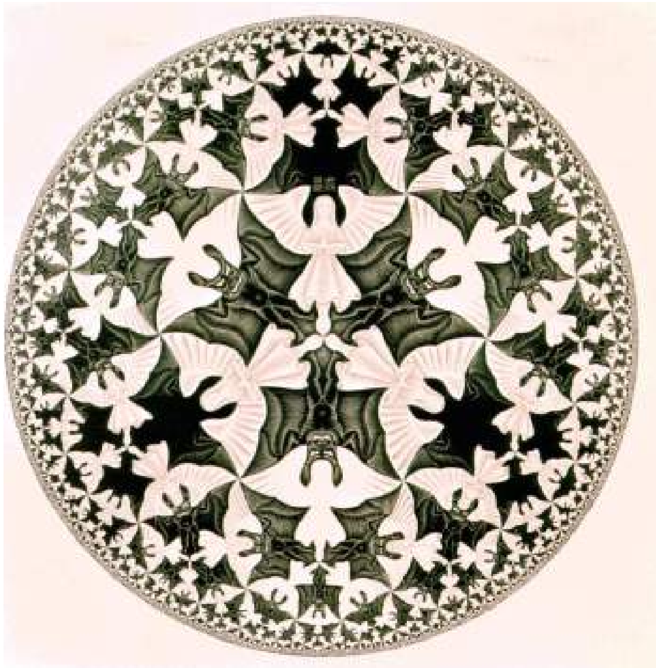


- This metric is the Poincaré metric (model of hyperbolic geometry)

$$z = \frac{m}{\sqrt{2}} + i.\sigma \qquad \omega = \frac{z-i}{z+i} \quad \left(|\omega| < 1\right) \qquad \Rightarrow ds^2 = 8.\frac{|d\omega|^2}{\left(1-|\omega|^2\right)^2}$$

$$\boxed{\begin{gathered} d^2\left(\{m_1,\sigma_1\},\{m_2,\sigma_2\}\right) = 2.\left(\log\frac{1+\delta(\omega^{(1)},\omega^{(2)})}{1-\delta(\omega^{(1)},\omega^{(2)})}\right)^2 \\ \text{with} \quad \delta(\omega^{(1)},\omega^{(2)}) = \left|\frac{\omega^{(1)}-\omega^{(2)}}{1-\omega^{(1)}\omega^{(2)*}}\right| \end{gathered}}$$

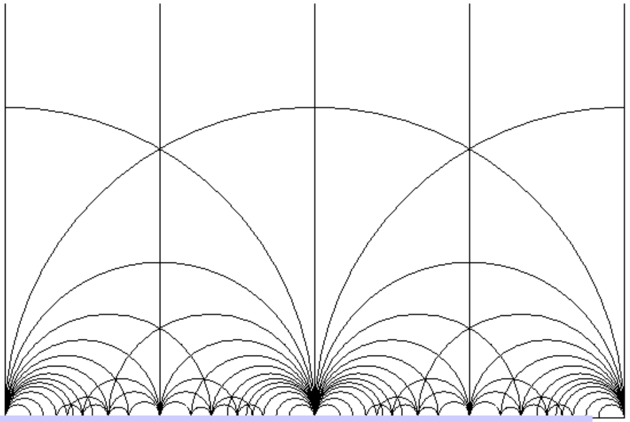
Poincaré Space in Art (Escher, Irène Rousseau)

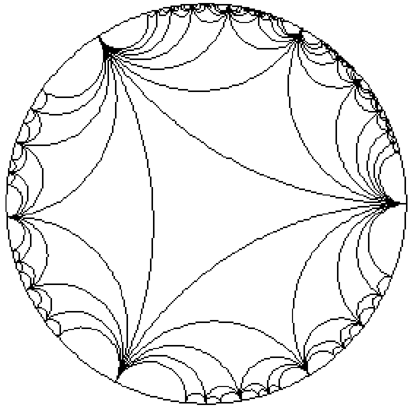


Henri Poincaré dans son cabinet de travail.

THALES

Poincaré & Siegel Upper Half Plane & Disk





Poincaré Upper Half Plane

$$w = \frac{z-i}{z+i}$$

Poincaré Unit disk

$$ds^2 = \frac{dx^2+dy^2}{y^2} = \frac{|dz|^2}{y^2}$$

$$ds^2 = y^{-1} dz y^{-1} dz^*$$

avec $z = x + iy$ and $y > 0$

$$ds^2 = \frac{|dw|^2}{\left(1-|w|^2\right)^2}$$

$$ds^2 = \left(1-ww^*\right)^{-1} dw \left(1-ww^*\right)^{-1} dw^*$$

Siegel Upper Half Plane

$$W = (Z-iI)(Z+iI)^{-1}$$

Siegel Unit disk

$$ds^2 = Tr\left(Y^{-1} dZ Y^{-1} d\overline{Z}\right)$$

with $Z = X + iY$

$X \in Herm(n,C)$ and $Y \in HPD(n,C)$

$$ds^2 = Tr\left[\left(I-WW^+\right)^{-1} dW \left(I-W^+W\right)^{-1} dW^+\right.$$

THALES

Siegel Space

L.K. Hua

C.L. Siegel

F. Berezin

Siegel Upper Half Space

$$SH_n = \{ Z = X + iY \in Sym(n, C) \,/\, \mathrm{Im}(Z) = Y > 0 \}$$

$$\det\left(R_1^{-1/2}.R_2.R_1^{-1/2} - \lambda.I\right) = 0$$

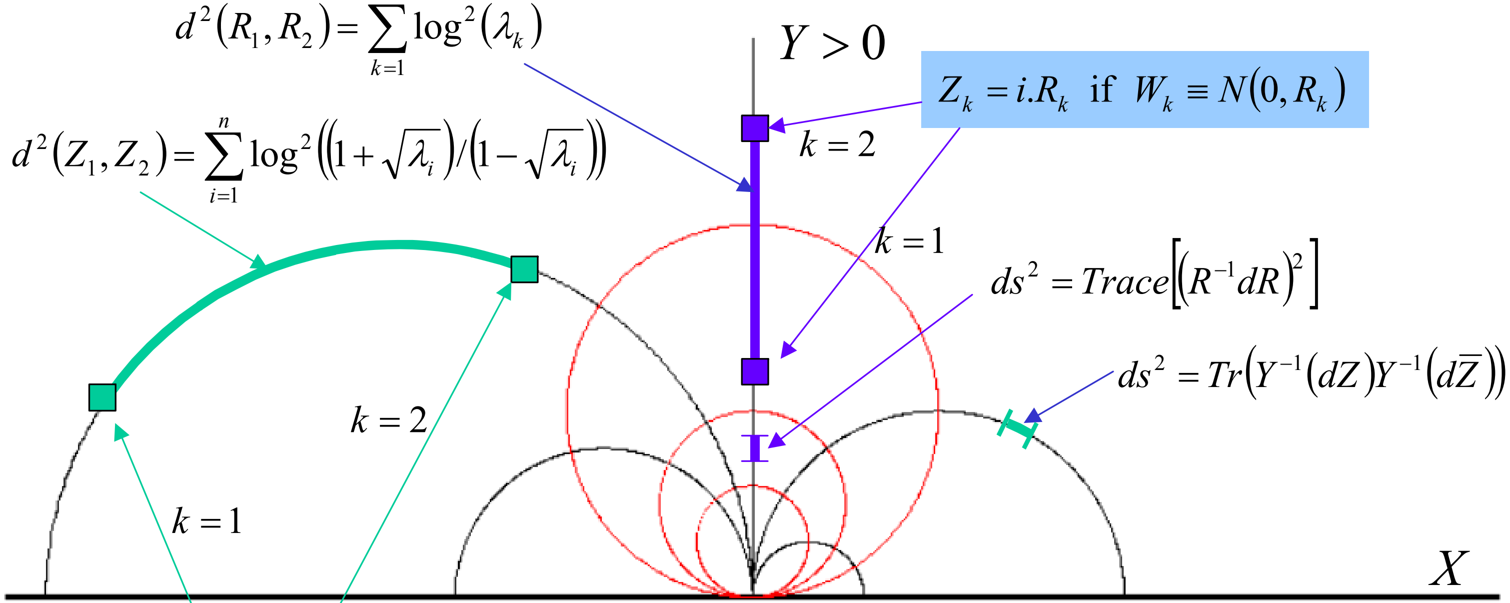
$$d^2(R_1, R_2) = \sum_{k=1}^{n} \log^2(\lambda_k)$$

$$d^2(Z_1, Z_2) = \sum_{i=1}^{n} \log^2\left(\left(1+\sqrt{\lambda_i}\right)/\left(1-\sqrt{\lambda_i}\right)\right)$$

$$Z_k = i.R_k \ \text{ if } \ W_k \equiv N(0, R_k)$$

$$ds^2 = Trace\left[\left(R^{-1}dR\right)^2\right]$$

$$ds^2 = Tr\left(Y^{-1}(dZ)Y^{-1}(d\bar{Z})\right)$$



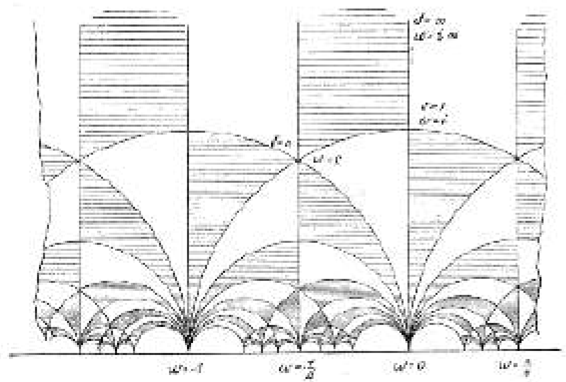
$$Z_k = X_k + i.Y_k$$

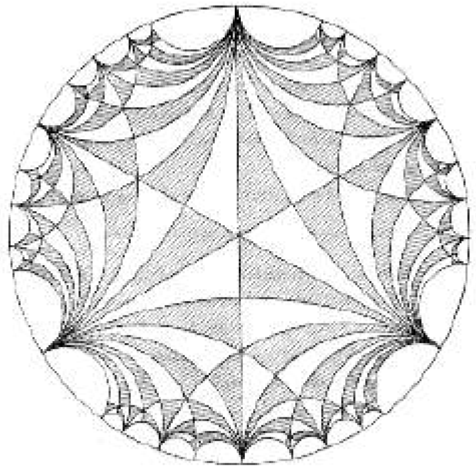
$$R(Z_1, Z_2) = (Z_1 - Z_2)(Z_1 - \bar{Z}_2)^{-1}(\bar{Z}_1 - \bar{Z}_2)(\bar{Z}_1 - Z_2)^{-1}$$

$$\det\left(R(Z_1, Z_2) - \lambda.I\right) = 0$$

259 Letters between Mittag-Leffler & Poincaré

- « Acta Mathematica » was founded by Gösta Mittag-Leffler in 1882
- Henri Poincaré has published paper on Fuchsian Group in first volume of Acta Mathematica: ***Henri Poincaré (1882) "Théorie des Groupes Fuchsiens", Acta Mathematica v.1, p.1.***





Information Geometry based on Poincaré's hyperbolic geometry

- ***Henri Poincaré (1882) "Théorie des Groupes Fuchsiens", Acta Mathematica v.1, p.1., 1882 published by*** Mittag-Leffler

THÉORIE DES GROUPES FUCHSIENS

PAR H. POINCARÉ

à PARIS.

Dans une série de mémoires présentés à l'Académie des Sciences j'ai défini certaines fonctions nouvelles que j'ai appelées fuchsiennes, kleinéennes, thétafuchsiennes et zétafuchsiennes. De même que les fonctions elliptiques et abéliennes permettent d'intégrer les différentielles algébriques, de même les nouvelles transcendantes permettent d'intégrer les équations différentielles linéaires à coëfficients algébriques. J'ai résumé succinctement les résultats obtenus dans une note insérée aux *Mathematische Annalen*. Ayant l'intention de les exposer en détail, je commencerai, dans le présent travail, par étudier les propriétés des groupes fuchsiens, me réservant de revenir plus tard sur leurs conséquences au point de vue de la théorie des fonctions.

§ 1. **Substitutions réelles.**

Soit z une variable imaginaire définie par la position d'un point dans un plan; t une fonction imaginaire de cette variable définie par la relation:

$$t = \frac{az + b}{cz + d}. \tag{1}$$

Je supposerai, ce qui ne restreint pas la généralité, que l'on a:

$$ad - bc = 1.$$

Si le point z décrit deux arcs de courbe se coupant sous un certain angle α, le point t décrira de son côté deux arcs de courbe se coupant

THALES

Carl Ludwig siegel

Information Geometry : Multivariate Gaussian Case, m=0

Information Geometry for Multivariate Gaussian Laws (cas m=0)

- **Geodesic :** $d(R_X,\gamma(t))=t.d(R_X,R_Y)$ with $t\in[0,1]$

$$\gamma(t)=R_X^{1/2}e^{t\log\left(R_X^{-1/2}R_YR_X^{-1/2}\right)}R_X^{1/2}=R_X^{1/2}\left(R_X^{-1/2}R_YR_X^{-1/2}\right)^tR_X^{1/2}$$

$$\gamma(0)=R_X \quad , \quad \gamma(1)=R_Y \text{ and } \gamma(1/2)=R_X\circ R_Y$$

Properties of this space

- **Symetric Space as studied by Elie Cartan : Existence of bijective geodesic isometry**

$$G_{(A,B)}X=\left(A\circ B\right)\mathrm{X}^{-1}(A\circ B) \quad \text{avec} \quad A\circ B=A^{1/2}\left(A^{-1/2}BA^{-1/2}\right)^{1/2}A^{1/2}$$

- **Bruhat-Tits Space : semi-parallelogram inequality**

$$\forall x_1,x_2 \quad \exists z \text{ tel que } \forall x$$

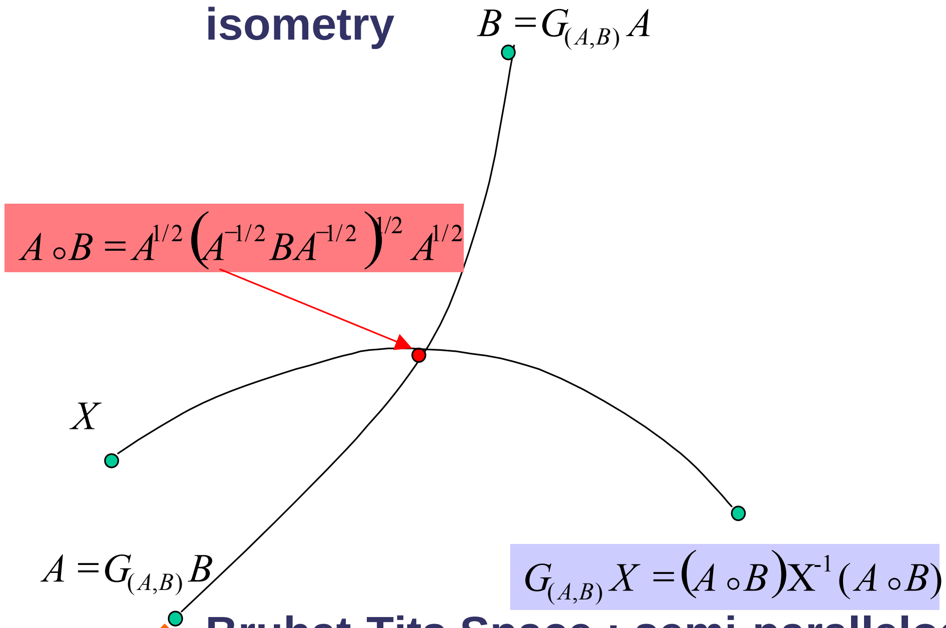
$$d(x_1,x_2)^2+4d(x,z)^2\leq 2d(x,x_1)^2+2d(x,x_2)^2 \quad \forall x\in X$$

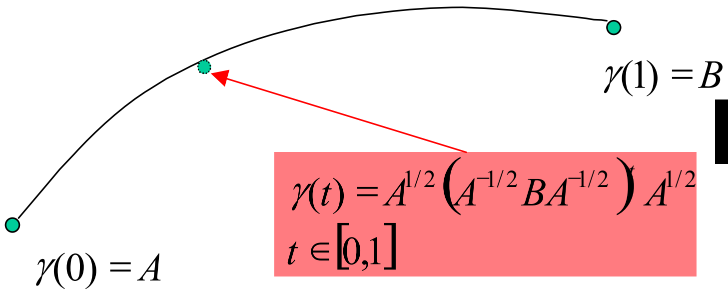
- **Cartan-Hadamard Space (Complete, simply connected with negative sectional curvature Manifold)**

THALES

Information Geometry : Multivariate Gaussian Case, m=0

- **Symetric Space as studied by Elie Cartan : Existence bijective geodesic isometry**



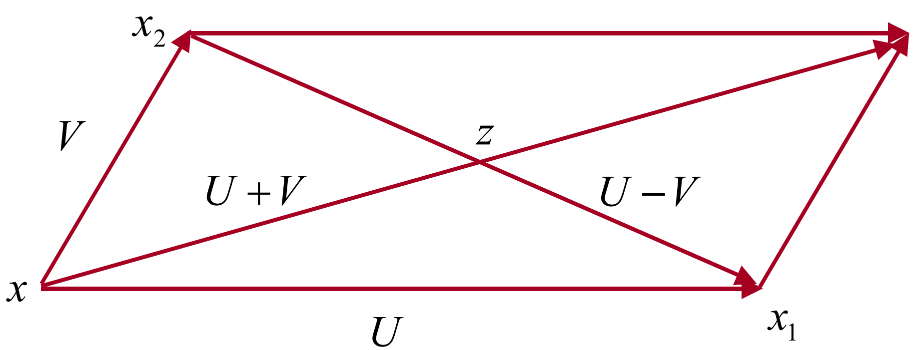



E. J. Cartan


M. Berger

- **Bruhat-Tits Space : semi-parallelogram inequality**

$$d(x_1,x_2)^2 + 4d(x,z)^2 \leq 2d(x,x_1)^2 + 2d(x,x_2)^2$$



$$\|U-V\|^2 + \|U+V\|^2 = 2\|U\|^2 + 2\|V\|^2$$

Information Geometry & Fréchet-Karcher Gradient Flow

THALES

OS-HD-Doppler-CFAR

For detection of Slow & Stealth/Small Target in inhomogeneous clutter, we need simultaneously :

- **High Doppler Resolution with short Bursts**
- **Robust CFAR in inhomogeneous clutter & closely separated targets**

Proposed Solution : OS-HR-Doppler-CFAR

- **Avoid drawbacks of Doppler Filters / FFT in case of short bursts**
- **Take advantages of Robust Ordered Statistic of OS-CFAR (Ordered Statistic CFAR, Median-CFAR)**

Challenges to define OS-HR-Doppler-CFAR :

- **Can we order « Doppler Spectrums » :** NO
  - there is no total order of covariance matrices $R_1>R_2>\ldots>R_n$
  - There is only Partial « Lowner » order : $R_1>R_2$ if and only if R1-R2 Positive Definite
- **Can we define « Median » of « Doppler Spectrums » :** YES !!!
  - In a « Metric Space », the median is defined as the point that minimizes the « geodesic » distance to each point (compared to the mean that minimizes the square distance to each point)
  - We can define a deterministic or stochastic gradient flow that converges fastly to « median spectrum » (Modified Karcher Flow : **THALES Patent**)

THALES

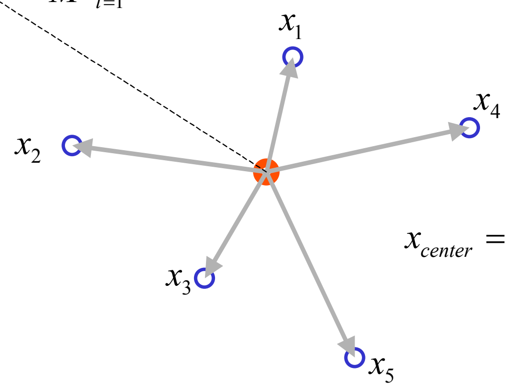
Center of Mass : Arithmetic Mean and Median in R^n

In R^n, the center of mass is defined for finite set of points $\{x_i\}_{i=1,\dots,M}$

- Arithmetic mean: $x_{center} = \frac{1}{M}\sum_{i=1}^{M} x_i$
- This point minimizes the function of distances: $x_{center} = \arg\underset{x}{Min}\sum_{i=1}^{M} d^2(x,x_i)$
- The median (Fermat-Weber Point) minimizes : $x_{median} = \arg\underset{x}{Min}\sum_{i=1}^{M} d(x,x_i)$

$$m_{mean} = \underset{m}{Min}\, E\left[\|x-m\|^2\right]$$

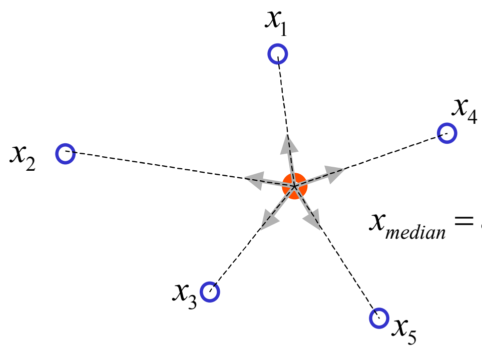
$$x_{center} = \frac{1}{M}\sum_{i=1}^{M} x_i$$



$$x_{center} = \arg\underset{x}{Min}\sum_{i=1}^{M} \overrightarrow{xx_i}$$

$$\boxed{x_{center} = \arg\underset{x}{Min}\sum_{i=1}^{M} d^2(x,x_i)}$$

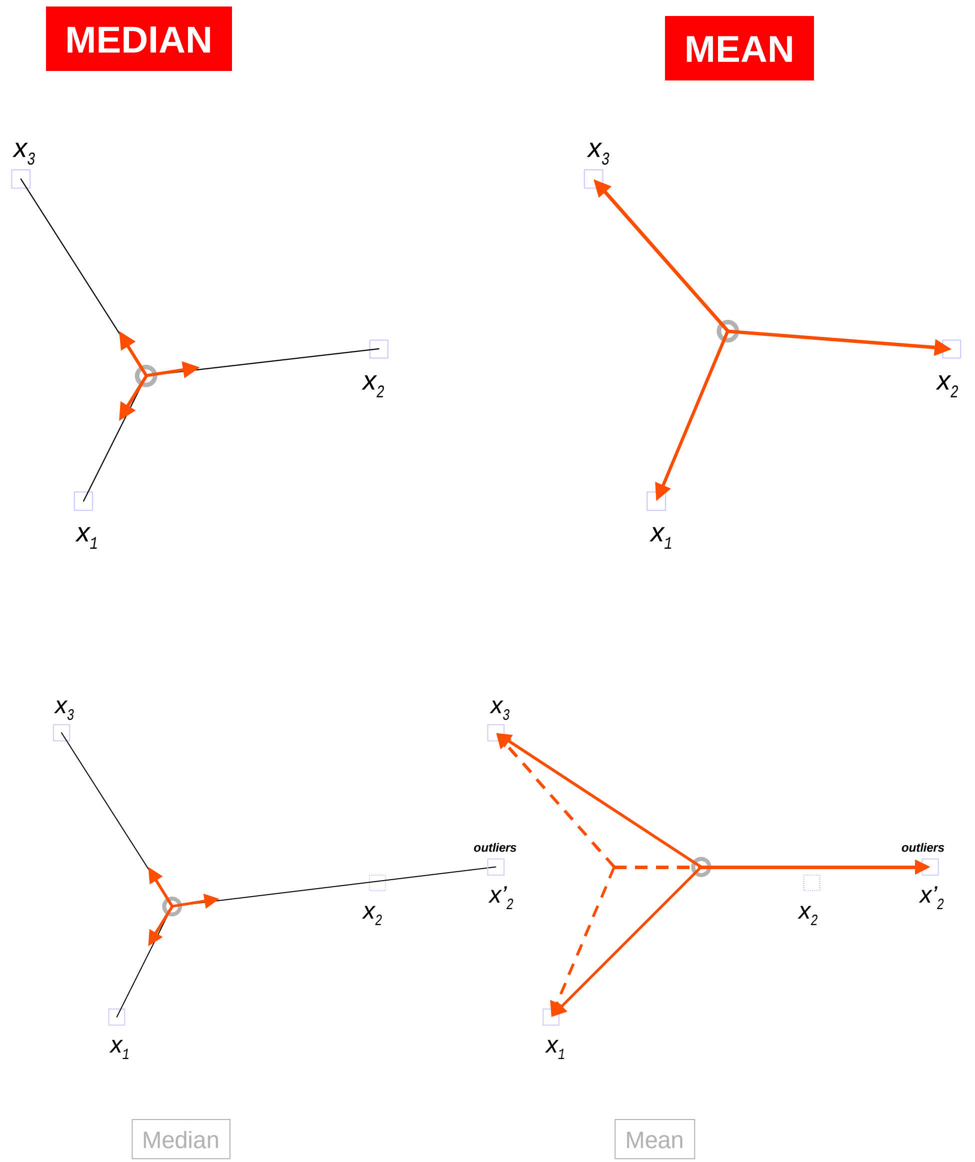
$$m_{median} = \underset{m}{Min}\, E\left[\|x-m\|\right]$$



$$x_{median} = \arg\underset{x}{Min}\sum_{i=1}^{M} \frac{\overrightarrow{xx_i}}{\left\|\overrightarrow{xx_i}\right\|}$$

$$\boxed{x_{median} = \arg\underset{x}{Min}\sum_{i=1}^{M} d(x,x_i)}$$

Sensitivity to outliers : Median versus Mean

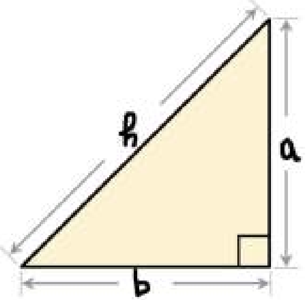


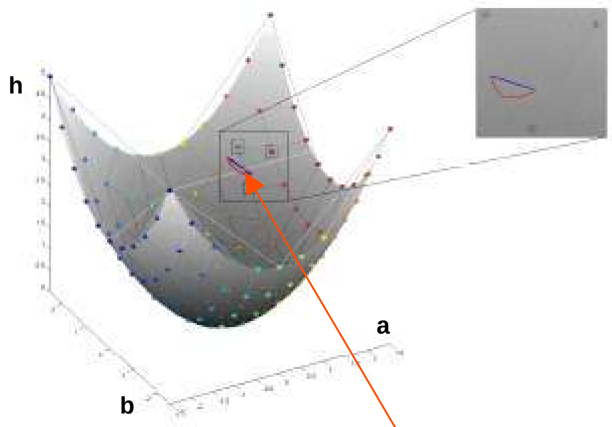
THALES

Extension of barycenter in metric space

Right Triangle

$h^2=a^2+b^2$





One Right Triangle could be represented by 1 point on surface $h^2=a^2+b^2$

Naive Mean of N Right Triangles :

- **Let N Right triangles :**

$$\{a_i, b_i, h_i\}_{i=1}^{N} \quad \text{with} \quad h_i^2 = a_i^2 + b_i^2$$

- **« Arithmetic » Mean is not a Right triangle**

$$\left\{ \frac{1}{N}\sum_{i=1}^{N} a_i, \frac{1}{N}\sum_{i=1}^{N} b_i, \frac{1}{N}\sum_{i=1}^{N} h_i \right\} = \{A, B, H\}$$

$$H^2 \neq A^2 + B^2$$

Solution : Fréchet Mean (Center of Mass)

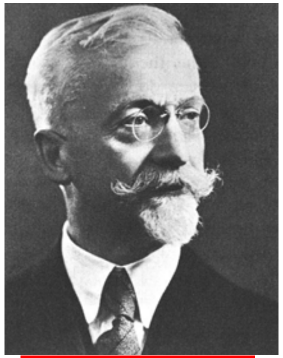
- **Consider the surface $h^2=a^2+b^2$ in Space of coordinates (a,b,h)**
- **Fréchet Mean/Barycenter/Center-of-Mass**

$$\{A, B, H\} = \arg \underset{\{A,B,H\}}{Min} \sum_{i=1}^{N} d^2_{geodesic}\left(\{a_i, b_i, h_i\}, \{A, B, H\}\right)$$

$$d^2_{geodesic}(.,.) \text{ defined on surface } \left\{(a,b,h) / h^2 = a^2 + b^2\right\}$$

Cartan Center of Mass and Karcher Flow

Cartan Center of Mass



E. J. Cartan

- **Elie Cartan has proved that the following functional :**

$$f : m \in M \mapsto \int_A d^2(m,a)da$$

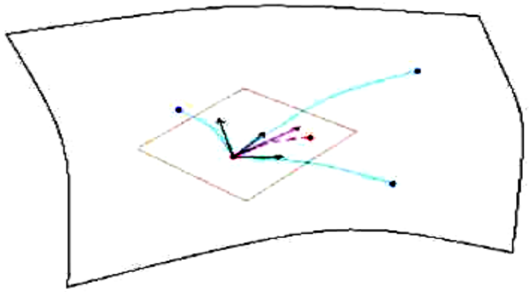
is strictly convexe and has only one minimum (center of mass of *A* for distribution *da*) for a manifold of negative curvature

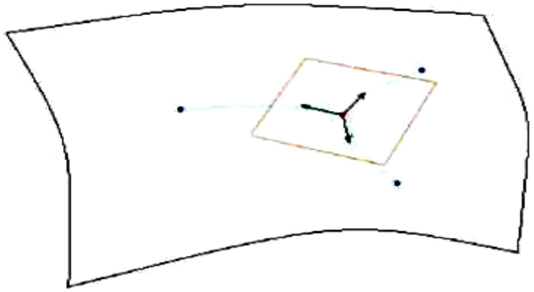
Karcher Flow

H. Karcher

- **Hermann Karcher has proved the convergence of the following flow to the Center of Mass :**

$$m_{n+1} = \gamma_n(t_n) = \exp_{m_n}\left(-t_n.\nabla f(m_n)\right) \text{ avec } \dot{\gamma}_n(0) = -\nabla f(m_n)$$





$$\nabla f = -\int_A \exp_m^{-1}(a)da$$

probability on a manifold : Emery's exponential barycenter

- **Maurice René Fréchet, inventor of Cramer-Rao bound in 1939, has also introduced the entire concept of** Metric Spaces Geometry **and functional theory on this space (any normed vector space is a metric space by defining but not the contrary). On this base, Fréchet has then extended probability in abstract spaces.** $d(x,y)=\|y-x\|$

M. R. Fréchet, "Les éléments aléatoires de nature quelconque dans un espace distancié", Annales de l'Institut Henri Poincaré, n°10, pp.215-310, 1948

- **In this framework, expectation** $b=E[g(x)]$ **of an abstract probabilistic variable** $g(x)$ **where** x **lies on a manifold is introduced by Emery as an exponential barycenter :**

$$\int_M \exp_b^{-1}(g(x))P(dx)=0$$

- **In Classical Euclidean space, we recover classical definition of Expectation** ***E[.]*** **:**

$$p,q\in R^n \Rightarrow \exp_p^{-1}(q)=q-p \Rightarrow E[g(x)]=\int_{R^n} g(x)P(dx)=\int_{R^n} g(x)p_X(x)dx$$

M. Emery & G. Mokobodzki, "Sur le barycentre d'une probabilité sur une variété", Séminaire de Proba. XXV, Lectures note in Math. 1485, pp.220-233, Springer, 1991

Mean & Median of N matrices HPD(n)

Mean of N Hermitian Positive Definite Matrices HPD(n)

- **Solution given by Karcher Flow with Information Geometry metric**

$$X_{moyenne} = \underset{X}{Arg\ Min} \sum_{k=1}^{N} d^2(X, B_k) = \sum_{k=1}^{N} \left\| \log\left(X^{-1/2}.B_k.X^{-1/2}\right) \right\|^2$$

$$X_{n+1} = X_n^{1/2} e^{\varepsilon \sum_{k=1}^{N} \log\left(X_n^{-1/2} B_k X_n^{-1/2}\right)} X_n^{1/2}$$

Median (Fermat-Weber Point) of N matrices HPD(n)

$$X_{mediane} = \underset{X}{Arg\ Min} \sum_{k=1}^{N} d(X, B_k) = \sum_{k=1}^{N} \left\| \log\left(X^{-1/2}.B_k.X^{-1/2}\right) \right\|$$

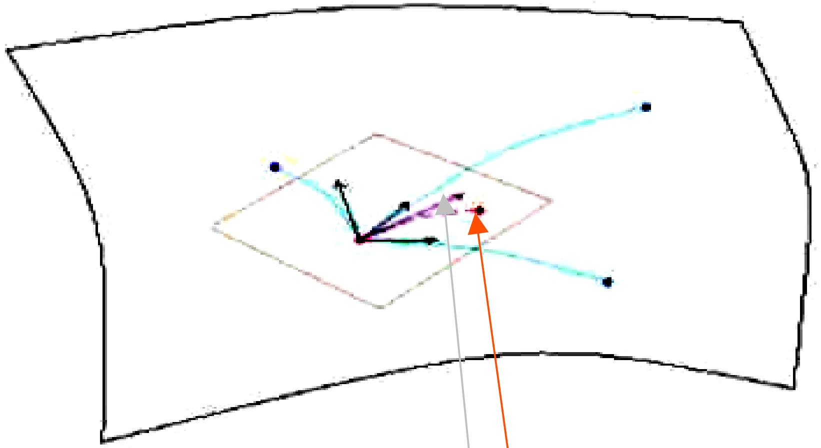
$$X_{n+1} = X_n^{1/2} e^{\varepsilon \sum_{k \in S_n} \frac{\log\left(X_n^{-1/2} B_k X_n^{-1/2}\right)}{\left\| \log\left(X_n^{-1/2} B_k X_n^{-1/2}\right) \right\|}} X_n^{1/2} \quad \text{with} \quad S_n = \left\{ k \,/\, X_n \neq B_k \right\}$$

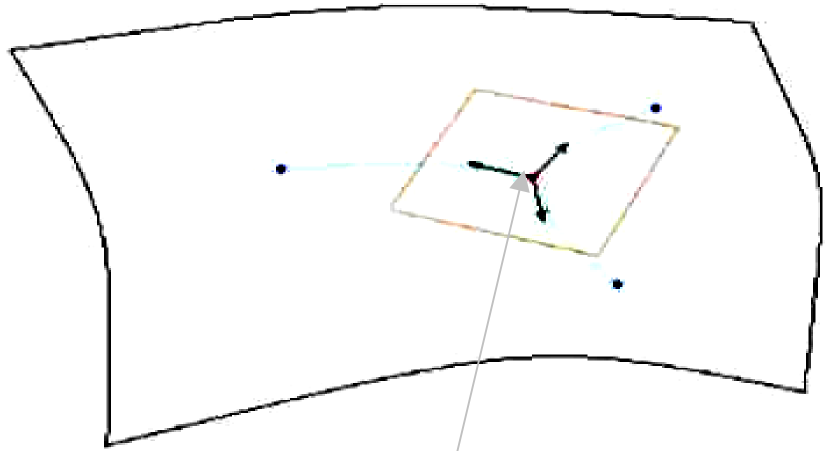
- **PhD Yang Le supervised by Marc Arnaudon (Univ. Poitiers/Thales)**

THALES

Median on a Manifold: Karcher-Cartan-Fréchet

- **Gradient flow on Surface/Manifold**





Gradient Flow :Pushed by Sum of Normalized Tangent vectors of Geodesics

Fréchet-Karcher Barycenter : Sum of Normalized Tangent vectors of Geodesics is equal to zero

$$h : m \in \mathrm{M} \mapsto \frac{1}{2}\int_A d(m,a)da \underset{Min}{\Rightarrow} \nabla h = -\boxed{\int_A \frac{\exp_m^{-1}(a)}{\left\|\exp_m^{-1}(a)\right\|}da}$$

Sum of Normalized Tangent Vectors

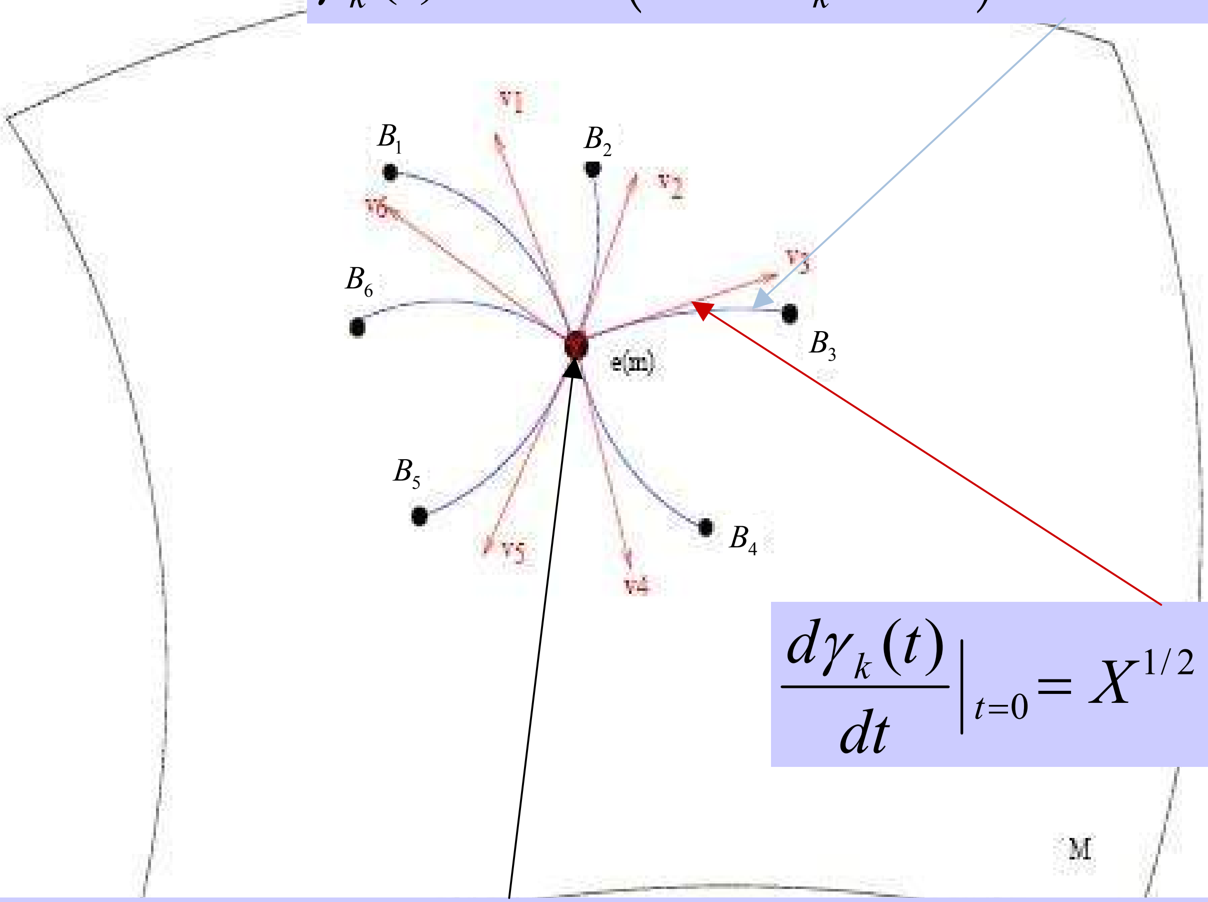
Compute point on the surface In the direction of sum of Normalized Tangent Vector

$$\boxed{m_{n+1} = \exp_{m_n}\left(t.\sum_{k=1}^{M}\frac{\exp_{m_n}^{-1}(x_k)}{\left\|\exp_{m_n}^{-1}(x_k)\right\|}\right)}$$

THALES

Mean : Karcher Barycenter

$$\gamma_k(t) = X^{1/2}\left(X^{-1/2}B_k X^{-1/2}\right)^t X^{1/2} = X^{1/2} e^{t\log\left(X^{-1/2}B_k X^{-1/2}\right)} X^{1/2}$$

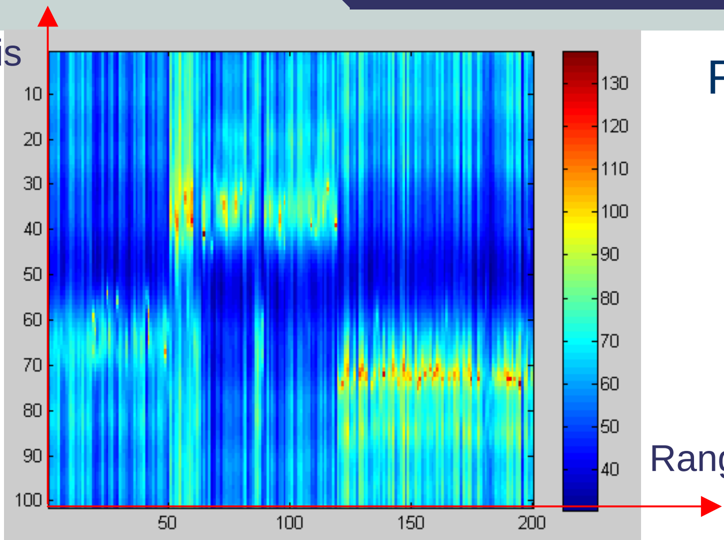


$$\left.\frac{d\gamma_k(t)}{dt}\right|_{t=0} = X^{1/2}\log\left(X^{-1/2}B_k X^{-1/2}\right)X^{1/2}$$

$$\sum_{k=1}^{N}\left.\frac{d\gamma_k(t)}{dt}\right|_{t=0} = X^{1/2}\left(\sum_{k=1}^{N}\log\left(X^{-1/2}B_k X^{-1/2}\right)\right)X^{1/2} = 0$$

Comparison of Mean & Median Doppler Spectrum

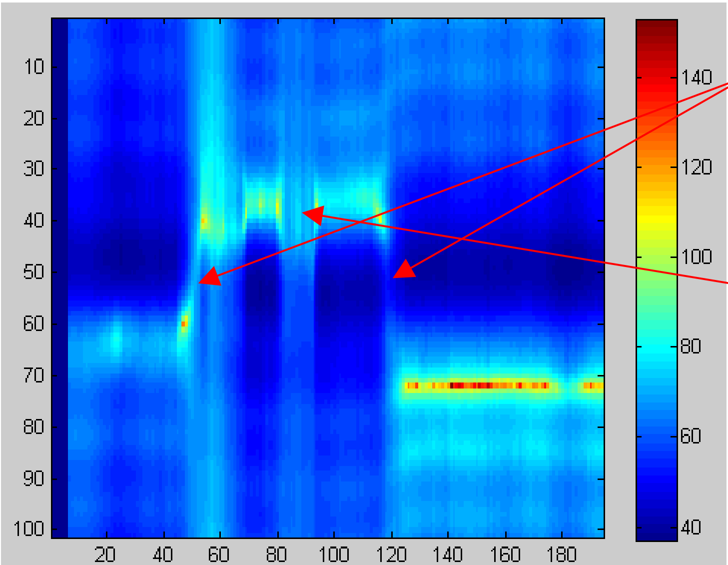
Doppler axis



Raw Doppler Spectrum

Range axis

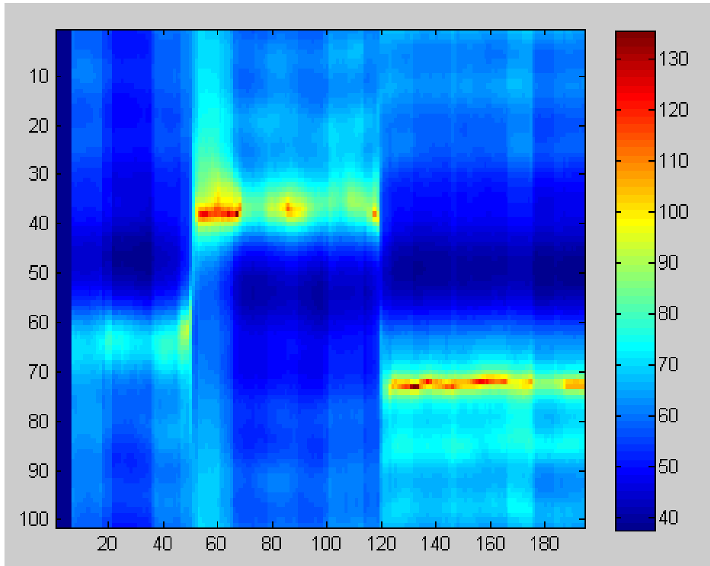
Mean Doppler Spectrum



No preservation of discontinuities

Perturbation by outlier

Median Doppler Spectrum



THALES

Other approach to smooth spectrum : Fourier Heat Equation

- **Diffusion Fourier Equation on 1D graph (scalar case)**
  - Approximation par un Laplacien discret :

J. Fourier

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}\Rightarrow\frac{\partial u}{\partial t}=\frac{1}{\nabla x}\left(\frac{u_{n+1}-u_n}{\nabla x}-\frac{u_n-u_{n-1}}{\nabla x}\right)=\frac{2}{\nabla x^2}\left(\hat{u}_n-u_n\right)$$

with arithmetic mean of adjacent points : $\hat{u}_n=\left(u_{n+1}+u_{n-1}\right)/2$

  - Dicrete Fourier Heat Equation for Scalar Values in 1D :

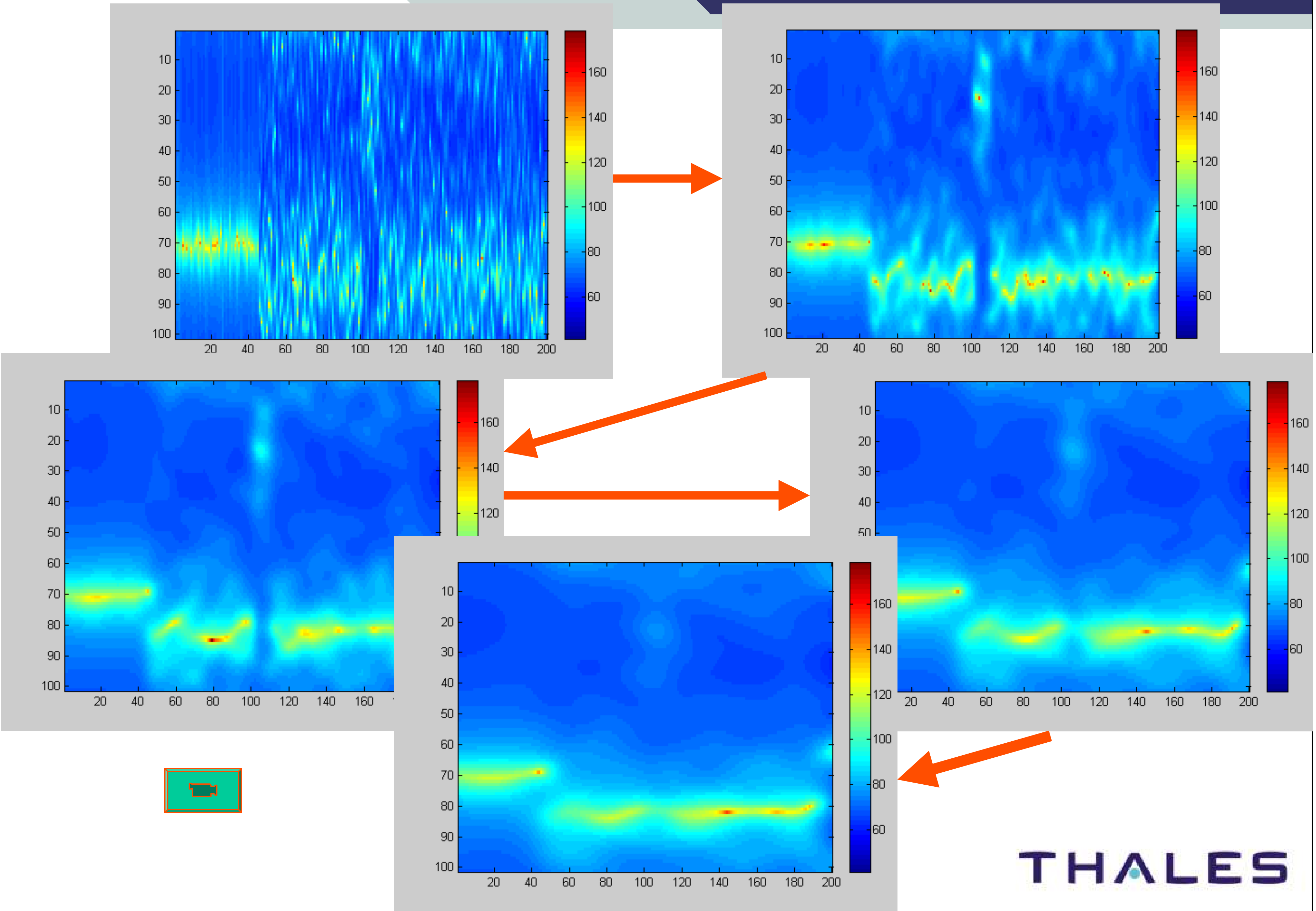
$$u_{n,t+1}=(1-\rho).u_{n,t}+\rho.\hat{u}_{n,t}\quad \text{with}\quad \rho=\frac{2.\nabla t}{\nabla x^2}$$

- **By Analogy, we can define Fourier Heat Equation in 1D grapf of HDP(n) matrices :**

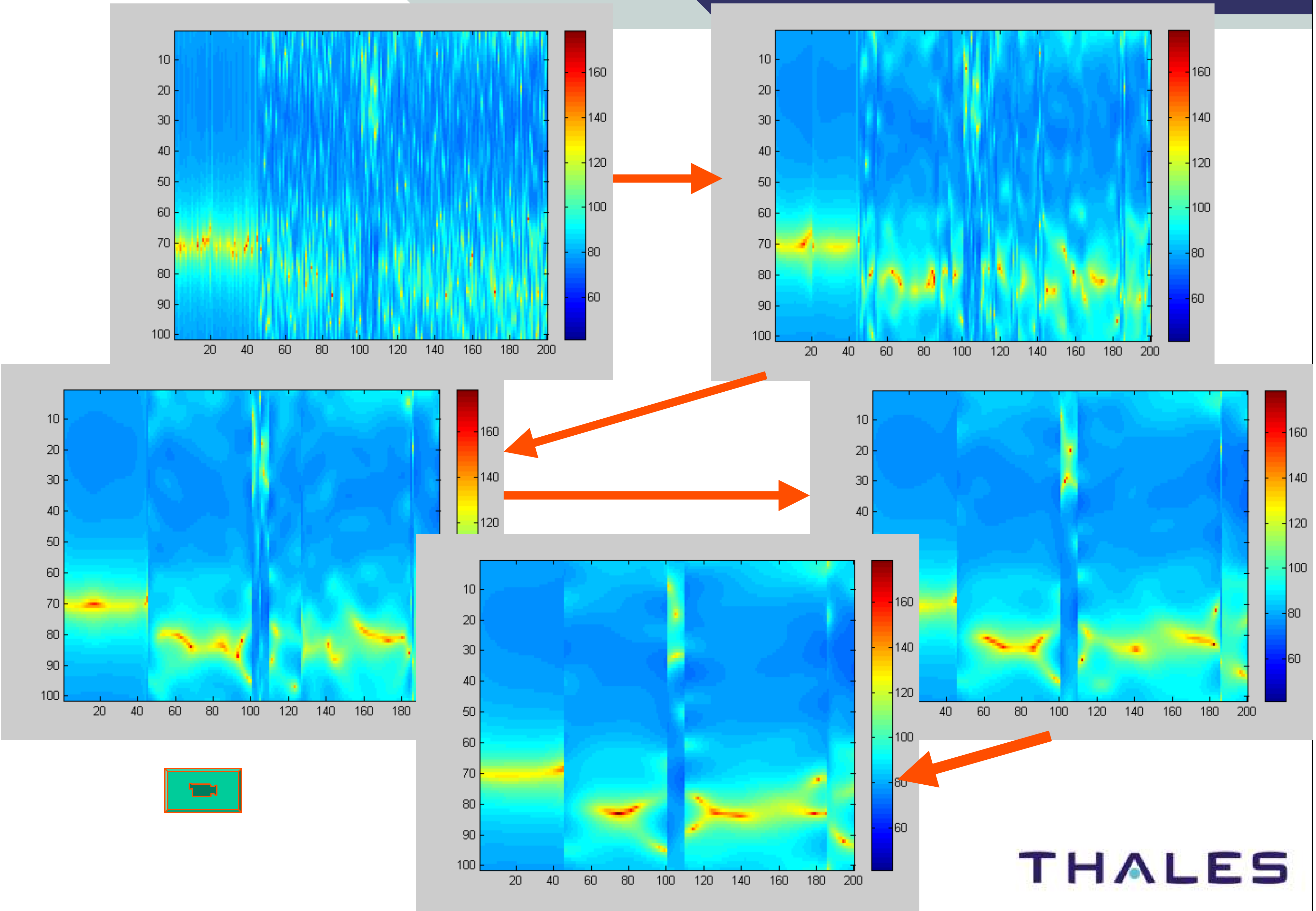
$$X_{n,t+1}=X_{n,t}^{1/2}e^{\rho\log\left(X_{n,t}^{-1/2}\hat{X}_{n,t}X_{n,t}^{-1/2}\right)}X_{n,t}^{1/2}=X_{n,t}^{1/2}\left(X_{n,t}^{-1/2}\hat{X}_{n,t}X_{n,t}^{-1/2}\right)^{\rho}X_{n,t}^{1/2}$$

$$\text{with}\quad \hat{X}_{n,t}=X_{n+1,t}^{1/2}\left(X_{n+1,t}^{-1/2}X_{n-1,t}X_{n+1,t}^{-1/2}\right)^{1/2}X_{n+1,t}^{1/2}=X_{n+1,t}\circ_{1/2}X_{n-1,t}$$

Isotropic Diffusion of Doppler Spectrum

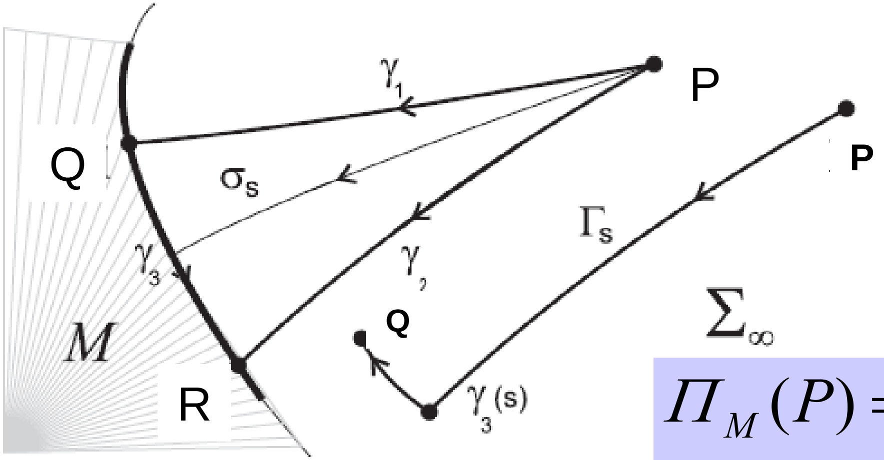


Anisotropic Diffusion of Doppler Spectrum



Geodesic Projection

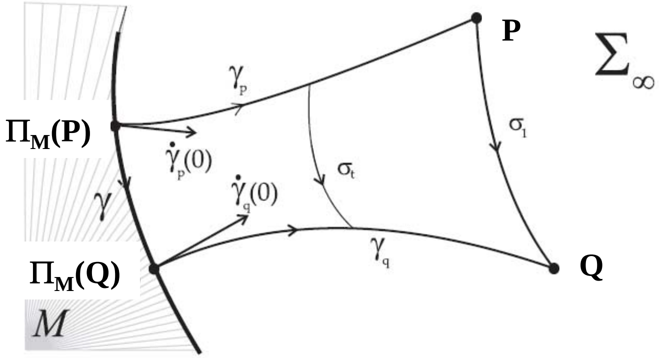
M : geodesically convex closed submanifold of Hadamard Manifold



$$\Pi_M(P) = \arg\min_{S\in M} dist(P,S)$$

- **Geodesic between matrix P and Geodesic between Q & R :**

$$\sigma_s(t) = \sigma(s,t) = P^{1/2}\left[P^{-1/2}Q^{1/2}\left(Q^{-1/2}RQ^{-1/2}\right)^s Q^{1/2}P^{-1/2}\right]^t P^{1/2}$$



- **The geodesic projection is a contraction :**

$$dist\left(\Pi_M(P), \Pi_M(Q)\right) \le dist(P,Q)$$

THALES

Optimal Transport Theory & Gradient Flow

THALES

Wasserstein Distance for Multivariate Gaussian Laws

- **Fréchet-Wasserstein distance**

$$E\left[\|X-Y\|^2\right]=E\left[Tr\left((X-Y)^+(X-Y)\right)\right]$$

$$E\left[\|X-Y\|^2\right]=|E(X)-E(Y)|^2+Tr[R_X]+Tr[R_Y]-2Tr[R_{X,Y}]$$

- **Proof**
  - If we set $W=\begin{bmatrix} X \\ Y \end{bmatrix} \Rightarrow R_W=\begin{bmatrix} R_X & \Psi \\ \Psi^+ & R_Y \end{bmatrix} \geq 0$

$$\sup_{R_X-\Psi R_Y^{-1}\Psi^+\geq 0} Tr\left(\Psi+\Psi^+\right) \Rightarrow \Psi=R_X^{1/2}\left(R_X^{1/2}.R_Y.R_X^{1/2}\right)^{1/2}R_X^{-1/2}$$

  - Solution is given by:

$$Tr\left(\Psi+\Psi^+\right)=2.Tr\left[\left(R_X^{1/2}.R_Y.R_X^{1/2}\right)^{1/2}\right]$$

$$Y=R_X^{1/2}\left(R_X^{1/2}.R_Y.R_X^{1/2}\right)^{-1/2}R_X^{1/2}.X$$

THALES

Wasserstein Distance for Multivariate Gaussian Laws

- **Fréchet-Wasserstein distance**

$$d^2((m_X,R_X),(m_Y,R_Y)) = |m_X - m_Y|^2 + Tr[R_X] + Tr[R_Y] - 2.Tr\left[\left(R_X^{1/2} R_Y R_X^{1/2}\right)^{1/2}\right]$$

- **Geodesic**
  - If we set : $D_{X,Y} = R_X^{1/2}\left(R_X^{1/2} R_Y R_X^{1/2}\right)^{-1/2} R_X^{1/2} = R_X \circ R_Y^{-1}$

$$\begin{cases} m_{(t)} = (1-t)m_Y + t.m_X \\ R_{(t)} = \left((1-t)I_k + t.D_{X,Y}\right)R_Y\left((1-t)I_k + t.D_{X,Y}\right) \end{cases}$$

$$W_2\left[(m_{(s)},R_{(s)}),(m_{(t)},R_{(t)})\right] \leq (t-s).W_2\left[(m_{(0)},R_{(0)}),(m_{(1)},R_{(1)})\right]$$

- **Optimal Transport :**
  - $(I_k, \nabla\psi)_{\#} N(m_Y,R_Y)$ transport from $N(m_Y,R_Y)$ to $N(m_X,R_X)$

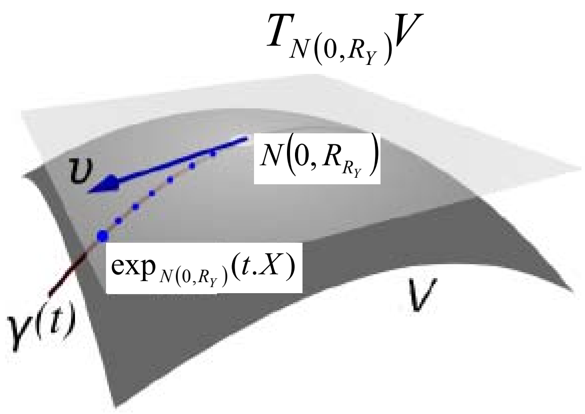
$$\psi(v) = \frac{1}{2}(v-m_Y)^{+} D_{X,Y}(v-m_Y) + (m_X - v)$$

$$x = \nabla\psi(y) = D_{X,Y}(y-m_Y) + m_X$$

$$(x-m_x)^{+} R_X^{-1}(x-m_X) = (y-m_Y)^{+} R_Y^{-1}(y-m_Y)$$

Optimal Transport : Fréchet-Wasserstein Distance

Characteristics of this space (case m=0)



- **Wasserstein metric :**

$$g_{N(0,R_Y)}(X,Y) = Tr\left(X.R_Y.Y\right)$$

- **Tangent Space and Exponential Map :**

$$\exp_{N(0,R_Y)}(t.X) = N\left(0, R_{R_Y,(t)}\right)$$

$$R_{R_Y,(t)} = \left((1-t)I_k + t.X\right)R_Y\left((1-t)I_k + t.X\right)$$

Properties of this space

- **Alexandrov space**

$$d\left(\alpha,\gamma(t)\right)^2 \geq (1-t).d\left(\alpha,\gamma(0)\right)^2 + t.d\left(\alpha,\gamma(t)\right)^2 - t(1-t).d\left(\gamma(0),\gamma(1)\right)^2$$

- **space Positive Sectional Curvature (geodesiquely convexe and simply connected)**

Optimal Transport : Fréchet-Wasserstein Barycenter

Wasserstein Barycenter for Multivariate Gaussian Laws (Case m=0)

- **Fréchet-Wasserstein Barycenter**

$$\underset{\mu}{Inf} \sum_{k=1}^{N} W_2^2(\mu, \nu_k) \quad \text{with} \quad W_2(\mu,\nu) = \underset{X,Y}{Inf}\, E\left[\left\|X - Y\right|^2\right]$$

- **Solution of Fréchet-Wasserstein Barycenter for N Multivariate Gaussian Laws of zero meams** $\left\{N\left(0, R_k\right)\right\}_{k=1}^{N}$
  - Constraint :

$$\sum_{k=1}^{N} \left(R^{1/2} R_k R^{1/2}\right)^{1/2} = R$$

- **Iterative Solution**
  - Iterative solution (convergence for d=2, d>3 convergence conditionaly to the initiation)

$$K^{(n+1)} = \left(\sum_{k=1}^{N} \left(K^{(n)} K_k^2 K^{(n)}\right)^{1/2}\right)^{1/2} \quad \text{with} \quad K_i = R_i^{1/2}$$

THALES

Stationnary Signal & Toeplitz Constraint on Covariance Matrix : Gradient Flow through Partial Iwasawa & CAR Model

Additional Constraint : Toeplitz Structure

- **Covariances matrices are** structured matrices **that verify the following constraints :**
  - Toeplitz Structure (for stationary signal) : $\forall n,\quad E\left[z_n z_{n-k}^*\right] = r_k$

$$R_n = \begin{bmatrix} r_0 & r_1^* & r_2^* & \cdots & r_{n-1}^* \\ r_1 & r_0 & r_1^* & \ddots & \vdots \\ r_2 & r_1 & \ddots & \ddots & r_2^* \\ \vdots & \ddots & \ddots & r_0 & r_1^* \\ r_{n-1} & \cdots & r_2 & r_1 & r_0 \end{bmatrix} = \left[\begin{array}{cccc|c} & & & & r_{n-1}^* \\ & R_{n-1} & & & \vdots \\ & & & & r_2^* \\ & & & & r_1^* \\ \hline r_{n-1} & \cdots & r_2 & r_1 & r_0 \end{array}\right]$$

  - Hermitian structure :

$$R_n^+ = R_n \quad \text{with} \quad + : \text{transposed} \quad \text{and} \quad \text{conjuguate}$$

  - Positive Definite Structure :

$$\forall Z \in C^n, Z^+ R_n Z > 0, \forall \lambda \text{ such that } \det\left(R_n - \lambda . I_n\right) = 0 \Rightarrow \lambda > 0$$

- **How to built a flow that preserves the Toeplitz structure ?**

If you remember first example on « Right Triangle », in this case Pythagore constraint is replaced by HPD & Toeplitz constraints

Preservation of Toeplitz Structure

We use Partial Iwasawa Decomposition = Complex AR model

$$z_n = -\sum_{k=1}^{N} a_k^{(N)} z_{n-k} + b_n \quad \text{with} \quad E\left[b_n b_{n-k}^*\right] = \delta_{k,0}\sigma^2 \quad \text{and} \quad A_N = \left[a_1^{(N)} \quad \cdots \quad a_N^{(N)}\right]$$

$$\Omega_n = (\alpha_n . R_n)^{-1} = W_n . W_n^+ = \left(1 - |\mu_n|^2\right) \begin{bmatrix} 1 & A_{n-1}^+ \\ A_{n-1} & \Omega_{n-1} + A_{n-1} . A_{n-1}^+ \end{bmatrix}$$

K. Iwasawa

$$\alpha_n^{-1} = \left[1 - |\mu_n|^2\right] \alpha_{n-1}^{-1} \qquad A_n = \begin{bmatrix} A_{n-1} \\ 0 \end{bmatrix} + \mu_n \begin{bmatrix} A_{n-1}^{(-)} \\ 1 \end{bmatrix} \qquad V^{(-)} = J.V^* \qquad P_0 = \alpha_0^{-1}$$

- **Information Geometry metric (metric = Hessian of Entropy)**

$$g_{ij} \equiv \frac{\partial^2 \tilde{\Phi}}{\partial \theta_i^{(n)} \partial \theta_j^{(n)*}} \quad \textbf{with} \quad \theta^{(n)} = \left[P_0 \quad \mu_1 \quad \cdots \quad \mu_{n-1}\right]^T$$

μ_k : reflection coefficient with $|\mu_k| < 1$

E. Kähler

$$\tilde{\Phi}(R_n, P_0) = \log\left(\det R_n^{-1}\right) - \log(\pi.e) = -\sum_{k=1} (n-k).\ln\left[1 - |\mu_k|^2\right] - n.\ln\left[\pi.e.P_0\right]$$

$$\boxed{ds_n^2 = d\theta^{(n)+}\left[g_{ij}\right]d\theta^{(n)} = n.\left(\frac{dP_0}{P_0}\right)^2 + \sum_{i=1}^{n-1}(n-i)\frac{|d\mu_i|^2}{\left(1 - |\mu_i|^2\right)^2}}$$

S. Bergman

Complex Autoregressive Model : Regularized Burg Algorithm

Regularized Burg Algorithm (THALES Patent)

. Initialisation :

$$\mathrm{f}_0(k) = b_0(k) = z(k) \quad , \quad k{=}1,\ldots,N \quad (\mathrm{N : nb.\ ech.})$$

$$P_0 = \frac{1}{N}.\sum_{k=1}^{N} \left| z(k) \right|^2$$

$$a_0^{(0)} = 1$$

. Step (n) : For $n = 1$ to M

$$\mu_n = -\frac{\dfrac{2}{N-n}\displaystyle\sum_{k=n+1}^{N} f_{n-1}(k).b_{n-1}^{*}(k-1) + 2.\sum_{k=1}^{n-1} \beta_k^{(n)}.a_k^{(n-1)}.a_{n-k}^{(n-1)}}{\dfrac{1}{N-n}\displaystyle\sum_{k=n+1}^{N} \left| f_{n-1}(k) \right|^2 + \left| b_{n-1}(k-1) \right|^2 + 2.\sum_{k=0}^{n-1} \beta_k^{(n)}.\left| a_k^{(n-1)} \right|^2} \quad \text{with} \quad \beta_k^{(n)} = \gamma_1.(2\pi)^2.(k-n)^2$$

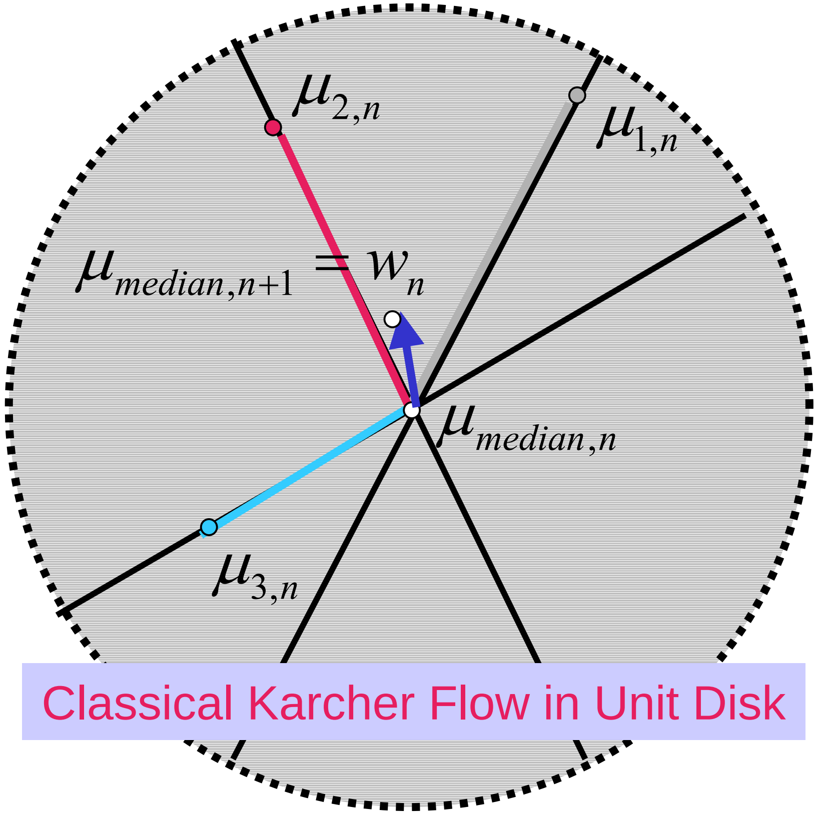
$$\begin{cases} a_0^{(n)} = 1 \\ a_k^{(n)} = a_k^{(n-1)} + \mu_n.a_{n-k}^{(n-1)*} \quad , \quad k{=}1,\ldots,n\text{-}1 \\ a_n^{(n)} = \mu_n \end{cases}$$

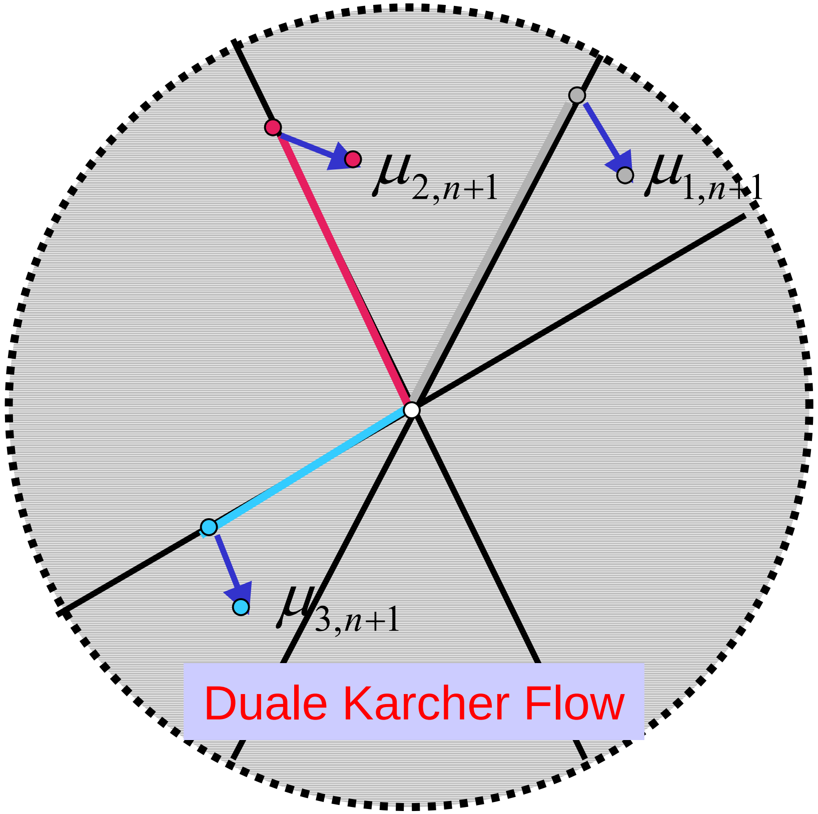
$$\begin{cases} f_n(k) = f_{n-1}(k) + \mu_n.b_{n-1}(k-1) \\ b_n(k) = b_{n-1}(k-1) + \mu_n^{*}.f_{n-1}(k) \end{cases}$$

Median AR model Through Median Reflection Coefficients μ_k

$$w_n = \gamma_n \sum_{\substack{k=1 \\ k \neq l}}^{m} \frac{\mu_{k,n}}{\left|\mu_{k,n}\right|} \quad avec \quad \left\{ l / \left|\mu_{l,n}\right| < \varepsilon \right\}$$

$$\mu_{k,n+1} = \frac{\mu_{k,n} - w_n}{1 - \mu_{k,n} . w_n^*}$$

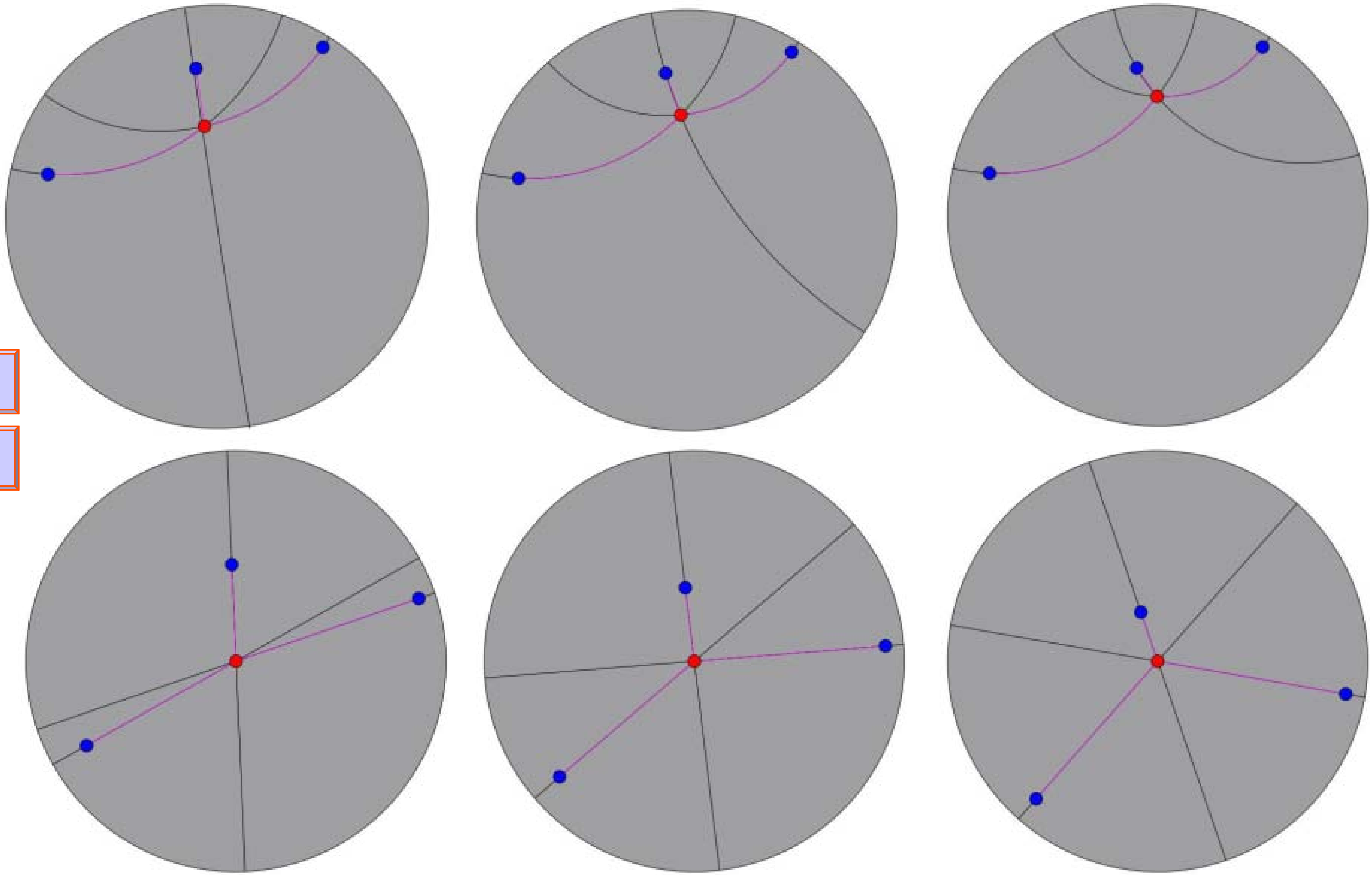




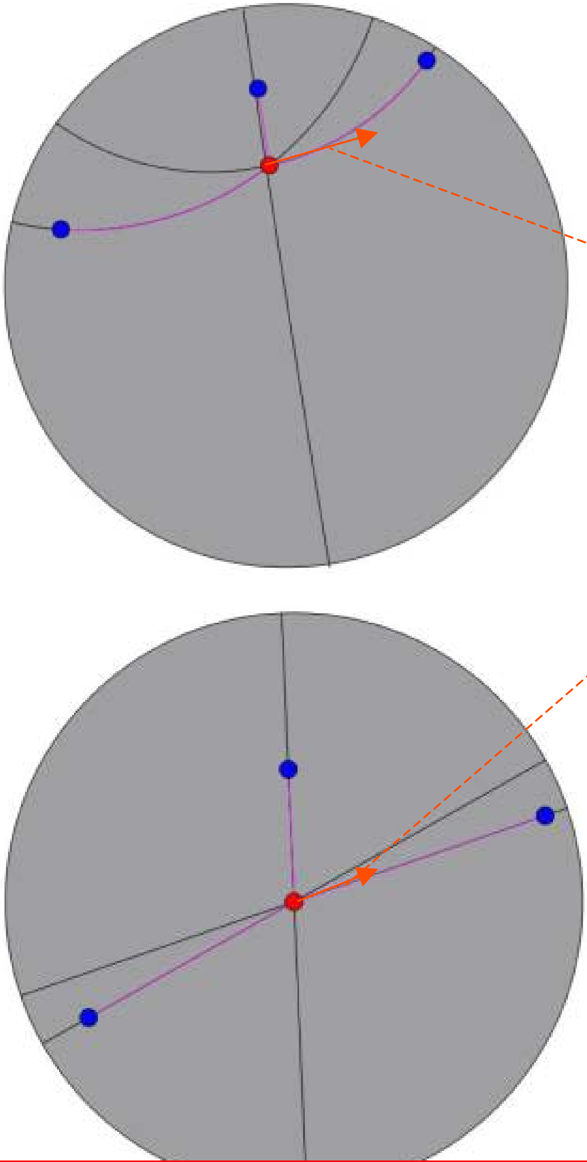
$$\mu_{median,n+1} = \frac{\mu_{median,n} + w_n}{1 + \mu_{median,n} w_n^*}$$

THALES

Median AR model Through Median Reflection Coefficients μ_k



Alternative to Karcher Flow : Arnaudon Stochastic Flow



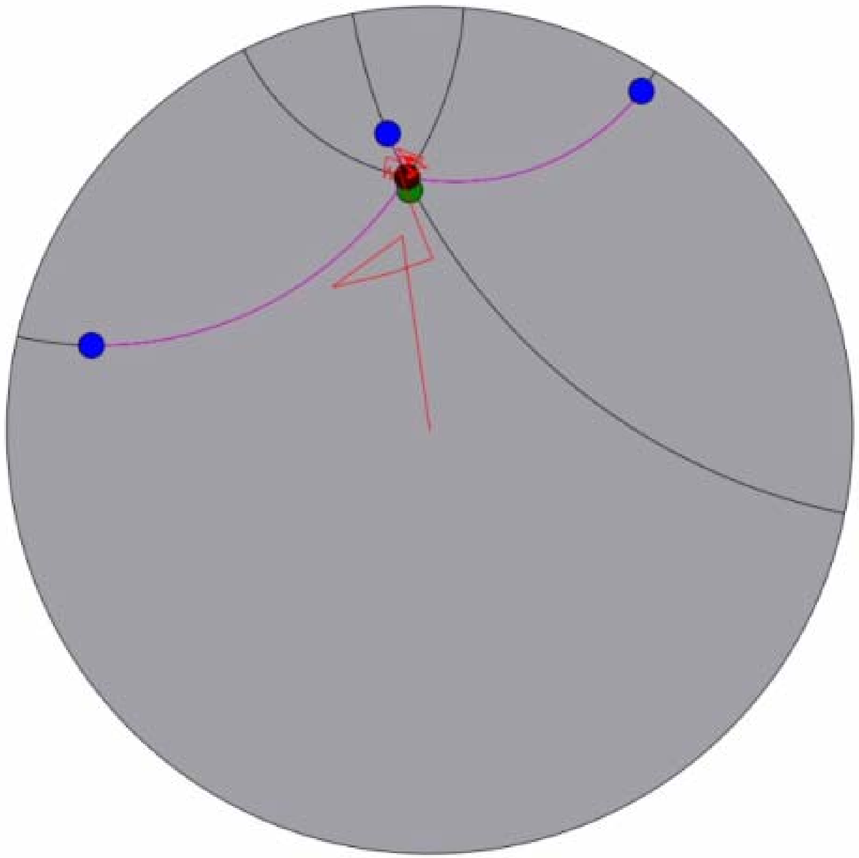
Random selection at each iteration of 1 point in the disk + evolution along the geodesic

$$w_n = \gamma_n \frac{\mu_{rand(n),n}}{\left|\mu_{rand(n),n}\right|}$$

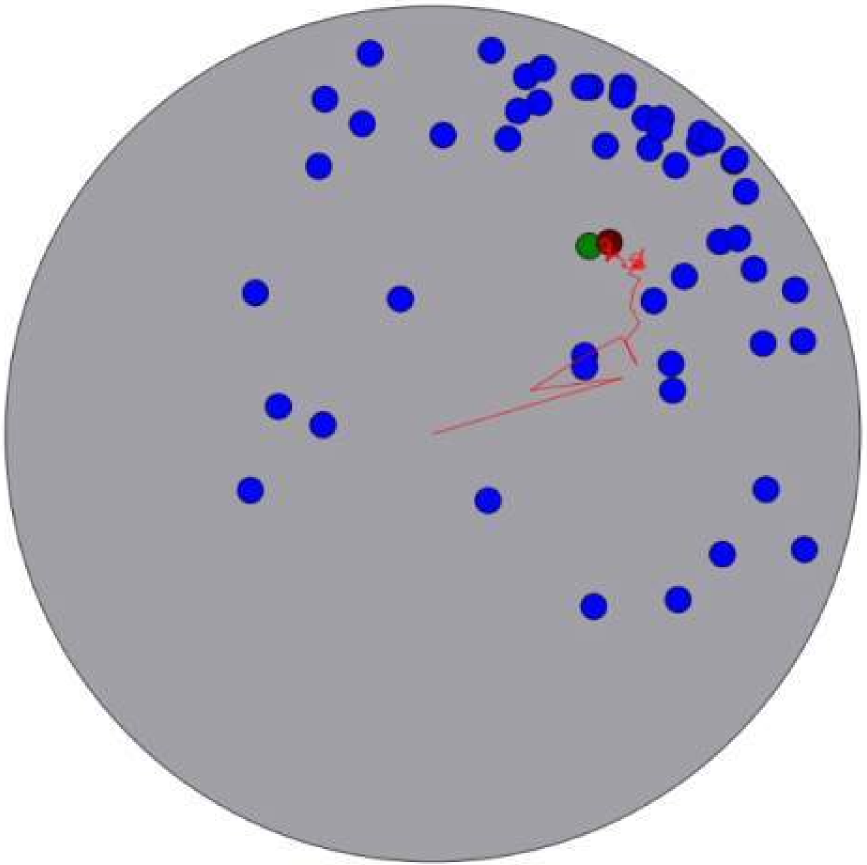
$$m_{n+1} = \exp_{m_n}\left(t_n \cdot \frac{\exp_{m_n}^{-1}\left(x_{rand(n)}\right)}{\left\|\exp_{m_n}^{-1}\left(x_{rand(n)}\right)\right\|} \right)$$

M. Arnaudon, C. Dombry, A.Phan, L.Yang, "Stochastic algorithms for computing means of probability measures", http://hal.archives-ouvertes.fr/hal-00540623

Arnaudon Stochastic Flow

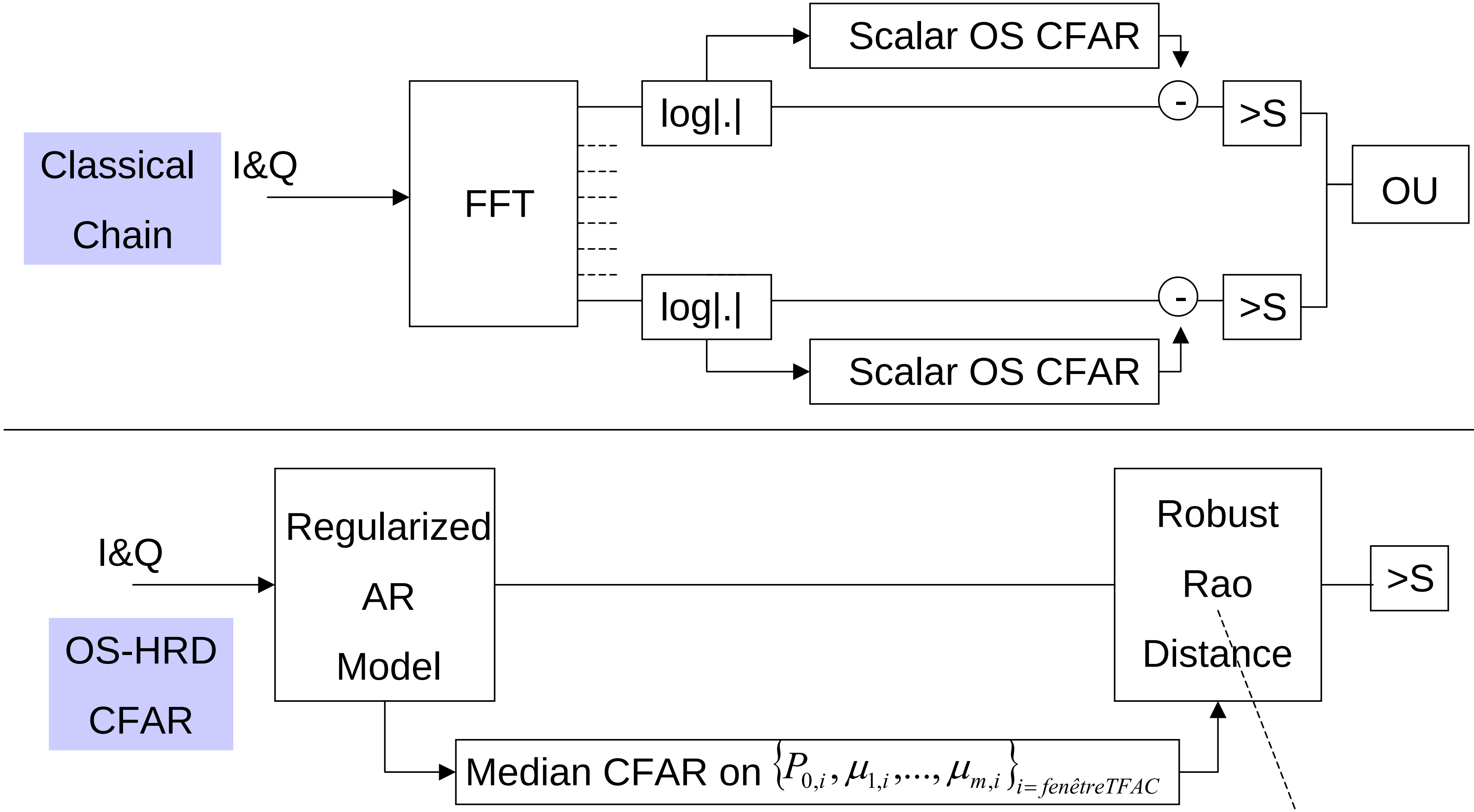


$$w_n = \gamma_n \frac{\mu_{rand(n),n}}{\left|\mu_{rand(n),n}\right|}$$



$$m_{n+1} = \exp_{m_n}\left(t_n \cdot \frac{\exp_{m_n}^{-1}\left(x_{rand(n)}\right)}{\left\|\exp_{m_n}^{-1}\left(x_{rand(n)}\right)\right\|} \right)$$

OS-HR-Doppler-CFAR



$$dist\left(\{P_{0,i}, \mu_{1,i}, \ldots, \mu_{m,i}\}, \{P_{0,median}, \mu_{1,median}, \ldots, \mu_{m,median}\}\right) = m \cdot \left[\log \frac{P_{0,median}}{P_{0,j}}\right]^2 + \sum_{n=1}^{m} (m-n) \left[\arg th\left(\left|\frac{\mu_{n,j} - \mu_{n,median}}{1 - [\mu_{n,j} . \mu_{n,median}^{*}]}\right|\right)\right]^2$$

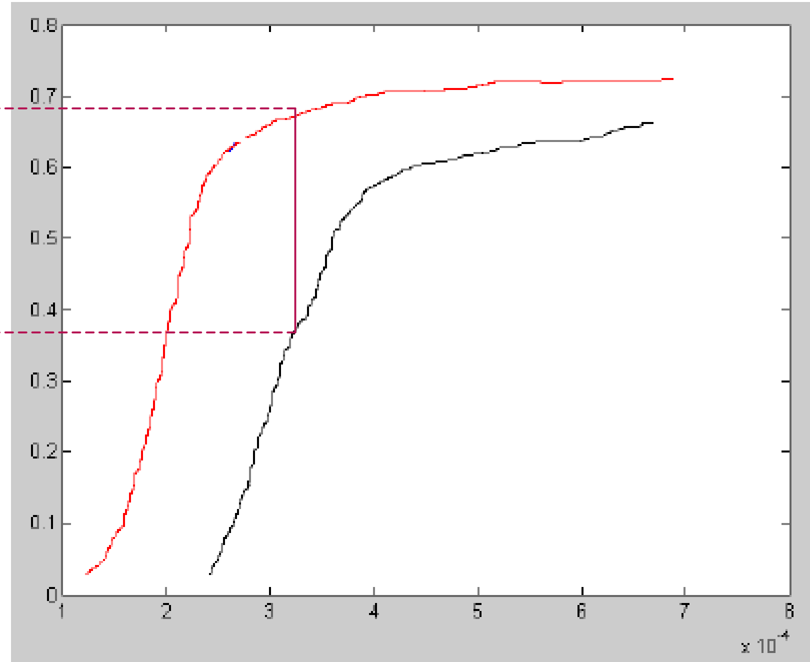
Figures of Merit (sea clutter)

- COR curve

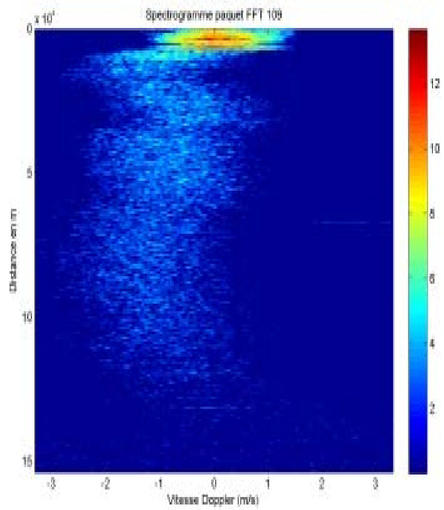
Gain of

New processing

chain



First tests on real recorded sea clutter



Red : OS-HR-DOPPLER-CFAR based on CAR median

Black : Classical Doppler Filter & OS-CFAR

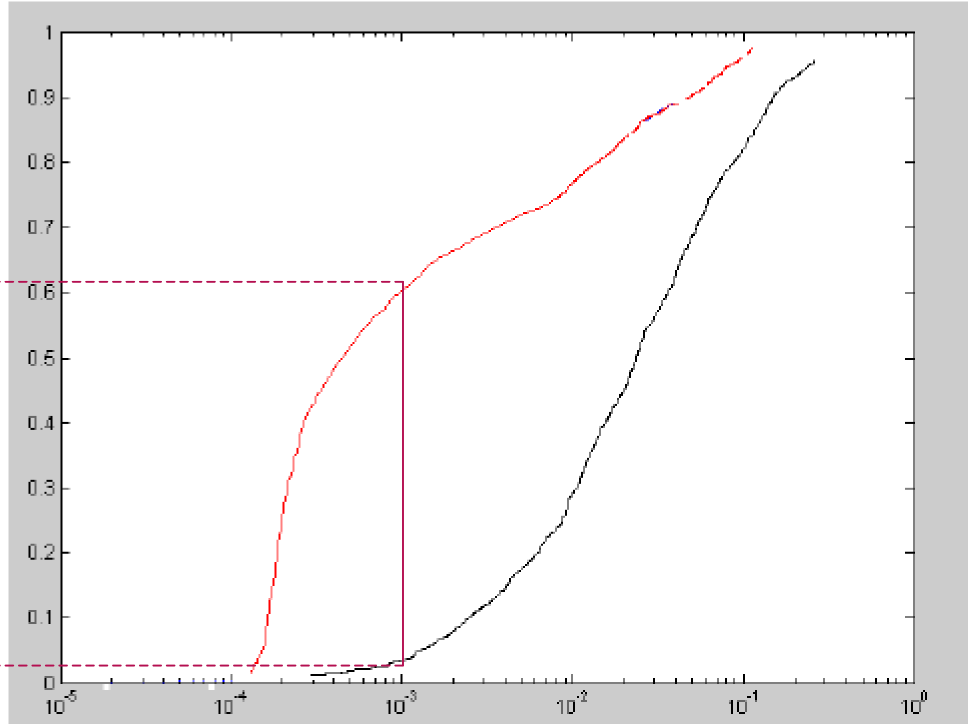
Figure of Merit for Closely Separated Targets (sea clutter)

- COR Curves

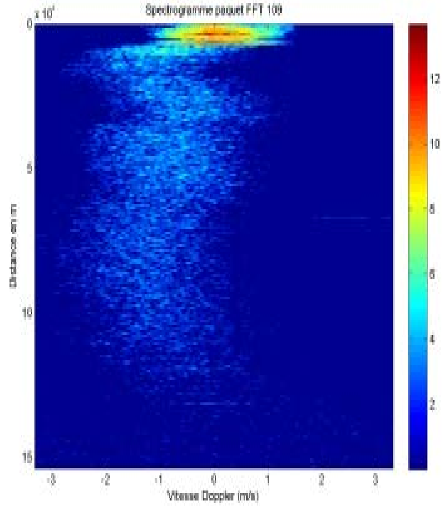
Gain of

New processing

Chain



First tests on real recorded sea clutter

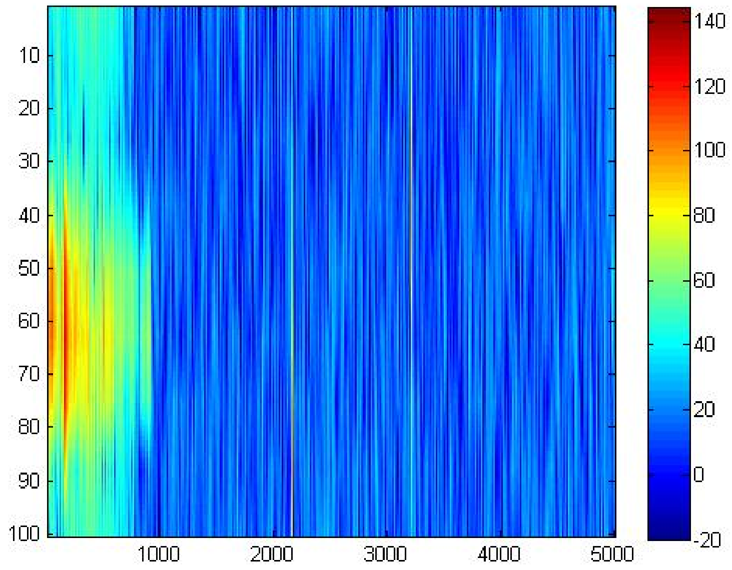


Red : OS-HR-DOPPLER-CFAR based on CAR median

Black : Classical Doppler Filter & OS-CFAR

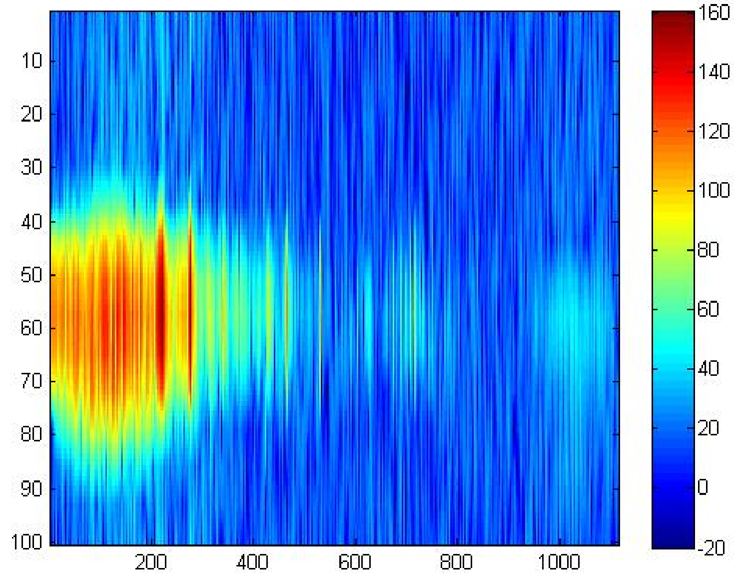
In Progress: tests on real Air Defense Radar Ground Clutter records

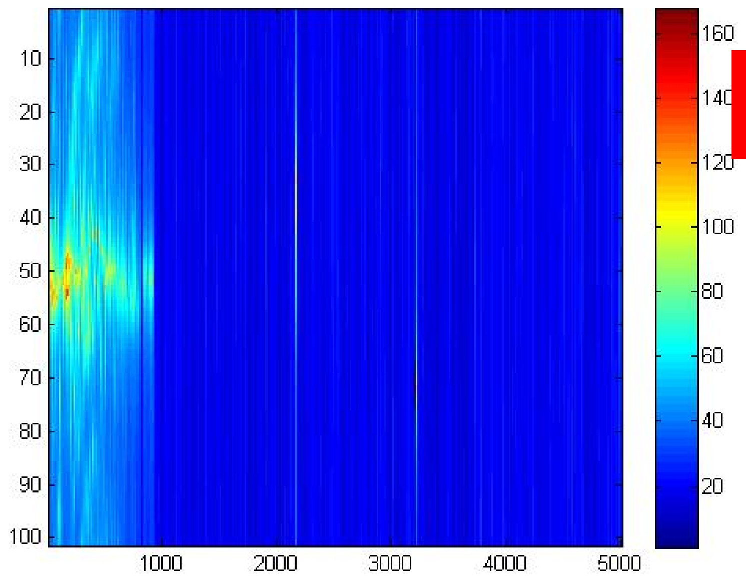
Waveform 1



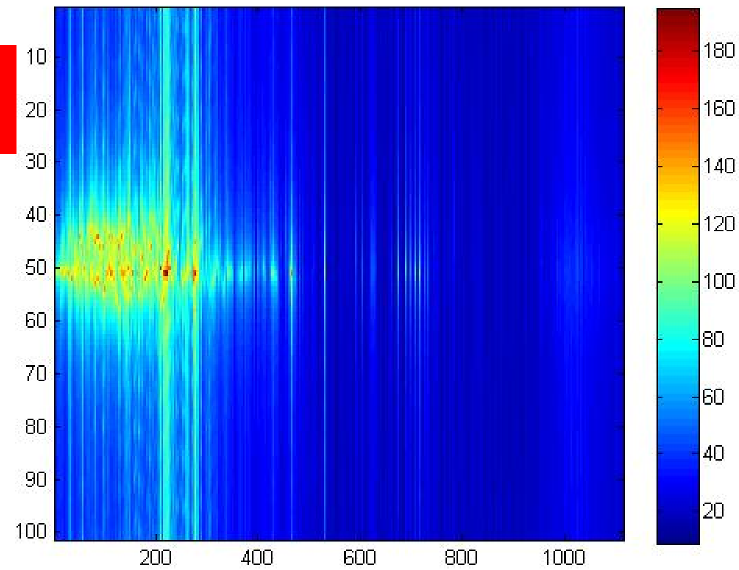
Doppler Filters

Waveform 2





High Resolution Doppler Filters



ROBUSTNESS of OS-HRD-CFAR

Robustness is given by :

- Metric & Distance between Doppler Spectrums take into account variances of estimation (Rao's metric of Information Geometry is given by « Fisher Information Matrix » used in Cramer-Rao Bound)
- HR Doppler estimation is « Regularized » (Regularized Burg Algorithm) because with short bursts, we cannot estimate AR model order or dimension of « signal space » (e.g. MUSIC)
- Detector is not based on whitening filters (sub-optimal because filter weights have high variances with short bursts) but on robust distance
- Median is a basic tool of Robust statistic : low sensitivity to outliers (Fréchet has studies main advantages of "Median" compared to "Mean")

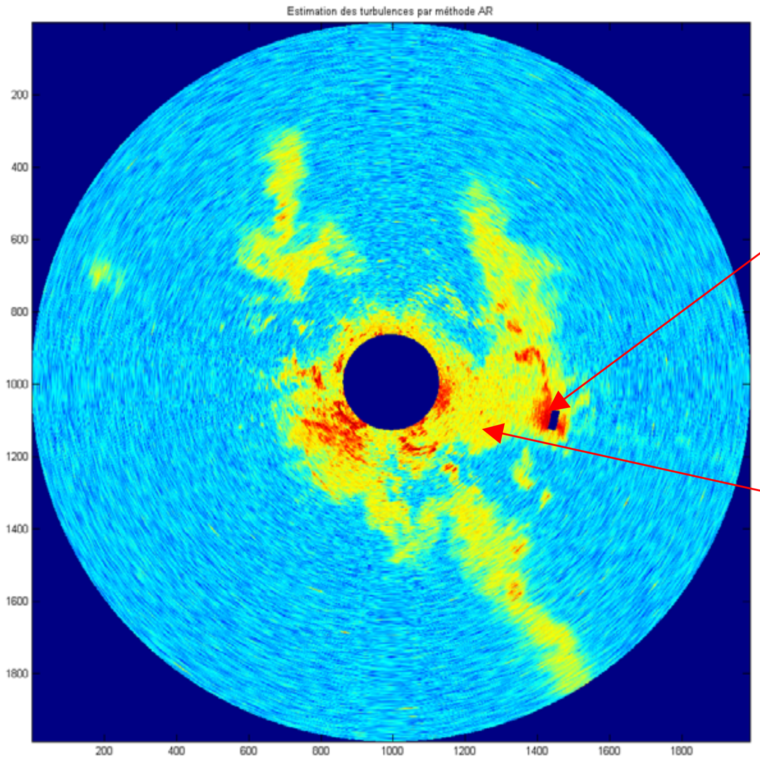
Jointly

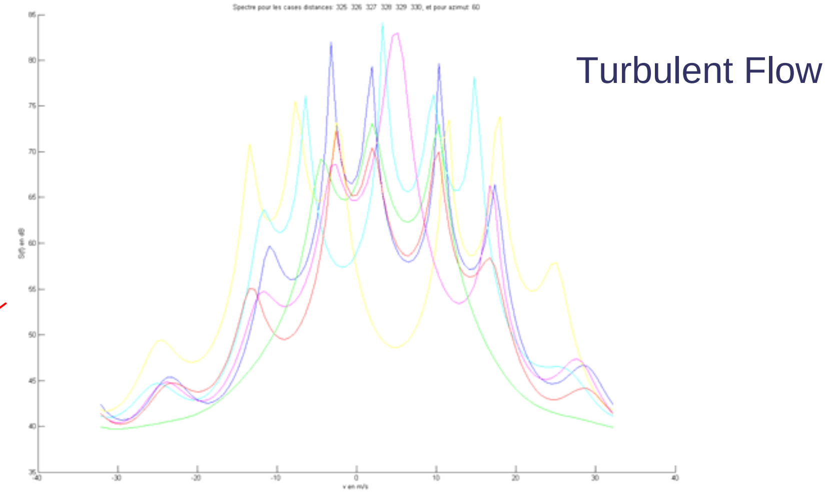
Robust Doppler Estimation ⇔ Robust Distance ⇔ Robust Statistics ⇔ Robust Detector

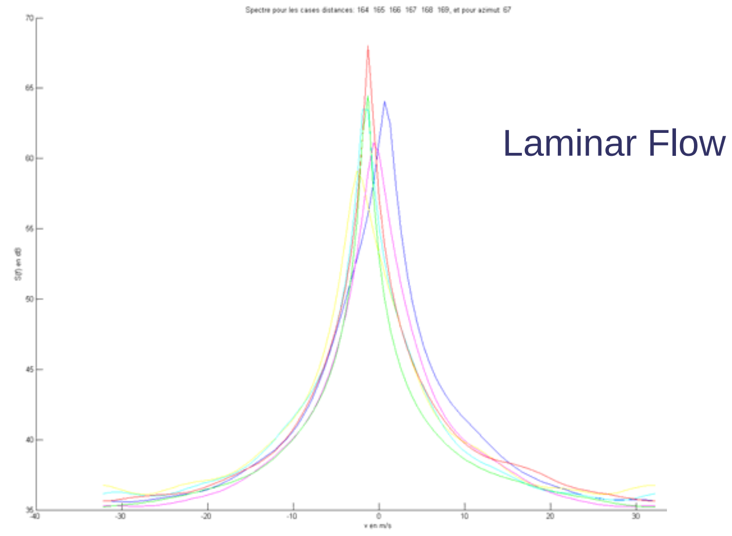
Information Geometry for turbulences monitoring

Monitoring of Atmospheric Turbulences

$$S = \sum_{k=2}^{n-1} (n-k)\left(\frac{1}{2}\ln\left(\frac{1+|\mu_k|}{1-|\mu_k|}\right)\right)^2$$





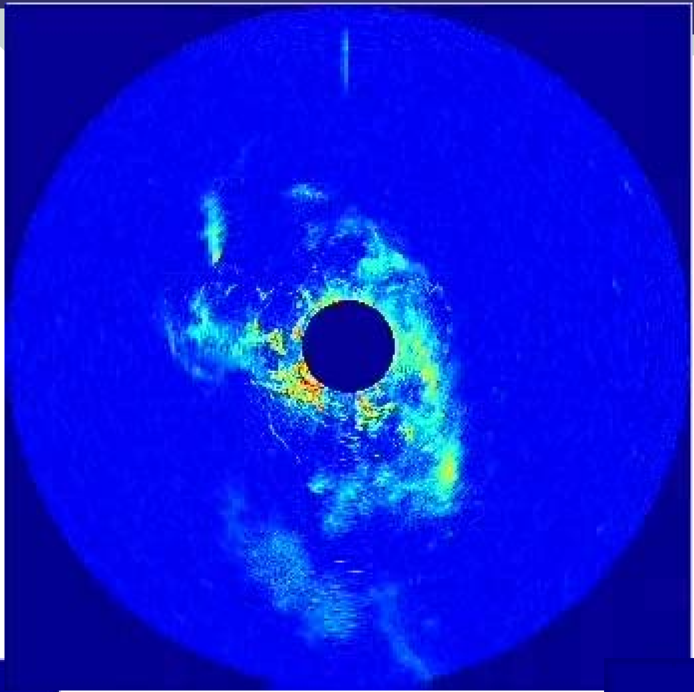


Distance between AR model of order 1 and Regularized AR model of maximum order

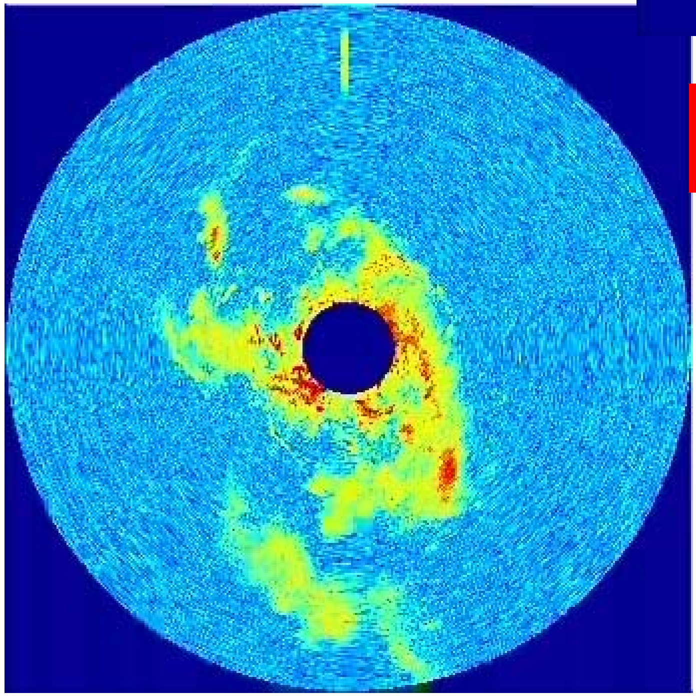
THALES

Upgrade of STAR2000 ATC Radar Weather Channel

STAR2000 (I&Q)

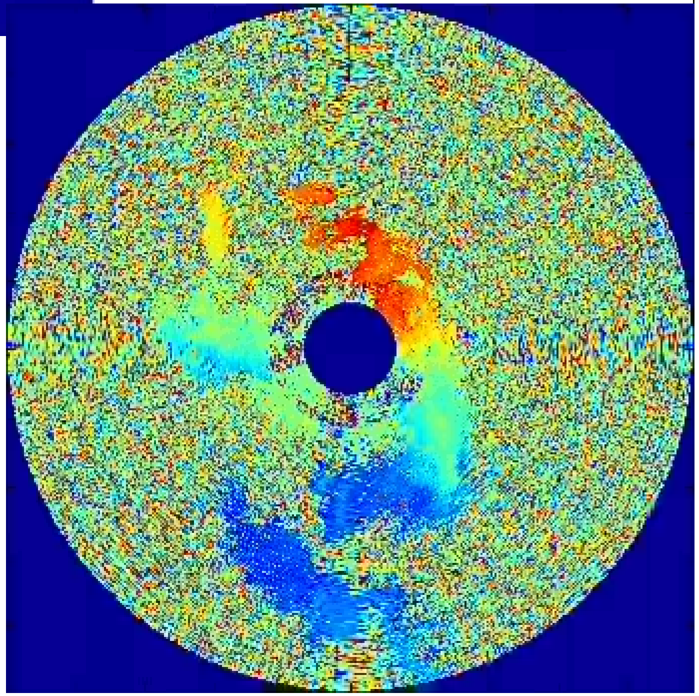


Rain Rate

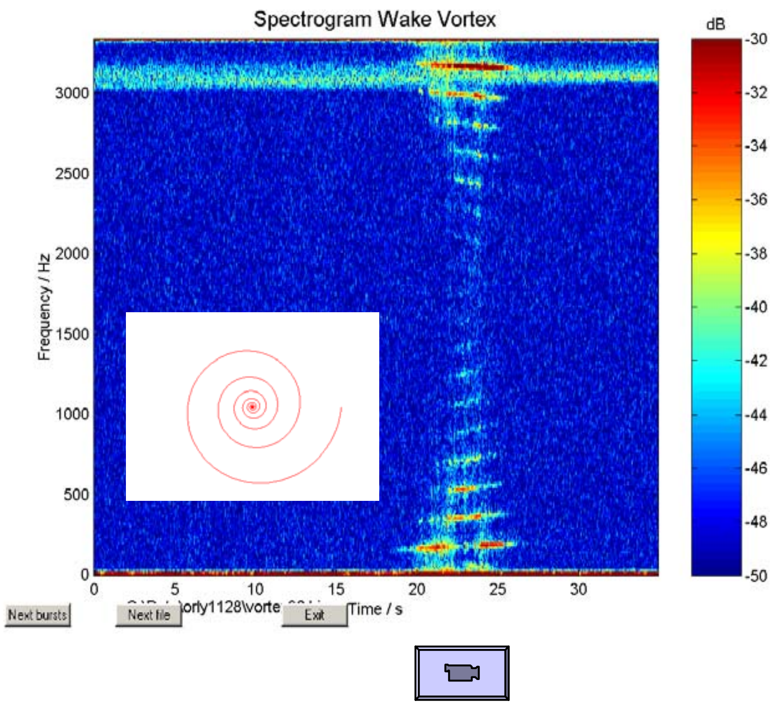


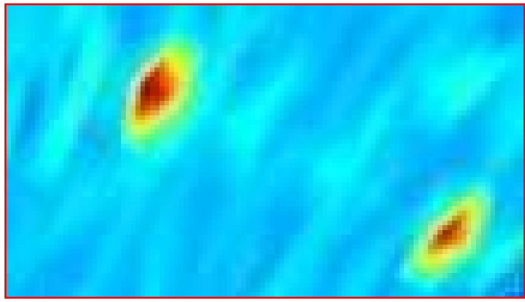
Atmospheric Turbulence

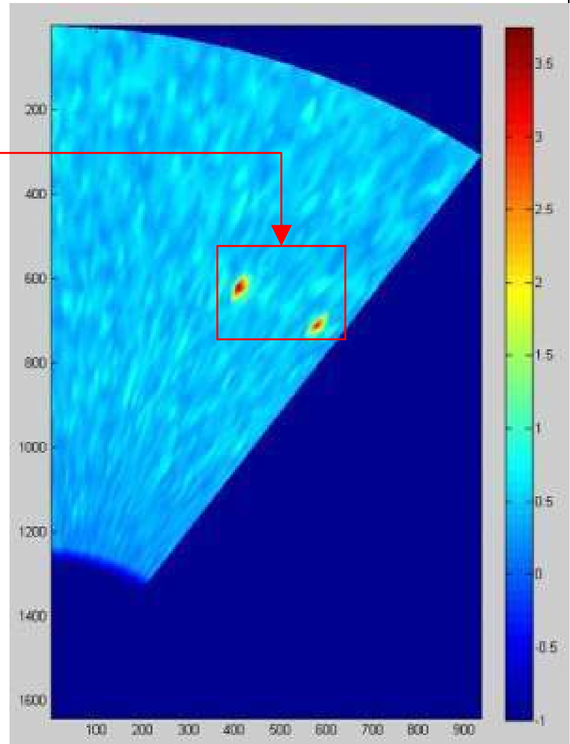
Wind Radial Speed



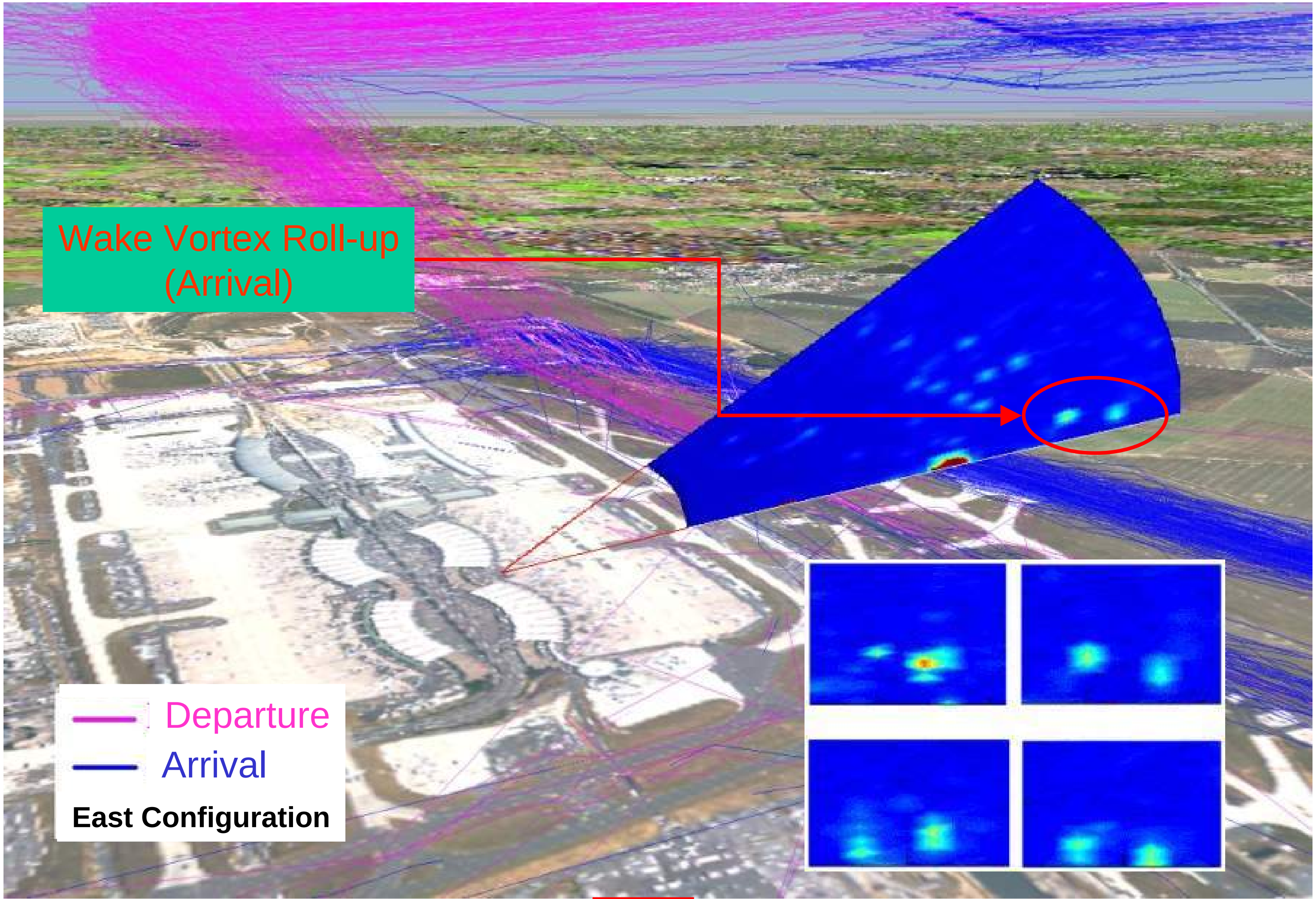
Wake-Vortex Monitoring



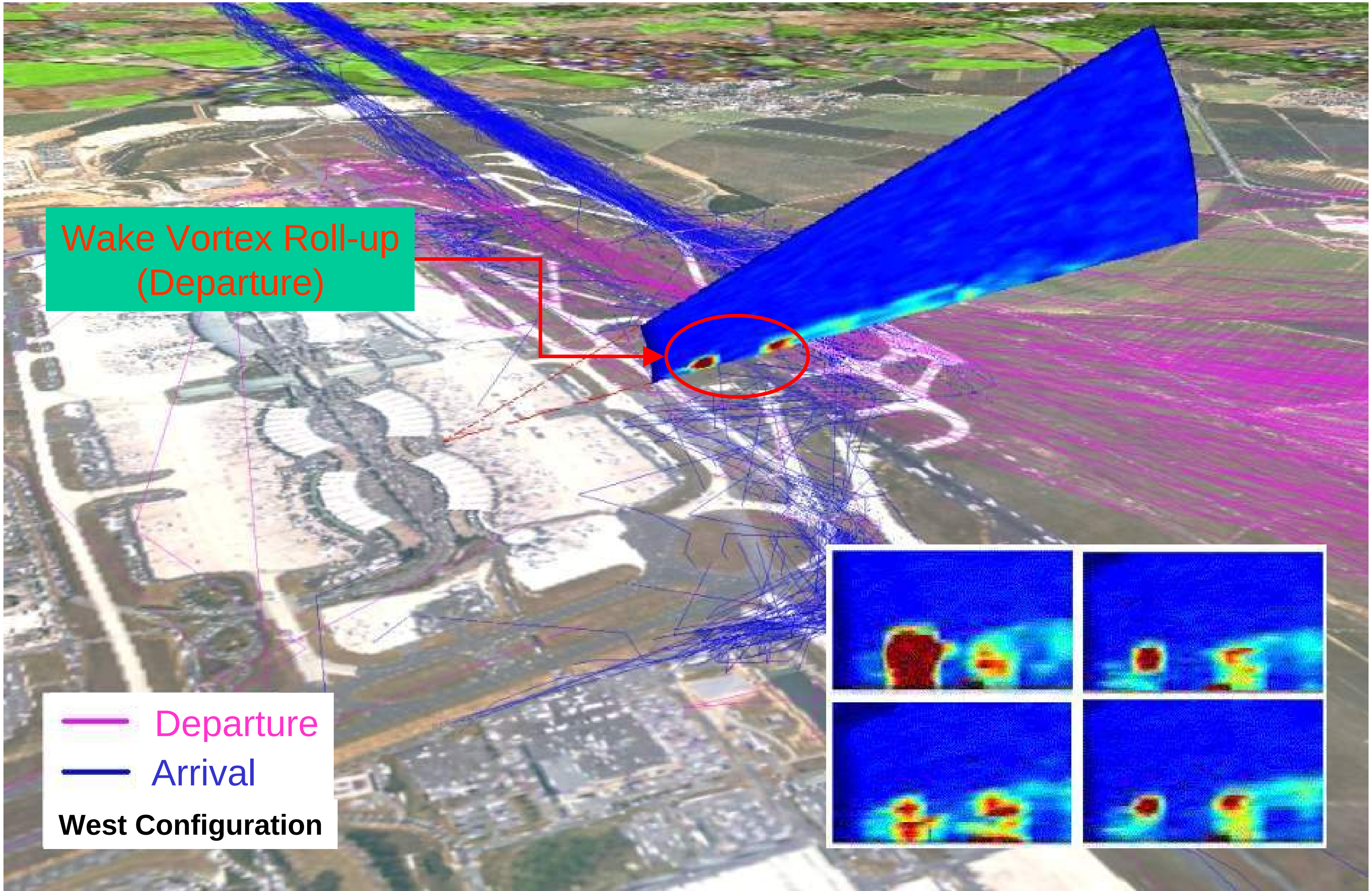




Wake-Vortex Monitoring : Arrival

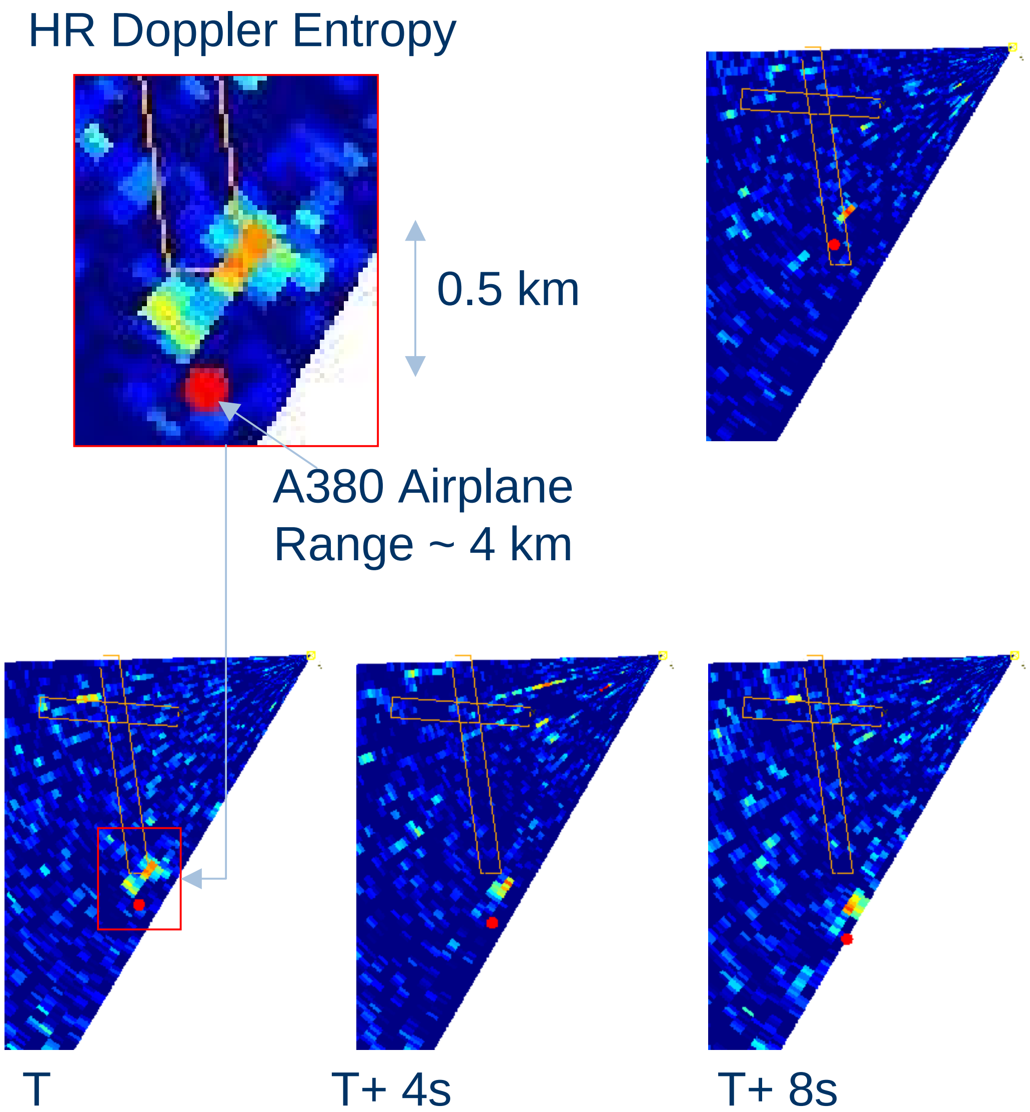


Wake-Vortex Monitoring : Departure



THALES

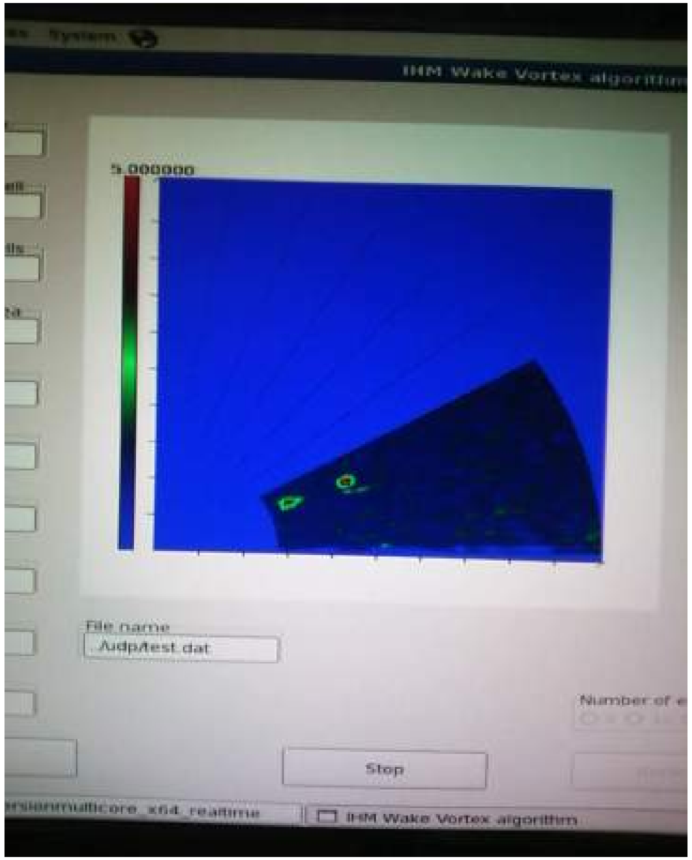
Opportunity Trials : ATC PSR Radar records of A380

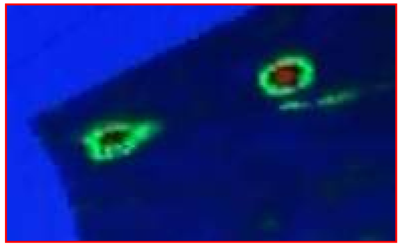


BOR-A Installation : 06/05/2011

THALES

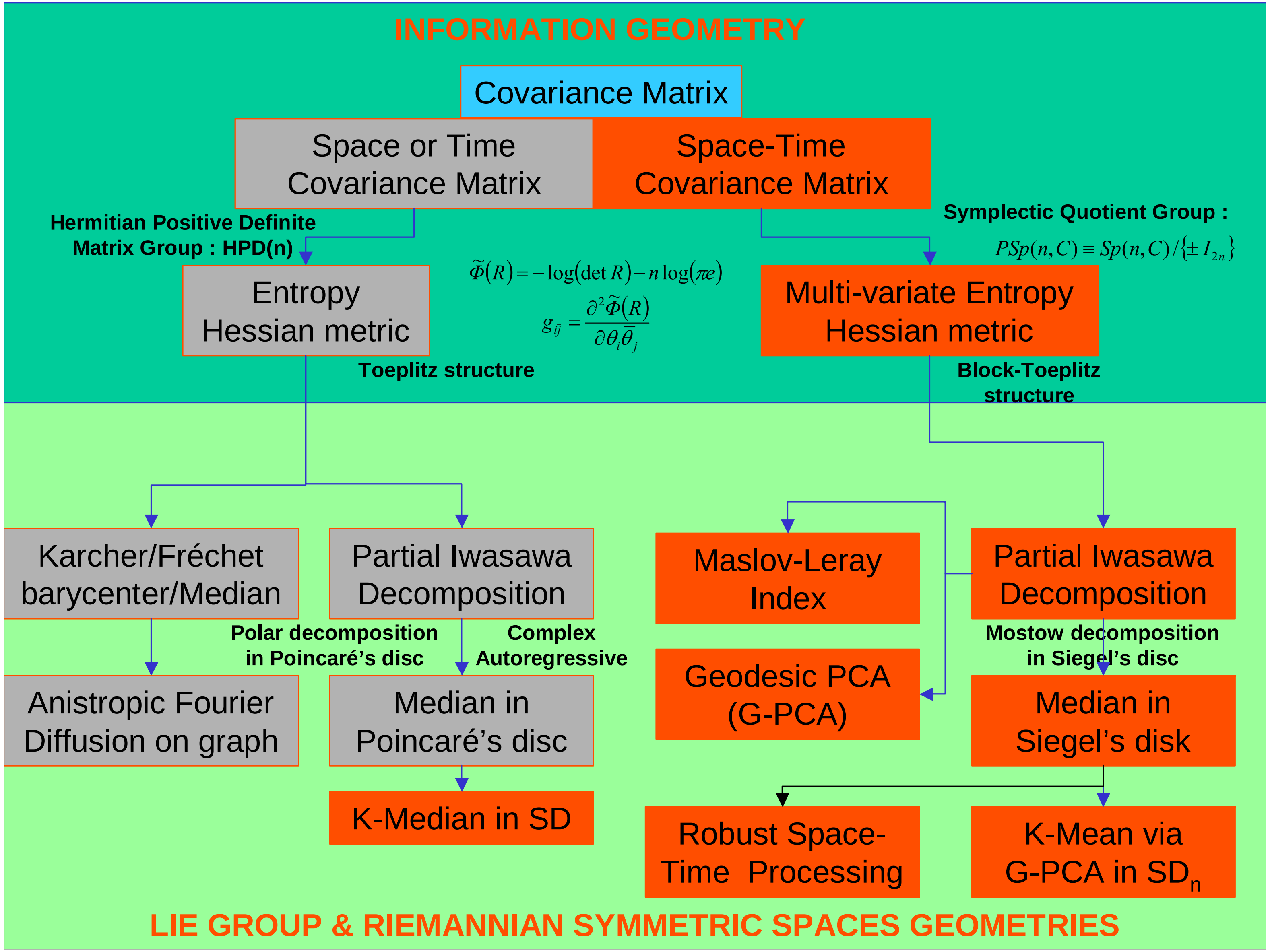
BOR-A First tests





SPACE-TIME COVARIANCE MATRIX MEDIAN COMPUTATION: STAP PROCESSING

Extension to STAP : Toeplitz-Block-Toeplitz Matrices



Extension to STAP : Toeplitz-Block-Toeplitz Matrices

Previous results can be extented to Block-Toeplitz Matrices :

$$R_{p,n+1} = \begin{bmatrix} R_0 & R_1 & \cdots & R_n \\ R_1^+ & R_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & R_1 \\ R_n^+ & \cdots & R_1^+ & R_0 \end{bmatrix} = \begin{bmatrix} R_{p,n} & \widetilde{R}_n \\ \widetilde{R}_n^+ & R_0 \end{bmatrix}$$

$$\widetilde{R}_n = V \begin{bmatrix} R_1 \\ \vdots \\ R_n \end{bmatrix}^* \quad \text{with} \quad V = \begin{bmatrix} 0 & \cdots & 0 & J_p \\ \vdots & \dots & \dots & 0 \\ 0 & J_p & \dots & \vdots \\ J_p & 0 & \cdots & 0 \end{bmatrix}$$

THALES

Extension to STAP : Toeplitz-Block-Toeplitz Matrices

From Burg-like parameterization, we can deduced this inversion of Toeplitz-Block-Toeplitz matrix :

$$R_{p,n+1}^{-1} = \begin{bmatrix} \alpha_n & \alpha_n . \widehat{A}_n^{+} \\ \alpha_n . \widehat{A}_n & R_{p,n}^{-1} + \alpha_n . \widehat{A}_n . \widehat{A}_n^{+} \end{bmatrix}$$

$$R_{p,n+1} = \begin{bmatrix} \alpha_n^{-1} + \widehat{A}_n^{+} . R_{p,n} . \widehat{A}_n & -\widehat{A}_n^{+} . R_{p,n} \\ -R_{p,n} . \widehat{A}_n & R_{p,n} \end{bmatrix}$$

$$\text{with} \quad \alpha_n^{-1} = \left[1 - A_n^n A_n^{n+}\right] . \alpha_{n-1}^{-1}, \quad \alpha_0^{-1} = R_0$$

$$\text{and} \quad \widehat{A}_n = \begin{bmatrix} A_1^1 \\ \vdots \\ A_n^n \end{bmatrix} = \begin{bmatrix} \widehat{A}_{n-1} \\ 0_p \end{bmatrix} + A_n^n . \begin{bmatrix} J_p A_{n-1}^{n-1*} J_p \\ \vdots \\ J_p A_1^{n-1*} J_p \\ I_p \end{bmatrix}$$

THALES

Extension to STAP : Toeplitz-Block-Toeplitz Matrices

Kähler potential defined by Hessian of multi-channel/Multi-variate entropy :

$$\tilde{\Phi}\left(R_{p,n}\right) = -\log\left(\det R_{p,n}\right) + cste = -Tr\left(\log R_{p,n}\right)\mu + cste$$

$$\Rightarrow g_{i\bar{j}} = Hess\left[\phi\left(R_{p,n}\right)\right]$$

$$\tilde{\Phi}\left(R_{p,n}\right) = \sum_{k=1}^{n-1}(n-k).\log\det\left[I_n - A_k^k A_k^{k+}\right] + n.\log\left[\pi.e.\det R_0\right]$$

$$ds^2 = n.Tr\left[\left(R_0^{-1}dR_0\right)^2\right] + \sum_{k=1}^{n-1}(n-k)Tr\left[\left(I_n - A_k^k A_k^{k+}\right)^{-1} dA_k^k \left(I_n - A_k^{k+} A_k^k\right)^{-1} dA_k^{k+}\right]$$

THALES

Multi-Channel Burg Algorithm :

$$\left[A_1^n, A_2^n, \ldots, A_n^n\right] = \left[A_1^{n-1}, A_2^{n-1}, \ldots, A_{n-1}^{n-1}, 0\right] + A_n^n\left[JA_{n-1}^{n-1*}J, JA_{n-2}^{n-1*}J, \ldots, JA_1^{n-1*}J, I\right]$$

$$\begin{cases} \varepsilon_n^f(k) = \sum_{l=0}^{n} A_l^n(k) Z(k-l) \\ \varepsilon_n^b(k) = \sum_{l=0}^{n} JA_l^n(k)^* JZ(k-n+l) \end{cases} \quad \text{with } J = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix} \text{ and } A_0^n = I$$

$$\begin{cases} \varepsilon_{n+1}^f(k) = \varepsilon_n^f(k) + A_{n+1}^{n+1}\varepsilon_n^b(k-1) \\ \varepsilon_{n+1}^f(k) = \varepsilon_n^b(k-1) + JA_{n+1}^{n+1*}J\varepsilon_n^f(k) \end{cases} \quad \text{with } \varepsilon_0^f(k) = \varepsilon_0^b(k) = Z(k)$$

$$\underset{A_{n+1}^{n+1}}{Min}\, Tr\left[P_n^f + JP_n^{b*}J\right] \Rightarrow A_{n+1}^{n+1} = -2\left[P_n^{fb} + JP_n^{fbT}J\right]\left[P_n^b + JP_n^{f*}J\right]^{-1}$$

$$A_{n+1}^{n+1} = -2\left[\sum_{k=1}^{N+n} \varepsilon_n^f(k)\varepsilon_n^b(k-1)^+\right]\left[\sum_{k=1}^{N+n} \varepsilon_n^f(k)\varepsilon_n^f(k)^+ + \sum_{k=1}^{N+n} \varepsilon_n^b(k)\varepsilon_n^b(k)^+\right]^{-1}$$

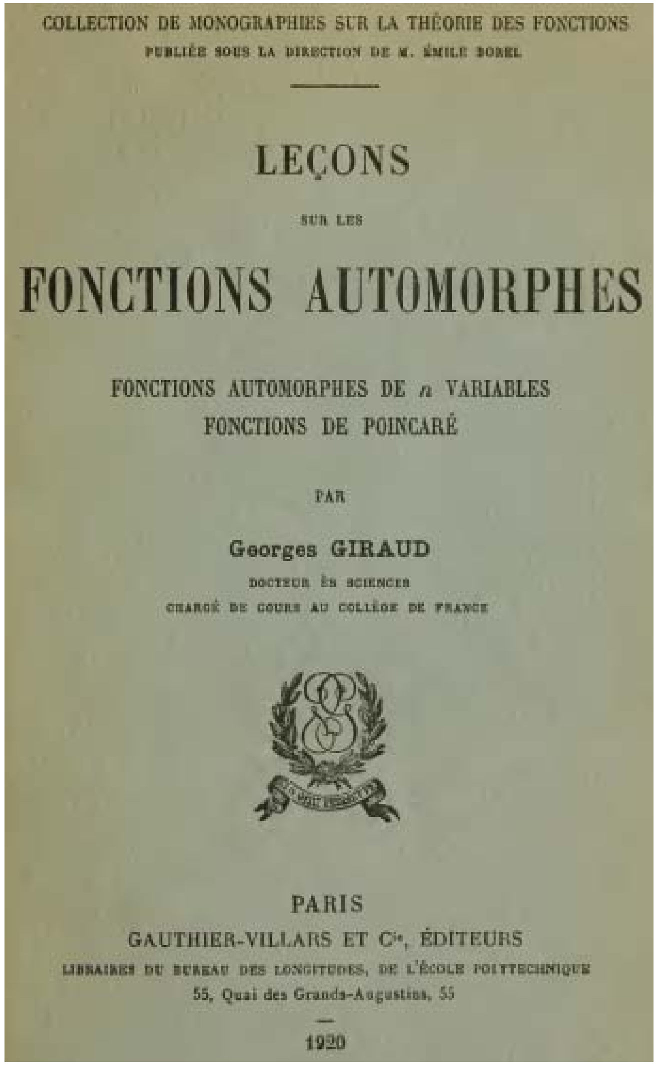
$$\text{with } \begin{cases} P_n^{fb} = E\left[\varepsilon_n^f(k)\varepsilon_n^b(k-1)^+\right] \\ P_n^f = E\left[\varepsilon_n^f(k)\varepsilon_n^f(k)^+\right] \\ P_n^b = E\left[\varepsilon_n^b(k)\varepsilon_n^b(k)^+\right] \end{cases}$$

$$A_{n+1}^{n+1}.\left(A_{n+1}^{n+1}\right)^+ < I_{n+1} \Rightarrow A_{n+1}^{n+1} \in Disk_{Siegel}$$

Mostow Decomposition & Berger Fibration

Georges Daniel Mostow

(Yale University & US Academy of Sciences)



COLLECTION DE MONOGRAPHIES SUR LA THÉORIE DES FONCTIONS
PUBLIÉE SOUS LA DIRECTION DE M. ÉMILE BOREL

LEÇONS
SUR LES
FONCTIONS AUTOMORPHES

FONCTIONS AUTOMORPHES DE n VARIABLES
FONCTIONS DE POINCARÉ

PAR

Georges GIRAUD
DOCTEUR ÈS SCIENCES
CHARGÉ DE COURS AU COLLÈGE DE FRANCE

PARIS
GAUTHIER-VILLARS ET C^ie, ÉDITEURS
LIBRAIRES DU BUREAU DES LONGITUDES, DE L'ÉCOLE POLYTECHNIQUE
55, Quai des Grands-Augustins, 55

1920

Marcel Berger

(IHES & French Academy of Sciences)

Mostows decomposition may be found in Georges Giraud's paper of 1921

THALES

Mostow Decomposition (& Berger Fibration)

Mostow Theorem :

Every matrix M of $GL(n, C)$ can be decomposed :

$$\boxed{M = Ue^{iA}e^{S}}$$

where

- **U is unitary**
- **A is real antisymmetric**
- **S is real symmetrix**

Can be deduce from

Lemma : Let A and B two positive definite hermitian matrices, there exist a unique positive definite hermitian matrix X such that : $XAX = B$

Corollary : if M is Hermitian Positive Definite, there exist a unique real symmetric matrix S such that :

$$M^{*} = e^{S}M^{-1}e^{S}$$

Mostow Decomposition

Mostow Theorem :

All matrix M of $GL(n,C)$ can be decomposed in :

$$M = Ue^{iA}e^{S}$$

U **is unitary,** A **is real antisymmetric réelle and** S **is real symmetric**

Proof :

$$M = Ue^{iA}e^{S} \Rightarrow P = M^{+}M = e^{S}e^{2iA}e^{S}$$

$$\Rightarrow P^{*} = e^{S}e^{-2iA}e^{S} = e^{2S}\left(e^{-S}e^{-2iA}e^{-S}\right)e^{2S}$$

$$\Rightarrow P^{*} = e^{2S}P^{-1}e^{2S}$$

$$\text{Lemma} + \text{corrolary}: P^{*} = e^{2S}P^{-1}e^{2S} \Rightarrow e^{2S} = P^{1/2}\left(P^{-1/2}P^{*}P^{-1/2}\right)^{1/2}P^{1/2}$$

$$\Rightarrow \boxed{S = \frac{1}{2}.\log\left(P^{1/2}\left(P^{-1/2}P^{*}P^{-1/2}\right)^{1/2}P^{1/2}\right) \quad \text{with} \quad P = M^{+}M}$$

$$\text{exponentielle injectivity}: e^{2iA} = e^{-S}Pe^{-S}$$

$$\Rightarrow \boxed{A = \frac{1}{2i}\log\left(e^{-S}Pe^{-S}\right) \quad \text{with} \quad P = M^{+}M}$$

$$\boxed{U = Me^{-S}e^{-iA}}$$

THALES

Automorphism of Siegel Disc SD_n given by :

$$\Sigma = \Phi_{Z_0}(Z) = \left(I - Z_0 \overline{Z}_0\right)^{-1/2} \left(Z - Z_0\right)\left(I - \overline{Z}_0 Z\right)^{-1} \left(I - \overline{Z}_0 Z_0\right)^{1/2}$$

All automorphisms given by :

$$\forall \Psi \in Aut(SD_n), \exists U \in U\left(n, C\right) / \Psi(Z) = U \Phi_{Z_0}(Z) U^t$$

Distance given by:

$$\forall Z, W \in SD_n, d\left(Z, W\right) = \frac{1}{2} \log\left(\frac{1 + \left\|\Phi_Z(W)\right\|}{1 - \left\|\Phi_Z(W)\right\|}\right)$$

Inverse automorphism given by :

$$G = \left(I - Z_0 \overline{Z}_0\right)^{1/2} \Sigma \left(I - \overline{Z}_0 Z_0\right)^{-1/2} = \left(Z - Z_0\right)\left(I - \overline{Z}_0 Z\right)^{-1}$$

$$\Rightarrow \begin{cases} Z = \Phi_{Z_0}^{-1}(\Sigma) = (G \overline{Z}_0 + I)^{-1} \left(G + Z_0\right) \\ \text{with } \; G = \left(I - Z_0 \overline{Z}_0\right)^{1/2} \Sigma \left(I - \overline{Z}_0 Z_0\right)^{-1/2} \end{cases}$$

THALES

Iterated Computation of Median in Siegel disk

$\{Z_1,\dots,Z_m\}$ in Siegel Half - Plane

For $i=1,\dots,m\ :\ W_i=(Z_i-iI)(Z_i+iI)^{-1}$

Initialisation :

$$W_{median,0}=0 \quad et \quad \{W_{1,0},\dots,W_{m,0}\}=\{W_1,\dots,W_m\}$$

Iterate on n until $\|G_n\|_F<\varepsilon$

$$W_{k,n}=U_{k,n}e^{iA_{k,n}}e^{S_{k,n}}\Rightarrow H_{k,n}=U_{k,n}e^{iA_{k,n}}=W_{k,n}e^{-S_{k,n}}=e^{-\frac{S_{k,n}}{2}}W_{k,n}e^{-\frac{S_{k,n}}{2}} \quad with:$$

$$S_{k,n}=1/2.\log\left(P_{k,n}^{1/2}\left(P_{k,n}^{-1/2}P_{k,n}^{*}P_{k,n}^{-1/2}\right)^{1/2}P_{k,n}^{1/2}\right) \quad \text{with} \quad P_{k,n}=W_{k,n}^{+}W_{k,n}$$

$$G_n=\gamma_n\sum_{\substack{k=1\\k\neq l}}^{m}H_{k,n} \quad \text{with} \quad \left\{l/\|H_{k,n}\|_F<\varepsilon\right\}$$

For $k=1,\dots,m$ then $W_{k,n+1}=\Phi_{G_n}(W_{k,n})$

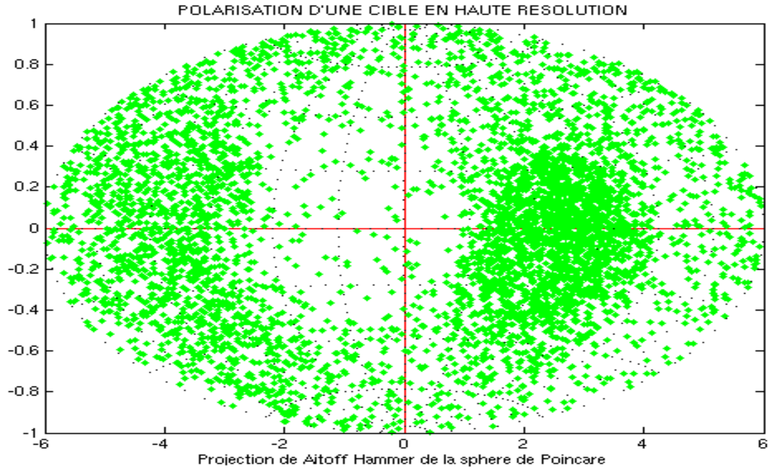
$$W_{k,n+1}=\left(I-G_n\overline{G}_n\right)^{-1/2}\left(W_{k,n}-G_n\right)\left(I-\overline{G}_nW_{k,n}\right)^{-1}\left(I-\overline{G}_nG_n\right)^{1/2}$$

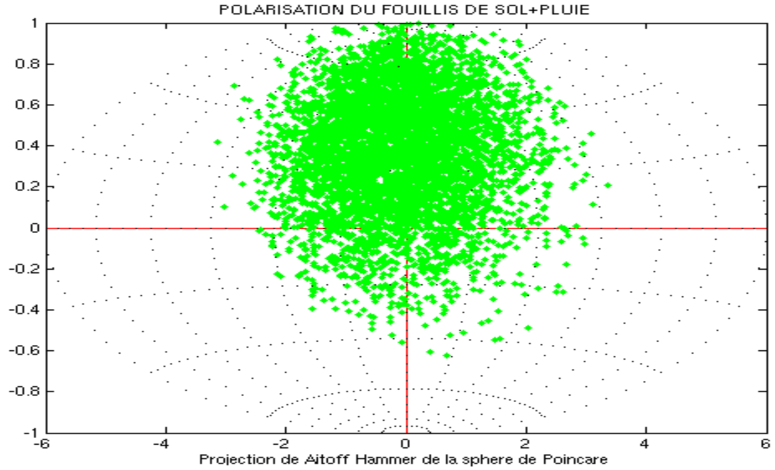
$$W_{median,n+1}=\Phi_{G_n}^{-1}\left(W_{median,n}\right)\Rightarrow\begin{cases}W_{median,n+1}=\Phi_{G_n}^{-1}(W_{median,n})=(G\overline{G}_n+I)^{-1}\left(G+G_n\right)\\ G=\left(I-G_n\overline{G}_n\right)^{1/2}W_{median,n}\left(I-\overline{G}_nG_n\right)^{-1/2}\end{cases}$$

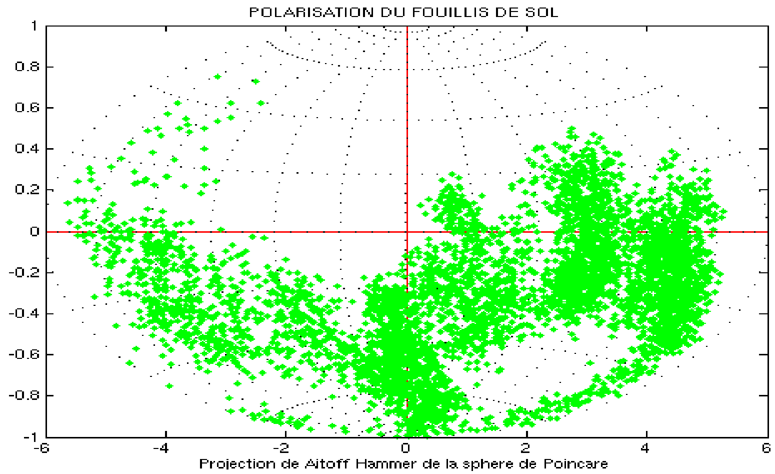
Extension for Polarimetric Data

THALES

STOKES VECTORS





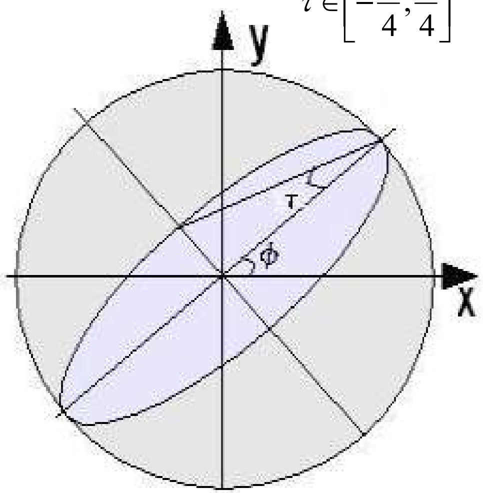


$$\vec{E} = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$\begin{cases} s_0 & = & |z_1|^2 + |z_2|^2 \\ s_1 & = & |z_1|^2 - |z_2|^2 \\ s_2 & = & 2Re(z_2 z_1^*) \\ s_3 & = & 2\,\mathrm{Im}(z_2 z_1^*) \end{cases}$$

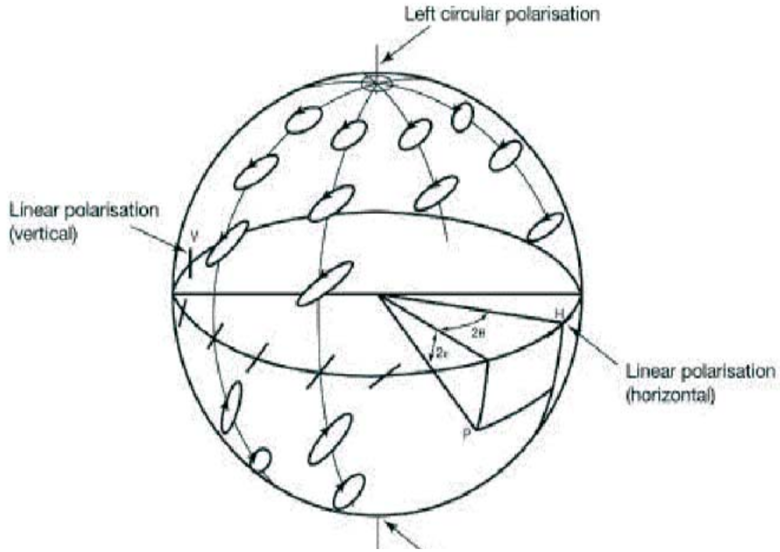
POLARIMETRY : POINCARE UNIT SPHERE

$$\tau \in \left[-\frac{\pi}{4};\frac{\pi}{4}\right]$$

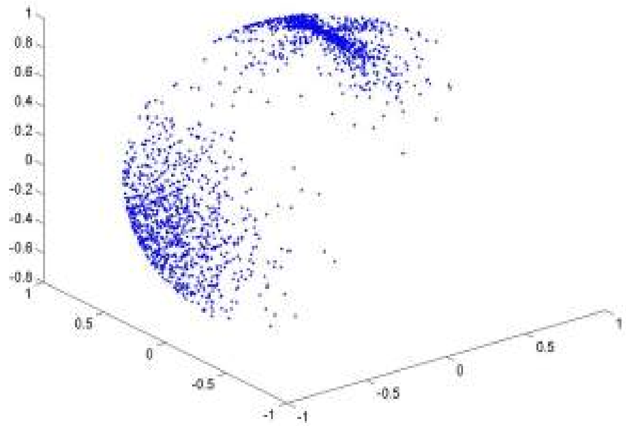


$$\phi \in \left[-\frac{\pi}{2};\frac{\pi}{2}\right[$$

$$\boldsymbol{s}_0^2 \geq \boldsymbol{s}_1^2 + \boldsymbol{s}_2^2 + \boldsymbol{s}_3^2$$



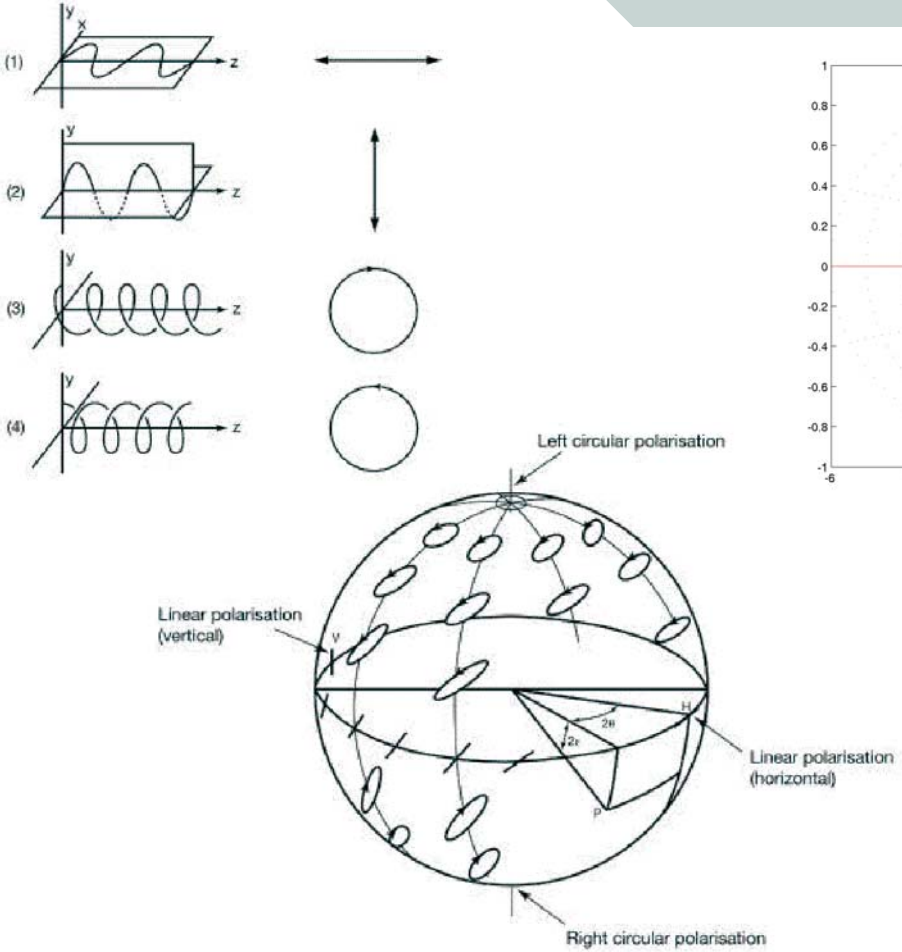
$$\vec{S} = \begin{bmatrix} \boldsymbol{s}_0 \\ \boldsymbol{s}_1 \\ \boldsymbol{s}_2 \\ \boldsymbol{s}_3 \end{bmatrix} = \begin{bmatrix} |\boldsymbol{z}_1|^2 + |\boldsymbol{z}_2|^2 \\ |\boldsymbol{z}_1|^2 - |\boldsymbol{z}_2|^2 \\ 2\,\mathrm{Re}(\boldsymbol{z}_2\boldsymbol{z}_1^*) \\ 2\,\mathrm{Im}(\boldsymbol{z}_2\boldsymbol{z}_1^*) \end{bmatrix} = \begin{bmatrix} \boldsymbol{s}_0 \\ \boldsymbol{s}_0 \cos 2\boldsymbol{\tau} \cos 2\boldsymbol{\phi} \\ \boldsymbol{s}_0 \cos 2\boldsymbol{\tau} \sin 2\boldsymbol{\phi} \\ \boldsymbol{s}_0 \sin 2\boldsymbol{\tau} \end{bmatrix}$$

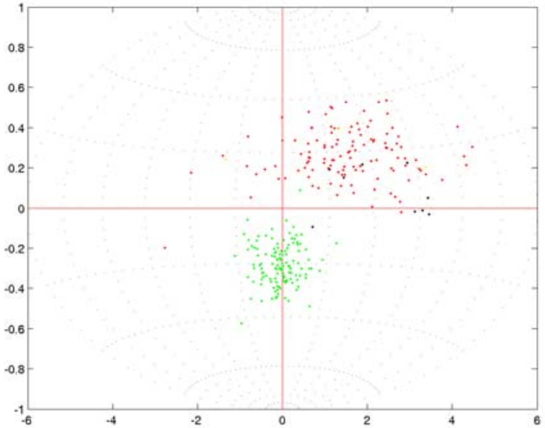


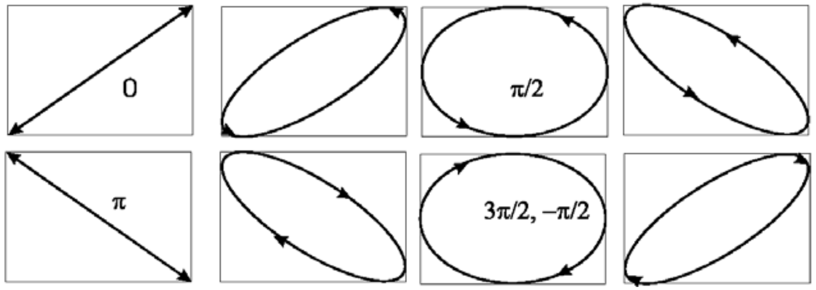
$$\boldsymbol{\phi} = \frac{\arctan\left(\dfrac{\boldsymbol{s}_2}{\boldsymbol{s}_1}\right)}{2} \qquad \boldsymbol{\tau} = \frac{\arctan\left(\dfrac{\boldsymbol{s}_3}{\sqrt{\boldsymbol{s}_1^2 + \boldsymbol{s}_2^2}}\right)}{2}$$

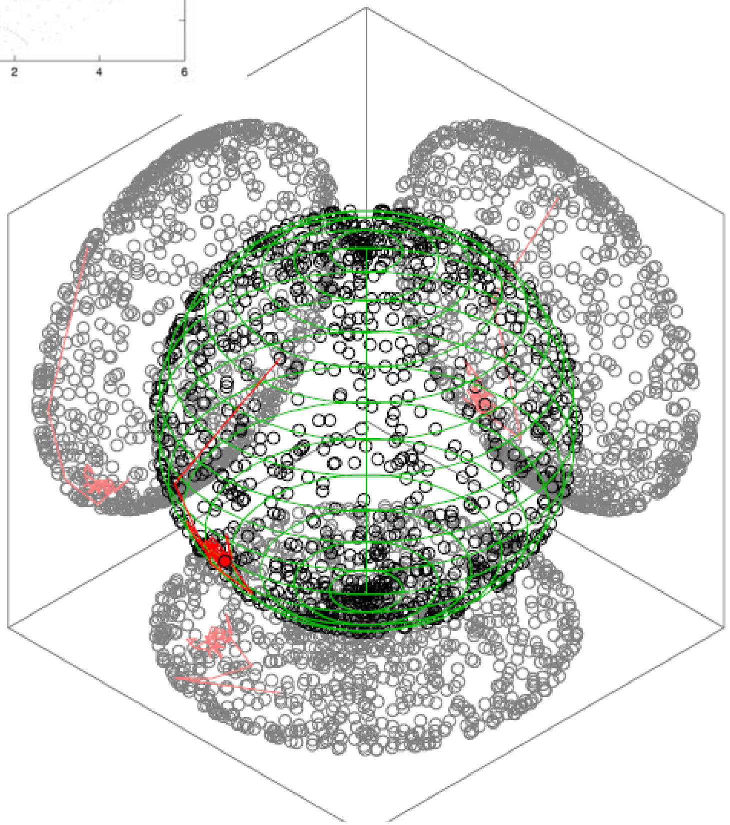
THALES

Extension : Arnaudon Flow for Polar Data









Arnaudon Stochastic Flow on Sphere

Conclusion

THALES

Conclusion

What are the benefits of Information Geometry

- **High Doppler Resolution Property**
  - Improve detection of slow targets
  - Increase monitoring accuracy of turbulences
- **Robustness Property**
  - very short Burst (few number of pulses)
  - with “Median” to be robust to outliers and clutter edges
  - With metric that take into account parameters correlations/variances

Future Extension

- **For Robust STAP**
  - Robust Estimation of Secundary Data Covariance Matrix
- **For Polarimetric Data Processing**
  - Stochastic Flow on Unit Poincaré Sphere

Other actors

- **EADS, MBDA, ONERA in Europe**
- **MIT LL in US**
- **NUDT & BIT (China), Australian labs**

QUESTIONS ?

More Information :

. Séminaire Léon Brillouin sur les « Sciences géométriques de l'information »

http://www.informationgeometry.org/Seminar/seminarBrillouin.html

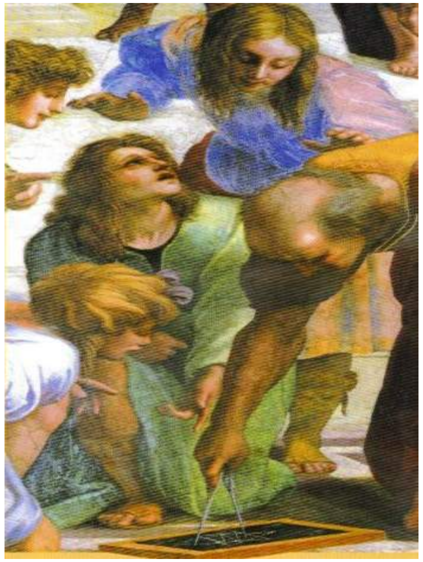
. Séminaire Franco-Indien CEFIPRA/THALES/Ecole-Polytechnique « Matrix Information Geometries »

http://www.informationgeometry.org/MIG/

http://www.informationgeometry.org/MIG/MIG-proceedings.pdf

http://www.lix.polytechnique.fr/~schwander/resources/mig/slides/

Mosaïque d'Irène Rousseau
(Le disque de Poincaré)



GENERAL THEORY OF COMPLEX SYMMETRIC SPACES

THALES

Classical Symmetric Bounded Domains

Symmetric bounded domains in C^n are particular cases of symmetric spaces of noncompact type.

Elie Cartan has proved that there is :

- **4 types of classical symmetric bounded domains**
- **2 exceptional types (group of motion E6 and E7)**

Classical Symmetric Bounded Domains (extension of Poincaré Disk)

Z : rectangular Complex Matrix

$\boxed{ZZ^+ < I}$ $\quad ({}^+ : transposed - conjugate)$

Type I : $\Omega^I_{p,q}$ Complex matrices with p rows and q columns

Type II : Ω^{II}_p Complex symmetric matrices of order p

Type III : Ω^{III}_p Complex skew-symmetric matrices of order p

Type IV : Ω^{IV}_n Complex matrices with n columns and 1 row such that :

$$\begin{cases} \left|ZZ^t\right| < 1 \\ 1 + \left|ZZ^t\right|^2 - 2ZZ^+ > 0 \end{cases}$$

Kernel function of Symmetric Bounded Domains

Luo-Geng Hua has computed the kernel functions for all classical domains :

$$\boxed{K\left(Z,W^*\right)=\frac{1}{\mu(\Omega)}\det\left(I-ZW^+\right)^{-\nu}}$$

$$\text{for}\begin{cases}\text{Type I}:\Omega^{I}_{p,q}\ ,\nu=p+q\\ \text{Type II}:\Omega^{II}_{p}\ ,\nu=p+1\\ \text{Type III}:\Omega^{III}_{p}\ ,\nu=p-1\end{cases}$$

$$K\left(Z,W^*\right)=\frac{1}{\mu(\Omega)}\left(1+ZZ^{t}W^*W^+-2ZW^*\right)^{-\nu}\ \text{for Type IV}:\Omega^{IV}_{n}\ ,\nu=n$$

where $\mu(\Omega)$ is the euclidean volume of the domain

Particular case (p=q=n=1) : Poincaré Unit Disk

$$\Omega^{I}_{1,1}=\Omega^{II}_{1}=\Omega^{III}_{1}=\Omega^{IV}_{1}=\left\{z\in C\,/\,zz^*<1\right\}$$

$$K\left(z,w^*\right)=\frac{1}{\left(1-zw^*\right)^2}$$

Analytic automorphisms of Bounded Domains

Groups of analytic automorphisms of these domains are locally isomorphic to the group of matrices which preserve following forms:

$$\text{Type I}: \Omega^{I}_{p,q}\ ,\ AHA^{*} = H, H = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}, \det A = 1$$

$$\text{Type II}: \Omega^{II}_{p}\ ,\ AHA^{*} = H, AKA^{t} = K, H = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}, K = \begin{pmatrix} 0 & I_p \\ -I_p & 0 \end{pmatrix}$$

$$\text{Type III}: \Omega^{III}_{p}\ ,\ AHA^{*} = H, ALA^{t} = L, H = \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}, L = \begin{pmatrix} 0 & I_p \\ I_p & 0 \end{pmatrix}$$

$$\text{Type IV}: \Omega^{IV}_{n}\ ,\ AHA^{*} = H, AHA^{t} = H, H = \begin{pmatrix} -I_2 & 0 \\ 0 & I_n \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \Rightarrow gZ = \left(A_{11}Z + A_{12}\right)\left(A_{21}Z + A_{22}\right)^{-1}}$$

THALES

Analytic automorphisms of Bounded Domains

All classical domains are circular following from Cartan's general theory, and the point 0 is distinguished for the potential :

$$\Phi\left(Z,Z^{*}\right)=\ln\left[\frac{K\left(Z,Z^{*}\right)}{K\left(0,0\right)}\right]=\log\det\left(I-ZZ^{+}\right)^{-\nu}$$

Berezin quantization is based on the construction of the Hilbert Space of functions analytic in Ω :

$$\left\langle f,g\right\rangle=c(h)\int f(Z)g(Z)\left[\frac{K(Z,Z^{*})}{K(0,0)}\right]^{-1/h}d\mu(Z,Z^{*})$$

$$c(h)^{-1}=\int\left[\frac{K(Z,Z^{*})}{K(0,0)}\right]^{-1/h}d\mu(Z,Z^{*})$$

$$K(gZ,gZ^{*})j(g,z)j(g,Z)^{*}=K(Z,Z^{*})\ \text{ with }\ j(g,Z)=\frac{\partial gZ}{\partial Z}$$

THALES

Berezin Quantification of Poincaré Unit Disk

The most elementary example of Berezian quantification is, in the case of complex dimension 1, given by the Poincaré unit Disk with volume element : $1/2i.(1-|z|^2)^{-2}dz \wedge dz^*$

$$D = \{z \in C / |z| < 1\} = SU(1,1)/S^1$$

$$g \in SU(1,1) \text{ with } g = \begin{bmatrix} a & b \\ b^* & a^* \end{bmatrix} \text{ where } |a|^2 - |b|^2 = 1$$

$$\text{Kähler potential}: \boxed{F(z) = -\ln\left(1-|z|^2\right)} \Rightarrow F(gz) = 2\operatorname{Re}\ln\left(b^* z + a^*\right) + F(z)$$

$$\Rightarrow \frac{\partial^2 F(gz)}{\partial z \partial z^*} = \frac{\partial^2 F(z)}{\partial z \partial z^*}$$

Map from path on D to automorphy factor :

$$g(0) = b\left(a^*\right)^{-1} \Rightarrow F(g(0)) = -\ln\left(1 - \left|b\left(a^*\right)^{-1}\right|^2\right) \underset{|a|^2 - |b|^2 = 1}{=} \ln\left(1 + |b|^2\right)$$

$$g^{-1} = \begin{pmatrix} a^* & -b \\ -b^* & a \end{pmatrix} \Rightarrow F(g^{-1}) = F(g)$$

THALES

Berezin Quantification of Siegel Unit Disk

Extension for Siegel Unit Disk :

$$SD_n = \{ Z / ZZ^+ < I \}$$

$$\text{with } \; g = \begin{bmatrix} A & B \\ B^* & B^* \end{bmatrix} \text{ and } g^t J g = J \; \text{ with } \; J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

$$\text{where } \begin{cases} A^+ A - B^t B^* = I \\ B^+ A - A^t B^* = 0 \end{cases}$$

$$g(Z) = (AZ + B)(B^* Z + A^*)^{-1}$$

$$\text{Kähler potential}: \boxed{F(z) = -\log\det\left(I - Z^+ Z\right) = -trace\ln\left(I - Z^+ Z\right)}$$

$$F(g(Z)) = F(Z) + 2\,\mathrm{Re}\,trace\ln\left(A^* + B^* Z\right)$$

$$\partial\partial^* F\left(g(Z)\right) = \partial\partial^* F(Z)$$

$$g(0) = B\left(A^*\right)^{-1} \Rightarrow F(g(0)) = \ln\det\left(I + B^+ B\right) = trace\ln\left(I + B^+ B\right)$$

THALES

Cartan Decomposition on Poincaré Unit Disk

Lemma of Cartan for radial coordinates in Poincaré Disk :

$$D = \{ z / |z| < 1 \}$$

$$g \in SU(1,1) \text{ with } g = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \text{ and } g(z) = \frac{az+b}{b^* z + a^*} \text{ where } |a|^2 - |b|^2 = 1$$

$$\text{Cartan Decomposition} : g = u_\Phi d_\tau u_\Psi$$

$$\text{with } u_\Phi = \begin{pmatrix} e^{i\Phi} & 0 \\ 0 & e^{-i\Phi} \end{pmatrix} \text{ and } d_\tau = \begin{pmatrix} ch(t) & sh(t) \\ sh(t) & ch(t) \end{pmatrix}$$

$$\Rightarrow \begin{cases} a = e^{i(\Phi+\Psi)} ch(\tau) \\ b = e^{i(\Phi-\Psi)} sh(\tau) \end{cases} \Rightarrow z = b\left(a^*\right)^{-1} = th(\tau) e^{i2\Phi}$$

$$ds^2 = 8\left(d\tau^2 + sh^2(2\tau) d\theta^2\right) \Rightarrow \Delta_{LB}. = \frac{\partial^2 .}{\partial \tau^2} + \coth(2\tau)\frac{\partial .}{\partial \tau} + \frac{1}{sh^2(\tau)} \frac{\partial^2 .}{\partial \Phi^2}$$

$$F(z) = -\ln\left(1 - |z|^2\right) = 2 \ln ch(\tau)$$

THALES

Iwasawa Decomposition on Poincaré Unit Disk

Iwasawa coordinates in Poincaré Disk :

$$D = \{ z / |z| < 1 \}$$

$$g \in SU(1,1) \text{ with } g = \begin{pmatrix} a & b \\ b^* & a^* \end{pmatrix} \text{ and } g(z) = \frac{az+b}{b^* z + a^*} \text{ where } |a|^2 - |b|^2 = 1$$

$$\text{Iwasawa Dec.}: g = h\left(K_\theta D_\tau N_\xi\right) \text{ with } h(g) = CgC^{-1} \;,\; C = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}$$

$$K_\theta = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2) \\ -\sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \;,\; D_\tau = \begin{pmatrix} ch(t) & sh(t) \\ sh(t) & ch(t) \end{pmatrix} \text{ and } N_\xi = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix}$$

$$\Rightarrow \begin{cases} a = e^{i\theta/2}\left(ch(\tau/2) + i\dfrac{u}{2}e^{-\frac{\tau}{2}} \right) \\ b = e^{i\theta/2}\left(sh(\tau/2) - i\dfrac{u}{2}e^{-\frac{\tau}{2}} \right) \end{cases} \text{ with } u = \xi e^{\tau}$$

THALES

Hua-Cartan Decomposition on Siegel Unit Disk

Lemma of Hua for radial coordinates in Siegel Disk (Hua-Cartan) :

$$\tau = \begin{bmatrix} \tau_1 & \tau_2 & \cdots & \tau_n \end{bmatrix} \text{ with } 0 \le \tau_n \le \tau_{n-1} \le \cdots \le \tau_1$$

$$A_0(\tau) = diag\begin{bmatrix} ch(\tau_1) & ch(\tau_2) & \cdots & ch(\tau_n) \end{bmatrix}$$

$$B_0(\tau) = diag\begin{bmatrix} sh(\tau_1) & sh(\tau_2) & \cdots & sh(\tau_n) \end{bmatrix}$$

$$g = \begin{bmatrix} A & B \\ B^* & A^* \end{bmatrix} \in Sp(n) \; , g = \begin{bmatrix} U^t & 0 \\ 0 & U^+ \end{bmatrix} \begin{bmatrix} A_0 & B_0 \\ B_0 & A_0 \end{bmatrix} \begin{bmatrix} V^* & 0 \\ 0 & V \end{bmatrix}$$

there exist U and V unitary complex matrices of order n

$$\begin{cases} A = U^t A_0(\tau) V^* \\ B = U^t B_0(\tau) V \end{cases} \Rightarrow \begin{pmatrix} A_0(\tau) & B_0(\tau) \\ B_0(\tau) & A_0(\tau) \end{pmatrix} = \exp\begin{pmatrix} 0 & Z_2(\tau) \\ Z_2(\tau) & 0 \end{pmatrix}$$

with $Z_2(\tau) = diag\begin{bmatrix} \tau_1 & \tau_2 & \cdots & \tau_n \end{bmatrix}$

Let $Z = B\left(A^*\right)^{-1} = U^t P U , P^2 = B_0^2 \left(A_0^{-1}\right)^2 = \mathrm{diag}\left[eigen\left(ZZ^+\right)\right]$

$$P = diag\begin{bmatrix} th(\tau_1) & th(\tau_2) & \cdots & th(\tau_n) \end{bmatrix}$$

THALES

Iwasawa Decomposition on Siegel Unit Disk

Iwasawa coordinates in Siegel Disk :

$$SD_n = \{Z / ZZ^+ < I\} \text{ and } g(Z) = (AZ + B)(B^* Z + A^*)^{-1}$$

$$g = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}, h(g) = CgC^{-1} \text{ with } C = \frac{1}{\sqrt{2}}\begin{pmatrix} I & -iI \\ I & iI \end{pmatrix}$$

$$K = \left\{ g / h(g) = \begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix}, U \text{ unitary order n} \right\} \Rightarrow g = \frac{1}{2}\begin{pmatrix} U + U^* & -i(U - U^*) \\ i(U - U^*) & U + U^* \end{pmatrix} = C^{-1}\begin{pmatrix} U & 0 \\ 0 & U^* \end{pmatrix} C$$

$$A = \left\{ A = \begin{pmatrix} A_0 + B_0 & 0 \\ 0 & A_0 - B_0 \end{pmatrix} = \begin{pmatrix} diag\left[e^{\tau_1} \quad \cdots \quad e^{\tau_n}\right] & 0 \\ 0 & diag\left[e^{-\tau_1} \quad \cdots \quad e^{-\tau_n}\right] \end{pmatrix} \right\}$$

$$N = \left\{ N / N = \begin{Bmatrix} I & S \\ 0 & I \end{Bmatrix}, S \text{ real matrix of order n} \right\}$$

$$h(A) = \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix}, \ h(N) = \begin{pmatrix} I + i/2.S & -i/2.S \\ i/2.S & I - i/2.S \end{pmatrix} \Rightarrow h(KAN) = \begin{pmatrix} A_1 & B_1 \\ B_1^* & A_1^* \end{pmatrix}$$

$$\text{with } \begin{cases} A_1 = U\left[A_0 + i(A_0 + B_0)\dfrac{1}{2} S \right] \\ B_1 = U\left[B_0 - i(A_0 + B_0)\dfrac{1}{2} S \right] \end{cases}$$

THALES

Iwasawa/Cartan Coordinates on Siegel Unit Disk

Iwasawa/Cartan coordinates relation in Siegel Disk :

$$M_S = \begin{pmatrix} I + i/2.S & -i/2.S \\ i/2.S & I - i/2.S \end{pmatrix} \ , \ \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix} M_S = M_{\tilde{S}} \begin{pmatrix} A_0 & B_0 \\ B_0 & A_0 \end{pmatrix}$$

$$\text{with } (A_0 + B_0)S = \tilde{S}(A_0 - B_0)$$

$$g = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix} \Rightarrow \begin{cases} \text{Cartan}: \begin{cases} A = U^t A_0 V^* \\ B = U^t B_0 V \end{cases} \\ \text{Iwasawa}: \begin{cases} A = U_1 \left[A_0 + i(A_0 + B_0)\frac{1}{2}S \right] \\ B = U_1 \left[B_0 - i(A_0 + B_0)\frac{1}{2}S \right] \end{cases} \end{cases}$$

$$Z = B\left(A^*\right)^{-1} = U_1 H U_1^t = U^t P U$$

$$\text{with } H = \left[B_0(A_0 + B_0) - \frac{i}{2}\tilde{S} \right] \left[A_0(A_0 + B_0) - \frac{i}{2}\tilde{S} \right]^{-1}$$

THALES

For every symmetric Riemannian space, there exist a dual space being compact. The isometry groups of all the compact symmetric spaces are described by block matrices (the action of the group in terms of special coordinates is described by the same formula as the action of the group of motions of the dual domain).

$$\Gamma = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \Rightarrow \Gamma(W) = \left(A_{11}W + A_{12}\right)\left(A_{21}W + A_{22}\right)^{-1}$$

$$\text{Isometry}: \Gamma = C\Gamma C^{-1} \text{ with } C = \frac{1}{\sqrt{2}}\begin{pmatrix} I & iI \\ iI & I \end{pmatrix}$$

- Berezin coordinates for Siegel domain :

$$\Gamma = \begin{pmatrix} A & B \\ B^* & A^* \end{pmatrix}, \Gamma^{-1} = \begin{pmatrix} A^+ & B^t \\ B^+ & A^t \end{pmatrix}$$

$$\text{or equivalently}: \Gamma\Gamma^+ = I \ , \ \Gamma L \Gamma^t = L \text{ with } L = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- Remark :

$$g(0) = B\left(A^*\right)^{-1} \Rightarrow F(g(0)) = \ln\det\left(I + BB^+\right) = trace \ln\left(I + BB^+\right)$$

Invariant metric in Special Berezin Coordinates

Let M be a classical complex compact symmetric space. The invariant volume and invariant metric in terms of special Berezin coordinates have the form :

$$d\mu\left(W,W^{*}\right)=F\left(W,W^{*}\right)\frac{d\mu_{L}\left(W,W^{*}\right)}{\pi^{n}}$$

$$ds^{2}=\sum_{\alpha,\beta}g_{\alpha,\beta}dW^{\alpha}dW^{\beta*} \quad \text{with} \quad g_{\alpha\beta}=-\frac{\partial^{2}\ln F\left(W,W^{*}\right)}{\partial W^{\alpha}\partial W^{\beta*}}$$

$$\text{where} \quad F(W,W^{*})=\det\left(I+WW^{+}\right)^{-\nu}$$

Link with : For arbitrary Kählerian homogeneous space, the logarithm of the density for the invariant measure is the potential of the metric

Information sur Séminaire Inter-disciplinaire Léon Nicolas Brillouin sur « les Sciences Géométriques de l’Information »

THALES

Corpus thématique : « **Science géométrique de l'information** »

- **Laboratoire d'accueil : IRCAM (Arshia Cont), Paris**
- **Animation : Arshia Cont (IRCAM), Frank Nielsen (Sony Research), F. Barbaresco (Thales)**
- **Les laboratoires**
  - IRCAM (Arshia Cont, Gerard Assayag, Arnaud Dessein)
  - Polytechnique (Schwanger)
  - Mines ParisTech (Pierre Rouchon, Silvere Bonnabel, Jesus Angulo)
  - Telecom ParisTech (Hugues Randriam)
  - SUPELEC (Mérouane Debbah, Romain Couillet)
  - UTT Troyes (Hichem Snoussi)
  - Univ. Poitiers (Marc Arnaudon, Le Yang)
  - Univ. Montpellier (Michel Boyom, Paul Bryand)
  - Observation de Nice (Cédric Richard)
  - INRIA (Jean-Paul Zolesio, Rama Cont, Xavier Pennec)
  - Thales (Frédéric Barbaresco, Jean-Francois Marcotorchino, François Gosselin)
  - …

« Science géométrique de l'information »

« La Science et la Théorie de l'Information », L. Brillouin, 1956

« Théorie scientifique de l'information d'une part, mais aussi application de la théorie de l'information à des problèmes de science pure. »

Science géométrique de l'information

- Géométrie discrète
- Géométrie algébrique & arithmétique
- Géométrie non commutative
- Géométrie complexe
- Géométrie symplectique
- Géométrie & Groupes de Lie
- Géométrie Riemannienne (symétrique, Kählerien, métrique)
- Probabilité & statistique
- Théorie & géométrie de l'information
- Physique Statistique
- Physique quantique
- Thermo-dynamique & Géométrie (Souriau, Ruppeiner)

Séminaire Brillouin : sciences géométriques de l'information

- Site web: http://www.informationgeometry.org/Seminar/seminarBrillouin.html

- 9 Juin 2011 : Session « stochastic geometry & information geometry »
  - New Perspectives in Stochastic Geometry **par** *Wilfrid S Kendall* **(univ. De Warwick, UK, w.s.kendall@warwick.ac.uk)**
  - Pecularities of the q-exponential function in phi-exponential functions **par** *Asuka Takatsu* **(IHES et Univ. De Nagoya, takatsu@math.nagoya-u.ac.jp)**

- 5 Juillet 2011 : Session Digiteo & EPFL
  - Minimisation de l'Entropie Géométrique **par** *Alfred Hero* **(Chaire DIGITEO lab, hero@eecs.umich.edu)**
  - *Christophe Vignat* **(Ecole Polytechnique de Lausanne)**

- Octobre 2011 : Session « Entropies et Divergences » (29 Avril)
  - Estimateurs statistiques par minimisation des ϕ-divergences duales **par** *Michel Broniatowski* **(Univ. UPMC/Paris-6, michel.broniatowski@upmc.fr)**

- Novembre 2011 : Session Franco-Italienne
  - Algebraic and Geometric Methods in Statistics **par** *Paolo Gibilisco* **(univ. De Rome, gibilisco@volterra.uniroma2.it)**
  - Learning the Fréchet Mean over the Manifold of Covariance Matrices **par** *Simone Fiori* **(Università Politecnica delle Marche, s.fiori@univpm.it)**

- Decembre 2011 : Session historique (date à définir)
  - Œuvre de Léon Brillouin **par** *Rémy Mosseri* **(UPMC Paris-6, remy.mosseri@upmc.fr)**
  - Léon Brillouin et les débuts de la théorie de l'information **par** *Philippe Jacquet* **(INRIA & X/LIX, philippe.jacquet@inria.fr)**
  - Le concept d'Entropie en Physique **par** *Roger Balian* **(CEA, roger.balian@cea.fr)**

·PEPS « Géométrie de l'Information »

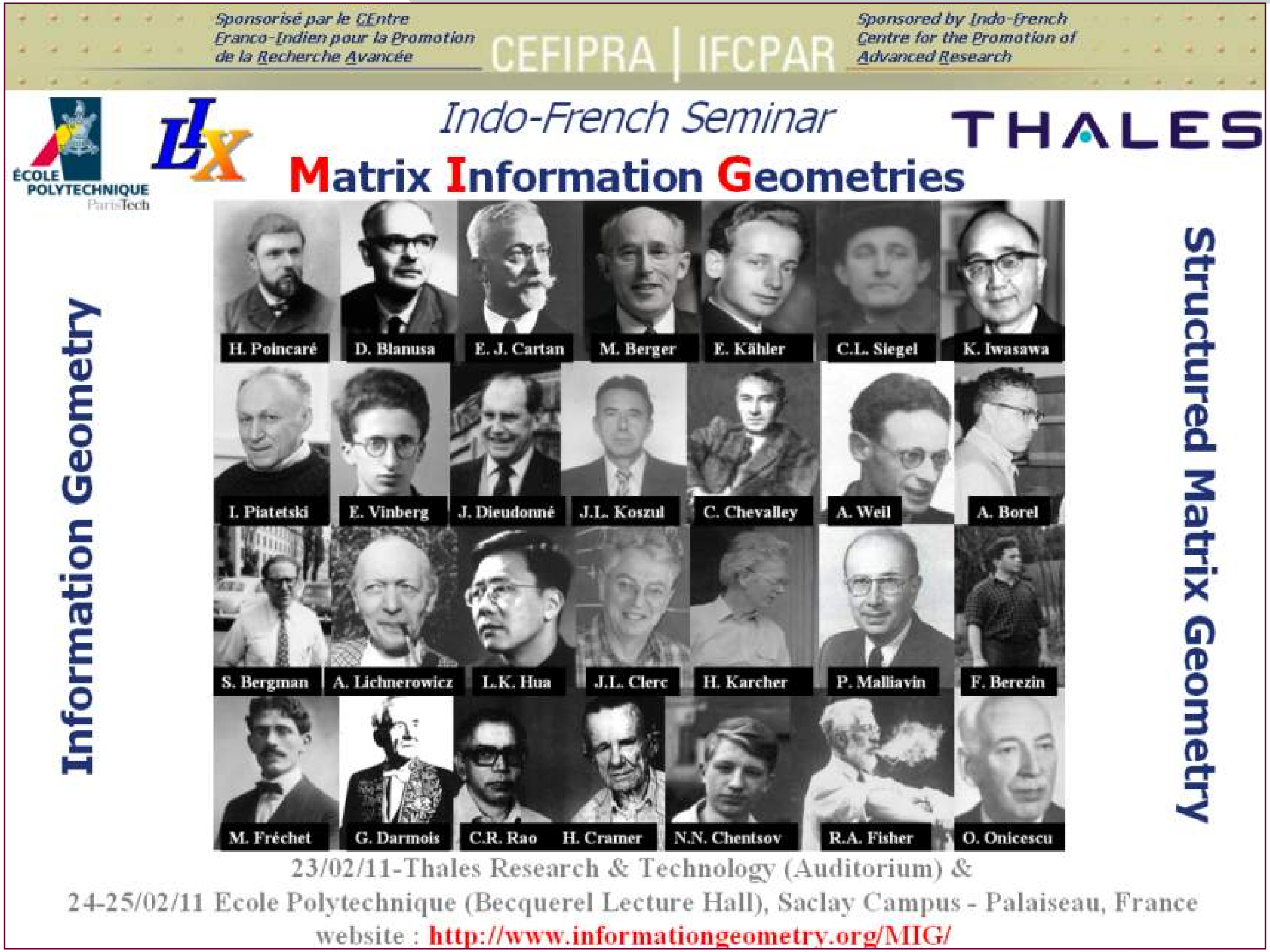
- **Titre long du Projet :**
  - La géométrie de l'information : un cadre général nouveau pour l'analyse et le traitement de signaux multiformes
- **Titre court du Projet :**
  - InfoGeo
- **Mots clés :**
  - Géométrie de l'information, analyse et traitement du signal, audio, image, radar, télécommunications, mathématiques appliquées.
- **Résumé du Projet :**
  - Ce projet centré autour de la géométrie de l'information cherche à regrouper des acteurs de différentes disciplines : audio,image, radar, télécommunications, mathématiques appliquées. L'objectif est d'amener les outils de la géométrie de l'information vers un nouveau champ d'applications : l'analyse et le traitement des signaux multiformes, pour lequel nous souhaitons formuler un cadre théorique générique. Afin de réunir les multiples compétences nécessaires, l'IRCAM, le LIX et Thales se sont associés et ont récemment initié un groupe de travail pluridisciplinaire autour de la géométrie de l'information avec des manifestations scientifiques prévues pour 2011. Le projet présenté permettra de soutenir le lancement de ce groupe de travail et de le consolider en étendant le réseau d'acteurs nationaux impliqués.
- **Coordinateur : Arshia cont (IRCAM)**
- **Equipes participantes : F. Nielsen (Sony Research), F. Barbaresco (Thales), A. Dessein (IRCAM)**

« Matrix & Information Geometries », MIG'11

- Lieu : Ecole Polytechnique et Thales Research & Technology
- Date : 23 au 25 Février 2011
- Site web : http://www.informationgeometry.org/MIG/
- Résumés : http://www.informationgeometry.org/MIG/MIG-proceedings.pdf
- Slides: http://www.lix.polytechnique.fr/~schwander/resources/mig/slides/
- Photos : http://www.lix.polytechnique.fr/~schwander/resources/mig/pictures/
- Financement :
  - **CEFRIPA**
    - **CEFRIPA (http://www.cefipra.org) : Programme bilatéral de coopération scientifique entre l'Inde et la France financé respectivement par :**
      - **le Département Science et Technologie (DST) du Ministère indien de la science et de la technologie**
      - **la Direction générale de la coopération internationale et du développement du Ministère français des affaires étrangères.**
- Laboratoires indiens :
  - **Dehli Indian Institute of Statistics (Prof. Rajendra Bhatia)**
  - **2 IITs : Guwahati (Prof. Amit Kumar Mishra), Kharagpur**
  - **Sociétés indiennes : The BuG Design, Honeywell Technology de Bangalore**
- Laboratoires français :
  - **Membres du séminaire Brillouin**

THALES

Séminaire Franco-Indien « Matrix Information Geometries »



Séminaire Franco-Indien « Matrix Information Geometries »

MATRIX INEQUALITIES

Some general principles

Rajendra Bhatia
Indian Statistical Institute

Marvelous fact

In $(\mathbb{P}, \delta_2)$ the natural parametric equation for the geodesic joining A, B is

$$\boxed{\begin{aligned} \gamma(t) &= A^{1/2}\left(A^{-1/2} B A^{-1/2}\right)^t A^{1/2} \\ &=: A \#_t B \qquad 0 \le t \le 1 \end{aligned}}$$

The geometric mean $A \#_{1/2} B$ is the midpoint of this!

GEOMETRY & GEOMETRIC MEAN

Rajendra Bhatia
Indian Statistical Institute

$$\begin{aligned} \delta_2(A,B) &= \delta_2(I, A^{-1/2} B A^{-1/2}) \\ &= \| \log A^{-1/2} B A^{-1/2} \|_2 \\ &\qquad \text{(EMI equality)} \\ &= \left[\sum \log^2 \lambda_i (A^{-1/2} B A^{-1/2})\right]^{1/2} \\ &= \left[\sum \log^2 \lambda_i (A^{-1} B)\right]^{1/2} \end{aligned}$$

- R. Bhatia, « Positive Definite Matrices », Princeton university Press, 2007

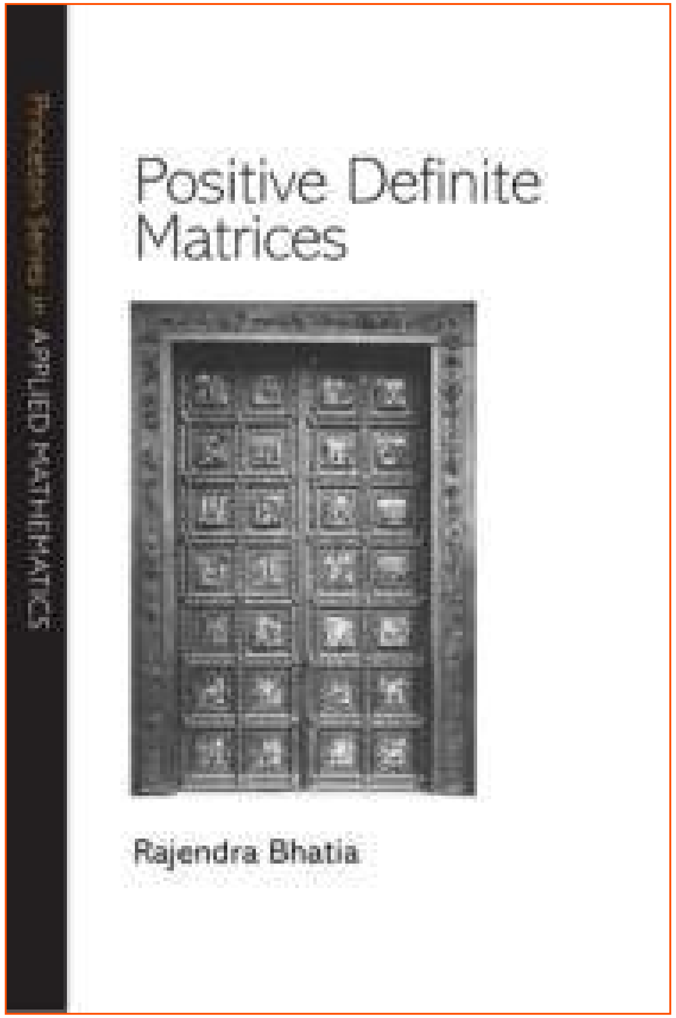


TABLE OF CONTENTS:

MATHS & ENTREPRISES

Semaine d'Etude Maths-Entreprises

Du 4 au 8 avril 2011 à l'Institut Henri Poincaré (Paris)

Lundi 4 avril : présentation des sujets par les industriels

13h30 **Marine Pichelin (Air liquide)**
Modèles de comparaison quantitative de matrices 3D
14h15 **Frédéric Barbaresco (Thalès)**
Géométrie des matrices de covariances pour le traitement de signaux radars
15h00 Pause café
15h30 **Eric Duceau (EADS)**
Modélisation et régulation des systèmes de climatisation
16h15 **Alain Fuser (GDF Suez)**
Optimisation du positionnement de parcs solaires
17h00 - 18h00 **Discussions avec les intervenants**

Mardi 5 - jeudi 7 avril : travail par groupe de jeunes chercheurs sur les 4 projets

Vendredi 8 avril : Présentation du travail des groupes

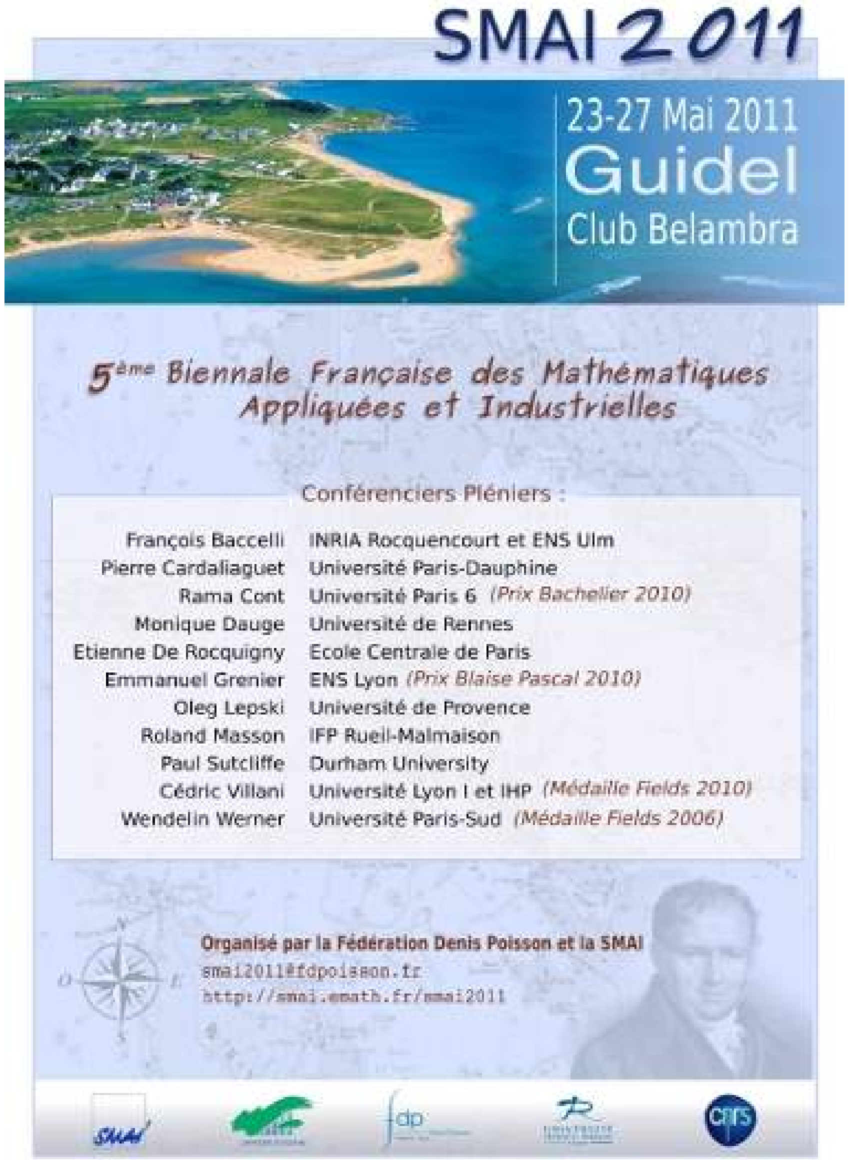
11h00 **Air Liquide**
13h30 **Thalès**
14h15 **EADS**
15h00 Pause café
15h30 **GDF Suez**
16h15 - 16h30 **Discussions avec les participants**

FMJH cnrs ihp Institut Henri Poincaré

THALES

·Mini-Symposia pour Congrès SMAI 2011, 5e Biennale Française des Mathématiques Appliquées et Industrielles

- Lieu : Guidel, Bretagne
- Date : 23 au 27 Mai 2011
- Site web : http://smai.emath.fr/smai2011/index.php
- Thème : Géométrie de l'Information
    - **Thème : La géométrie de l'information est un thème émergeant qui consiste à traiter des phénomènes aléatoires en regardant la géométrie différentielle des espaces de probabilités correspondants. Les applications sont nombreuses en machine learning, théorie de l'information, traitement du signal et aussi géométrie différentielle. Il s'agit ici de donner une introduction aux techniques et aux motivations de la géométrie de l'information. Ce thème intéresse tout autant des mathématiciens purs qu'appliqués, des informaticiens et des industriels.**
- Programme :
    - **Arshia Cont (IRCAM)**
    - **Frank Nielsen (Ecole Polytechnique, Sony)**
    - **Xavier Pennec (INRIA, Sophia)**
    - **Frédéric Barbaresco (THALES)**



THALES

Session spéciale SS2 « Science géométrique de l'information », au XXIIIème colloque GRETSI 2011

- Lieu : Bordeaux
- Date : 5 au 8 Septembre 2011
- Site web : http://www.gretsi2011.org/sessions-speciales.html
- Programme :
  - **Frédéric Barbaresco (THALES), « Science géométrique de l'Information : Géométrie des matrices de covariance, espace métrique de Fréchet et domaines bornés homogènes de Siegel »**
  - **Olivier SCHWANDER (Ecole Polytechnique), Frank NIELSEN (Sony), « Simplification de modèles de mélange issus d'estimateur par noyau »**
  - **Silvère Bonnabel (Ecole des Mines de Paris), « Convergence des méthodes de gradient stochastique sur les variétés riemanniennes »**
  - **Arnaud Dessein, Arshia Cont (IRCAM), « Segmentation statistique de flux audio en temps-réel dans le cadre de la géométrie de l'information »**
  - **V. Devlaminck (Université de Lille), « Modèles sous-jacents à certaines techniques d'interpolation géodésique dans l'espace des matrices de cohérence en optique de polarisation »**
  - **Pierre Formont (Supelec), Frédéric Pascal (Sondra), Jean-Philippe Ovarlez (ONERA), Gabriel Vasile (INPG Grenoble), Laurent Ferro-Famil (Université de Rennes), « Apport de la géométrie de l'information pour la classication d'images SAR polarimétriques**
  - **Xavier Pennec (INRIA), Marco Lorenzi (IRCCS, Italie), « Which parallel transport for the statistical analysis of longitudinal deformations? »**
  - **Hichem Snoussi (UTT Troyes), « Filtrage particulaire sur les vari´et´es riemanniennes »**
  - **Marc Arnaudon (Université de Poitiers), Yang Le, « Algorithmes stochastiques pour calculer les p-moyennes de mesures de probabilité et géométrie des matrices de covariance Toeplitz »**

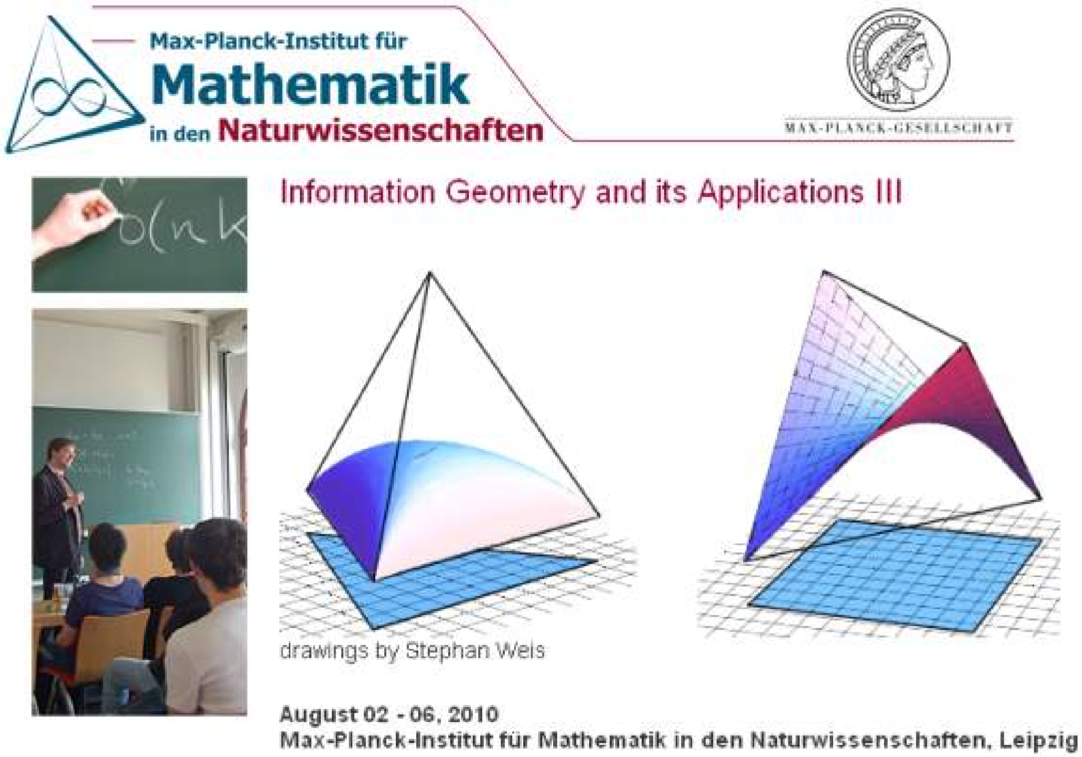
2 Sessions spéciales « New Generation of Advanced Radar Processing based on Information Geometry », au 12ème IRS 2011, International Radar Symposium

- Lieu : Leipzig, Allemagne
- Date : 7-9 Septembre 2011
- Site web : http://www.irs-2011.de
- Programme :
  - **F. Barbaresco (THALES) : Geometric Radar Processing based on Fréchet Distance : Information Geometry versus Optimal Transport Theory**
  - **M. Frasca (MBDA Italy) : Optimal Cramer-Rao estimators for dimensions greater than two**
  - **F. Opitz (EADS Gmbh) : Differential Geometry and Applications to Signal Processing and Tracking**
  - **J.P. Ovarlez (ONERA), P. Formont (SONDRA), F. Pascal (SONDRA), G. Vasile (GIPSA) and L. Ferro-Famil (Université de Rennes) : Contribution of Information Geometry for Polarimetric SAR Classification in Heterogeneous Areas**
  - **M. Arnaudon (Université de Poitiers), Le YANG & F. Barbaresco : Stochastic algorithms for computing p-means of probability measures, geometry of radar Toeplitz covariance matrices and applications to HR Doppler processing**
  - **Y. Cheng (NUDT, Chine), X. Wang, M. Morelande & Bill Moran (Université de Melbourne) : Bearings-Only Sensor Trajectory Optimization Using Accumulative Information**
  - **Y. Cheng (NUDT, Chine), X. Wang, M. Morelande & Bill Moran (Université de Melbourne) : Information Resolution of Joint Detection-Tracking Systems**
  - **F. Barbaresco (THALES) : Robust Statistical Radar Processing in Fréchet Metric Space: OS-HDR-CFAR and OS-STAP Processing in Siegel Homogeneous Bounded Domains**

THALES

Projet d’organisation du 4^{th} IGAIA’12 à Paris

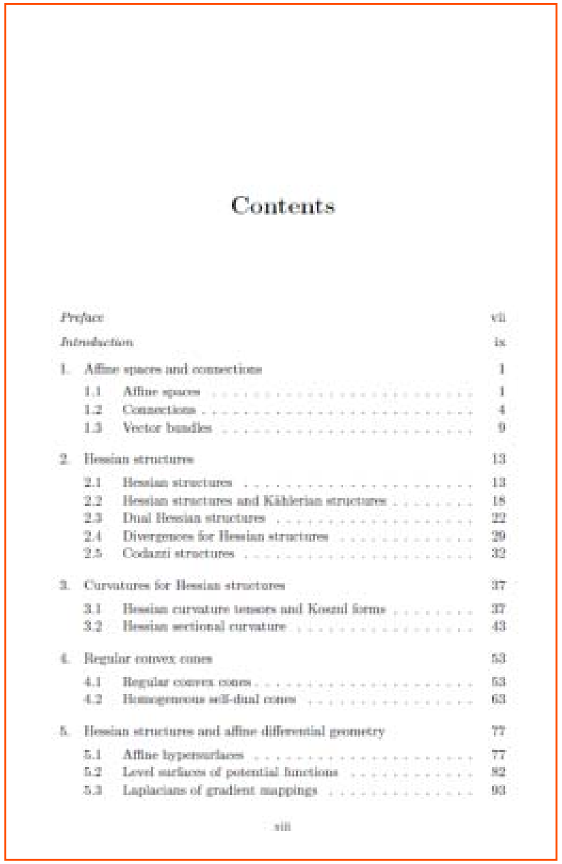
- **Organisateur : séminaire Brillouin (A. Dessein & A. Cont / IRCAM)**
- **Lieu : IRCAM, Paris**
- **Pour info sur 3^{rd} IGAIA’10 :**
  - http://www.mis.mpg.de/calendar/conferences/2010/infgeo.html
  - http://www.mis.mpg.de/calendar/conferences/2010/infgeo/slides.html



THALES

Géométrie Hessienne & travaux de Koszul

- Soutenance de thèse de P. Byande (Prof. Boyom), 7. Déc. 2011
- Livre de Hirohiko Shima, « Geometry of Hessian Structures », world Scientific Publishing 2007



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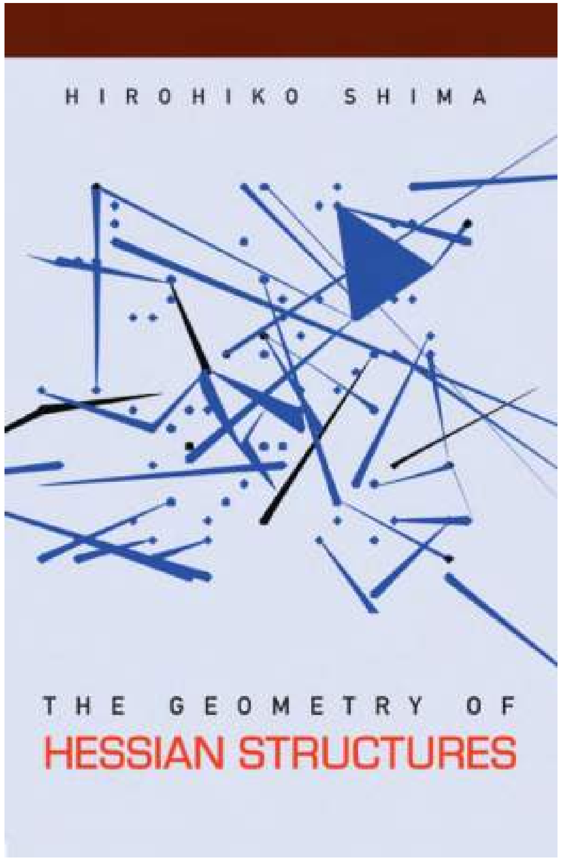

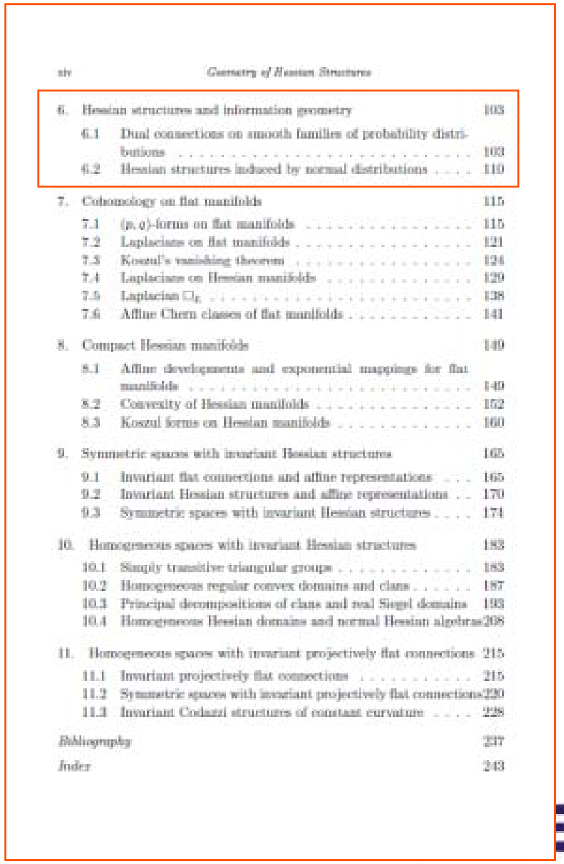

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G. Pistone : Algebraic and Geometric Methods in Statistics

THALES