GEOMETRIE STOCHASTIQUE ET THEORIE DE L'INFORMATION

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Structure of the Lecture

- Shannon Capacity and Error Exponents for Point Processes
  - Additive White Gaussian Noise AWGN
  - Additive Stationary Ergodic Noise ASEN
- Shannon Capacity and Error Exponents for Additive Noise Channels with Power Constraints

AWGN DISPLACEMENT OF A POINT PROCESS

- μ^n: (simple) stationary ergodic point process on $\mathbb{R}^n$.
- $\lambda_n = e^{nR}$: intensity of μ^n.
- $\{T_k^n\}$: points of μ^n (codewords).
- $\mathbb{P}_n^0$: Palm probability of μ^n.
- $\{D_k^n\}$: i.i.d. sequence of displacements, independent of μ^n:
$$D_k^n = (D_k^n(1), \ldots, D_k^n(n))$$
i.i.d. over the coordinates and $\mathcal{N}(0, \sigma^2)$ (noise).
- $Z_k^n = T_k^n + D_k^n$: displacement of the p.p. (received messages)

AWGN UNDER MLE DECODING

- $\{\mathcal{V}_k^n\}$: **Voronoi cell of** T_k^n **in** μ^n.
- **Error probability under MLE decoding:**

$$p_e(n) = \mathbb{P}_n^0(Z_0^n \notin \mathcal{V}_0^n) = \mathbb{P}_n^0(D_0^n \notin \mathcal{V}_0^n) = \lim_{A\to\infty} \frac{\sum_k 1_{T_k^n \in B^n(0,A)} 1_{Z_k^n \notin \mathcal{V}_k^n}}{\sum_k 1_{T_k^n \in B^n(0,A)}}$$

- **Theorem 1-wgn Poltyrev [94]**

1. If $R < -\frac{1}{2}\log(2\pi e\sigma^2)$, **there exists a sequence of point processes** μ^n **(e.g. Poisson) with intensity** e^{nR} **s.t.**

$$p_e(n) \to 0, \quad n \to \infty$$

2. If $R > -\frac{1}{2}\log(2\pi e\sigma^2)$, **for all sequences of point processes** μ^n **with intensity** e^{nR},

$$p_e(n) \to 1, \quad n \to \infty$$

Proof of 2 [AB 08]

– $V_n(r)$: **volume of the** n**-ball or radius** r**.**

– **By monotonicity arguments, if** $|\mathcal{V}_0^n| = V_n(\sqrt{n}L_n)$**,**

$$I\!P_n^0(D_0^n \notin \mathcal{V}_0^n) \ \geq \ I\!P_n^0(D_0^n \notin B^n(0, \sqrt{n}L_n)) = I\!P_n^0\left(\frac{1}{n}\sum_{i=1}^{n} D_0^n(i)^2 \geq L_n^2\right)$$

– **By the SLLN,**

$$I\!P_n^0\left(\left|\frac{1}{n}\sum_{i=1}^{n} D_0^n(i)^2 - \sigma^2\right| \geq \epsilon\right) = \eta_\epsilon(n) \rightarrow_{n\rightarrow\infty} 0$$

– **Hence**

$$\begin{aligned} I\!P_n^0(D_0^n \notin \mathcal{V}_0^n) \ &\geq \ I\!P_n^0(\sigma^2 - \epsilon \geq L_n^2) - \eta_\epsilon(n) \\ &= \ 1 - I\!P_n^0(V_n(\sqrt{n(\sigma^2 - \epsilon)}) < |\mathcal{V}_0^n|) - \eta_\epsilon(n) \end{aligned}$$

Proof of 2 [AB 08] (*continued*)

– **By Markov ineq.**

$$\mathbb{P}_n^0(|\mathcal{V}_0^n| > V_n(\sqrt{n(\sigma^2-\epsilon)})) \leq \frac{\mathbb{E}_n^0(|\mathcal{V}_0^n|)}{V_n(\sqrt{n(\sigma^2-\epsilon)})}$$

– **By classical results on the Voronoi tessellation**

$$\mathbb{E}_n^0(|\mathcal{V}_0^n|) = \frac{1}{\lambda_n} = e^{-nR}$$

– **By classical results**

$$V_n(r) = \frac{\pi^{\frac{n}{2}} r^n}{\Gamma(\frac{n}{2}+1)} \sim \frac{\pi^{\frac{n}{2}} r^n}{\sqrt{\pi n}\left(\frac{n}{2e}\right)^{\frac{n}{2}}}$$

– **Hence**

$$\frac{\mathbb{E}_n^0(|\mathcal{V}_0^n|)}{V(\sqrt{n(\sigma^2-\epsilon)})} \sim e^{-nR} e^{-\frac{n}{2}\log(2\pi e(\sigma^2-\epsilon))} \rightarrow_{n\to\infty} 0$$

since $R > -\frac{1}{2}\log(2\pi e\sigma^2)$.

AWN DISPLACEMENT OF A POINT PROCESS

- **Same framework as above concerning the p.p.** μ^n**.**
- $\{D_k^n\}$**: i.i.d. sequence of centered displacements, independent of** μ^n**.**
- $D_k^n = (D_k^n(1), \ldots, D_k^n(n))$**: i.i.d. coordinates with a density** f **with well defined differential entropy**

$$h(D) = -\int_{\mathbb{R}} f(x) \log(f(x)) dx$$

- **If** D **is** $\mathcal{N}(0, \sigma^2)$**,** $h(D) = \frac{1}{2} \log(2\pi e \sigma^2)$

AWN UNDER TYPICALITY DECODING

- **Aim: For all n, find a partition $\{\mathcal{C}_k^n\}$ of $\mathbb{R}^n$, jointly stationary with μ^n such that**

$$p_e(n) = \mathbb{P}_n^0(D_0^n \notin \mathcal{C}_0^n) \to_{n\to\infty} 0$$

- **Theorem 1-wn**

1. If $R < -h(D)$, there exists a sequence of point processes μ^n (e.g. Poisson) with intensity e^{nR} and a partition s.t.

$$p_e(n) \to 0, \quad n \to \infty$$

2. If $R > -h(D)$, for all sequences of point processes μ^n with intensity e^{nR}, for all jointly stationary partitions,

$$p_e(n) \to 1, \quad n \to \infty$$

Proof of 1

- **Let μ^n be a Poisson p.p. with intensity e^{nR} with $R+h(D)<0$.**
- **For all n and δ, let**

$$A_\delta^n = \left\{ (y(1),\ldots,y(n)) \in \mathbb{R}^n \ : \ \left| -\frac{1}{n}\sum_{i=1}^n \log f(y(i)) - h(D) \right| < \delta \right\}$$

- **By the SLLN,** $\mathbb{P}_0^n((D_0^n(1),\ldots,D_0^n(n)) \in A_\delta^n) \to_{n\to\infty} 1$

Proof of 1 (*continued*)

- $\mathcal{C}_k^n$ **contains**
  - **all the locations** x **which belong to the set** $T_k^n + \mathcal{A}_\delta^n$ **and to no other set of the form** $T_l^n + \mathcal{A}_\delta^n$**;**
  - **all the locations** x **that are ambiguous and which are closer to** T_k^n **than to any other point;**
  - **all the locations which are uncovered and which are closer to** T_k^n **than to any other point.**

Proof of 1 (*continued*)

- **Let** $\widetilde{\mu}^n = \mu^n - \epsilon_0$ **under** $\mathbb{P}_n^0$
- **Basic bound:**

$$\mathbb{P}_n^0(D_0^n \notin \mathcal{C}_0^n) \leq \mathbb{P}_n^0(D_0^n \notin A_\delta^n) + \mathbb{P}_n^0(D_0^n \in A_\delta^n, \widetilde{\mu}^n(D_0^n - A_\delta^n) > 0)$$

- **The first term tends to 0 because of the SLLN.**
- **For the second, use Slivnyak's theorem to bound it from above by**

$$\begin{aligned} \mathbb{P}(\mu^n(D_0^n - A_\delta^n) > 0) &\leq \mathbb{E}(\mu^n(D_0^n - A_\delta^n)) \\ &= \mathbb{E}(\mu^n(-A_\delta^n)) = e^{nR}|A_\delta^n| \end{aligned}$$

Proof of 1 (*continued*)

- **But**

$$1 \geq \mathbb{P}(D_0^n \in A_\delta^n) = \int_{A_\delta^n} \prod_{i=1}^{n} f(y(i))dy = \int_{A_\delta^n} e^{n\frac{1}{n}\sum_{i=1}^{n}\log(f(y(i)))}dy$$

$$\geq \int_{A_\delta^n} e^{n(-h(D)-\delta)}dy = e^{-n(h(D)+\delta)}|A_\delta^n|$$

so that

$$|A_\delta^n| \leq e^{n(h(D)+\delta)}$$

- **Hence the second term is bounded above by**

$$e^{nR}e^{n(h(D)+\delta)} \longrightarrow_{n\to\infty} 0$$

since $R + h(D) < 0$**.**

EXAMPLES

- **Examples of A_δ^n sets for white noise with variance σ^2:**
  - **Gaussian case: difference of two concentric L_2 n–balls of radius approximately $\sqrt{n}\sigma$.**
  - **Symmetric exponential case: difference of two concentric L_1 n–balls of radius approximately $n\frac{\sigma}{\sqrt{2}}$.**
  - **Uniform case: n-cube of side $2\sqrt{3}\sigma$.**

ADDITIVE STATIONARY AND ERGODIC DISPLACEMENT OF A POINT PROCESS

- **Setting**
  - **Same framework as above concerning the p.p.** μ^n.
  - $\{\mathcal{D}\}_k$: **i.i.d. sequence of centered, stationary and ergodic displacement processes, independent of the p.p.s.**
  - **For all** n, $D_k^n = (\mathcal{D}_k(1), \ldots, \mathcal{D}_k(n))$ **with density** f^n **on** $\mathbb{R}^n$.

ADDITIVE STATIONARY AND ERGODIC DISPLACEMENT OF A POINT PROCESS (*continued*)

- $\mathcal{D}$: **with well defined differential entropy rate** $h(\mathcal{D})$
  - $H(D^n)$ **differential entropy of** $D^n = (\mathcal{D}(1), \ldots, \mathcal{D}(n))$
  - $h(\mathcal{D})$ **defined by**

$$h(\mathcal{D}) = \lim_{n\to\infty} \frac{1}{n} H(D^n) = \lim_{n\to\infty} -\frac{1}{n} \int_{\mathbb{R}^n} \ln(f^n(x^n)) f^n(x^n) dx^n.$$

- **Typicality sets**

$$A_\delta^n = \left\{ x^n = (x(1), \ldots, x(n)) \in \mathbb{R}^n \ : \ \left| -\frac{1}{n} \log(f^n(x^n)) - h(\mathcal{D}) \right| < \delta \right\}.$$

ASEN UNDER TYPICALITY DECODING

- **Theorem** 1-sen

1. **If $R < -h(\mathcal{D})$, there exists a sequence of point processes μ^n (e.g. Poisson) with intensity e^{nR} and a partition s.t.**

$$p_e(n) \to 0, \quad n \to \infty$$

2. **If $R > -h(\mathcal{D})$, for all sequences of point processes μ^n with intensity e^{nR}, for all jointly stationary partitions,**

$$p_e(n) \to 1, \quad n \to \infty$$

- **Proof: similar to that of the i.i.d. case.**

COLORED GAUSSIAN NOISE EXAMPLE

- $\{\mathcal{D}\}$ **regular stationary and ergodic Gaussian process with spectral density** $g(\beta)$**, covariance matrix** Γ_n**:**

$$\mathbb{E}[\mathcal{D}(i)\mathcal{D}(j)] = \Gamma_n(i,j) = r(|i-j|)$$

and

$$\mathbb{E}[\mathcal{D}(0)\mathcal{D}(k)] = \frac{1}{2\pi}\int_0^{2\pi} e^{ik\beta} g(\beta)\, d\beta.$$

COLORED GAUSSIAN NOISE EXAMPLE (*continued*)

– **Differential entropy rate:**

$$h(\mathcal{D}) = \frac{1}{2}\ln\left(2e\pi\exp\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln(g(\beta))d\beta\right)\right).$$

– **Typicality sets:**

$$A_\delta^n = \left|\frac{1}{n}(x^n)^t\Gamma_n^{-1}x^n - 1 + d(n)\right| < 2\delta,$$

with

$$d(n) = \frac{1}{n}\ln(\mathrm{Det}(\Gamma_n)) - \left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln(g(\beta))d\beta\right) \rightarrow_{n\to\infty} 0.$$

MARKOV NOISE EXAMPLE

- **Assume that $\{\mathcal{D}_n\}$ is a stationary Markov chain with values in $I\!R$, stationary distribution $\pi(x)dx$, mean 0 and with transition kernel $P(dy \mid x) = p(y \mid x)dy$, where $p(y \mid x)$ is a density on $I\!R$.**
- **If $(\mathcal{D}_1, \mathcal{D}_2)$ has a well defined differential entropy then**

$$h(\mathcal{D}) \; = \; -\int_{I\!R^2} \pi(x)p(y \mid x) \ln(p(y \mid x))dxdy = h(\mathcal{D}_2|\mathcal{D}_1) \; ,$$

with $h(U|V)$ the conditional entropy of V given U.

REPRESENTATION OF AWGN MLE ERROR PROB.

- **Theorem 2-wgn. Assume WGN and MLE,**
  - **If μ^n is stationary and ergodic, the probability of success is**

$$p_s(n) = \int\limits_{r \geq 0} \int\limits_{\vec{v} \in \mathbb{S}^{n-1}} \mathbb{P}_0^n(\mu^n(B_n(r\vec{v}, r)) = 0) \frac{g_\sigma^n(r)}{A_{n-1}} d\vec{v} dr \ ,$$

withA_{n-1}, the area of the n-sphere of radius 1: $\mathbb{S}^{n-1}(1)$, and

$$g_\sigma^n(r) = 1_{r>0} e^{-\frac{r^2}{2\sigma^2}} \frac{1}{2^{n/2}} \frac{r^{n-1}}{\sigma^n} \frac{2}{\Gamma(n/2)}.$$

  - **If μ^n is Poisson,**

$$p_s(n) = \int\limits_0^\infty e^{-\lambda_n V_B^n(r)} g_\sigma^n(r) dr = \int\limits_0^\infty e^{-\lambda_n V_B^n(r\sigma)} g_1^n(r) dr \ ,$$

with $V_B^n(r)$ the volume of the ball $B^n(0, r)$.

PROOF

- $x \in \mathcal{V}_0^n$ **iff the open ball** $B^n(x, |x|)$ **has no point of** μ^n**:**

$$p_s(n) \;=\; \mathbb{P}_0^n\left(D_0^n \in \mathcal{V}_0^n\right) = \mathbb{P}_0^n\left(\mu^n(B^n(D_0^n, |D_0^n|)) = 0\right).$$

- $|D_0^n|$ **has for density** $g_\sigma^n(x) = g_1^n(x/\sigma)/\sigma$ **on** $I\!R^+$**.**
- **Given that** $|D_0^n| = r$**, the angle is uniform on** $\mathbb{S}_{n-1}$**.**
- **In the Poisson case, from Slyvniak's theorem,**

$$\mathbb{P}_0^n(\mu^n(B^n(r\vec{v}, r)) = 0) = \mathbb{P}(\mu^n(B^n(r\vec{v}, r)) = 0) = e^{-\lambda_n V_B^n(r)}.$$

REPRESENTATION OF ASEN MLE ERROR PROB.

- **Stun of** $s^n \in I\!R^n$

$$\mathbb{S}(s^n) = -\frac{1}{n}\ln(f^n(s^n)).$$

- **Key observation:**

$$\mathbb{S}(x^n - T_k^n) > \mathbb{S}(x^n),\ \forall k \neq 0 \quad \Leftrightarrow \quad (\mu^n - \epsilon_0)(F(x^n)) = 0$$

$$\begin{aligned} F(x^n) &= \{y^n \in I\!R^n \textbf{ s.t. } \mathbb{S}(x^n - y^n) \leq \mathbb{S}(x^n)\} \\ &= \{y^n \in I\!R^n \textbf{ s.t. } -\frac{1}{n}\ln(f^n(x^n - y^n)) \leq -\frac{1}{n}\ln(f^n(x^n))\}. \end{aligned}$$

- $\mathrm{Vol}(F(x^n))$ **only depends on** $\mathbb{S}(x^n) = -\frac{1}{n}\ln(f^n(x^n))$**. Let**

$$V_f^n(r) = \mathrm{Vol}\ \{y^n \in I\!R^n \textbf{ s.t. } -\frac{1}{n}\ln(f^n(y^n)) \leq r\}.$$

REPRESENTATION OF ASEN MLE ERROR PROB. (*continued*)

- **Theorem 2-sen.** **Assume MLE,**
  - **If μ^n is stationary and ergodic, the probability of success under MLE is**

$$p_s(n) \geq \int\limits_{x^n \in \mathbb{R}^n} \mathbb{P}_0^n((\mu^n - \epsilon_0)(F(x^n)) = 0) f^n(x^n) dx^n \ .$$

  - **If μ^n is Poisson, then**

$$p_s(n) \geq \int\limits_{r \in \mathbb{R}} \exp\left(-\lambda_n V_f^n(r)\right) \rho^n(dr) \ ,$$

where $\rho^n(dr)$ is the law of the random variable $-\frac{1}{n}\ln(f^n(D^n))$.

REPRESENTATION OF ASEN MLE ERROR PROB. (*continued*)

- **Terminology in relation with**

$$\int_{r\in \mathbb{R}} \exp\left(-\lambda_n V_f^n(r)\right) \rho^n(dr) \ ,$$

– **Normalized entropy density of D^n: RV $-\frac{1}{n}\ln(f^n(D^n))$**

– **Entropy spectrum of D^n: law $\rho^n(dr)$ on $\mathbb{R}$:**

– **Stun level sets of D^n:**

$$\mathcal{S}_f^n(r) = \{y^n \in \mathbb{R}^n \text{ s.t. } -\frac{1}{n}\ln(f^n(y^n)) \le r\}$$

– **Stun level volume for r: volume $V_f^n(r)$ of $\mathcal{S}_f^n(r)$.**

REPRESENTATION OF ASEN MLE ERROR PROB. (*continued*)

- **The Stun cell $\mathcal{L}_k^n(\mathcal{D})$ of point T_k^n:**

$$\begin{aligned}\mathcal{L}_k^n(\mathcal{D}) &= \{x^n \textbf{ s.t. } \mathbb{S}(x^n - T_k^n) < \inf_{l \neq k} \mathbb{S}(x^n - T_l^n)\} \\ &\quad \cup \{x^n \textbf{ s.t. } \mathbb{S}(x^n - T_k^n) = \mathbb{S}(x^n - T_l^n) \textbf{ for some } l \neq k\} \cap \mathcal{V}_k^n \ .\end{aligned}$$

  - **The locations x^n with a stun (w.r.t. f^n) to T_k^n smaller than that to any other point;**
  - **The locations x^n with an ambiguous stun (this includes the case where $\mathbb{S}(x^n - T_k^n) = \infty$ for all k) which are closer to T_k^n than to all other point.**

These cells form a decomposition of the Euclidean space.

ERROR EXPONENTS

– $\mathcal{D}$, **displacement process;**

– μ^n, **stationary and ergodic point process with intensity** e^{nR}**,** $R = -h(\mathcal{D}) - \ln(\alpha)$**,** $\alpha > 1$**;**

– $\mathcal{C}^n = \{\mathcal{C}_k^n\}_k$ **jointly stationary partition.**

- **Associated error probability:**

$$p_e^{pp}(n, \mu^n, \mathcal{C}^n, \alpha, \mathcal{D})$$

- **Optimal error:**

$$p_{e,opt}^{pp}(n, \alpha, \mathcal{D}) = \inf_{\mu^n, \mathcal{C}^n} p_e^{pp}(n, \mu^n, \mathcal{C}^n, \alpha, \mathcal{D})$$

ERROR EXPONENTS (*continued*)

- **Error exponents:**

$$\bar{\eta}(\alpha, \mathcal{D}) = \limsup_{n} -\frac{1}{n} \log p^{pp}_{e,opt}(n, \alpha, \mathcal{D}) \ ,$$

$$\underline{\eta}(\alpha, \mathcal{D}) = \liminf_{n} -\frac{1}{n} \log p^{pp}_{e,opt}(n, \alpha, \mathcal{D}) \ ,$$

ERROR EXPONENTS (*continued*)

- **Each particular point process sequence $\mu = \{\mu^n\}$ and partition $\mathcal{C} = \{\mathcal{C}^n\}$ provides a lower bound on $\underline{\eta}(\alpha, \mathcal{D})$:**

$$\underline{\pi}(\mu, \mathcal{C}, \alpha, \mathcal{D}) = \liminf_n -\frac{1}{n} \log p_e^{pp}(n, \mu^n, \mathcal{C}^n, \alpha, \mathcal{D})$$

- **Main focus in what follows:**

$$\underline{\eta}(\alpha, \mathcal{D}) \; \geq \; \underline{\pi}(\textbf{Poisson}, \mathcal{L}(\mathcal{D}), \alpha, \mathcal{D}) \quad \textbf{(random coding e-e)}$$

POISSON LOWER BOUNDS ON WGN ERROR EXPONENTS

Theorem 3-wgn-Poisson [Poltyrev 94]
In the $\mathcal{D}$=AWGN case,

1. **For** $1 < \alpha < \sqrt{2}$**:** $\underline{\pi}(\textbf{Poisson}, \mathcal{L}(\mathcal{D}), \alpha, \mathcal{D}) = \frac{\alpha^2}{2} - \frac{1}{2} - \log(\alpha)$
2. **For** $\alpha > \sqrt{2}$**:** $\underline{\pi}(\textbf{Poisson}, \mathcal{L}(\mathcal{D}), \alpha, \mathcal{D}) = \frac{1}{2} - \log(2) + \log(\alpha)$

- **Obtained from the Poisson–Voronoi case with**

$$R = -\frac{1}{2}\log(2\pi e\sigma^2\alpha^2), \quad \alpha > 1$$

Proof of Poltyrev's AWGN exponent in [AB 08]

- **From Palm representation of $p_e(n,\alpha)$:**

$$p_e(n,\alpha) = \int\limits_0^\infty \left(1 - e^{-\lambda_n V_n(r)}\right) g_n^\sigma(r)dr$$

with $g_n^\sigma(v\sigma\sqrt{n}) = e^{-n\left(\frac{v^2}{2}-\frac{1}{2}-\log(v)+o(1)\right)}$

- **Since** $\lambda_n = e^{\frac{n}{2}\log 2\pi e\sigma^2\alpha^2}$, $1 - e^{-\lambda_n V_n(v\sigma\sqrt{n})} = e^{-n\left((\log\alpha - \log v)^+ + o(1)\right)}$ **and**

$$p_e(n,\alpha) = \int\limits_0^\infty e^{-n(\frac{v^2}{2}-\frac{1}{2}-\log v + (\log\alpha-\log v)^+ + o(1))}dv$$

- **The result follows from the minimization of the function**

$$\frac{v^2}{2} - \frac{1}{2} - \log v + (\log\alpha - \log v)^+.$$

POISSON LOWER BOUNDS ON SEN ERROR EXPONENTS

- **Setting Stationary ergodic noise $\mathcal{D}$**

– $H(D^n)$ **differential entropy of** $D^n = (\mathcal{D}_1, \ldots, \mathcal{D}_n)$

– $h(\mathcal{D})$ **differential entropy rate of** $\{\mathcal{D}\}$**.**

We have

$$\begin{aligned} h(\mathcal{D}) &= \lim_{n\to\infty} \frac{1}{n} H(D^n) = \lim_{n\to\infty} -\frac{1}{n} \int_{\mathbb{R}^n} \ln(f^n(x^n)) f^n(x^n) dx^n \\ &= \lim_{n\to\infty} \frac{1}{n} \int_{\mathbb{R}} r \rho^n(dr) \,. \end{aligned}$$

POISSON LOWER BOUNDS ON SEN ERROR EXPONENTS (*continued*)

- **SEN Assumption:**
  The family of measures $\rho^n(.)$ satisfies an LDP with good rate function $I(x)$

- **Example 1: Gärtner-Ellis: if $\{-\ln(f^n(D^n)\}$ satisfies the conditions of the Gärtner-Ellis Theorem, namely if the limit**

$$\lim_{n\to\infty} \frac{1}{n} \ln \left(\mathbb{E} \left(\left(f^n(D^n) \right)^{-\theta} \right) \right) = G(\theta)$$

exists and satisfies appropriate continuity properties, then the family of measures $\rho^n(.)$ satisfies a LDP with good rate function

$$I(x) = \sup_{\theta} \left(\theta x - G(\theta) \right).$$

- **Example 2: LDP on empirical measures: wn case**
  - S **the support of** f
  - $K(\tau \mid\mid \phi)$ **relative entropy (or Kullback-Leibler divergence) of the probability law** $\tau(dx)$ **w.r.t. the probability law** $\phi(dx) = f(x)dx$**:**

$$K(\tau \mid\mid \phi) = \int_{\mathbb{R}} \ln(r(x))\, r(x) f(x) dx,$$

with $r = \frac{d\tau}{d\phi}$**. This is** ∞ **unless** τ **is absolutely continuous w.r.t.** ϕ**, i.e.** τ **admits a density** g **such that** $g(y) = 0$ **when** $f(y) = 0$ **for a.a.** y**. In this case,**

$$K(\tau \mid\mid \phi) = \int_{S} \ln\left(\frac{g(x)}{f(x)}\right) g(x) dx.$$

POISSON LOWER BOUNDS ON SEN ERROR EXPONENTS (*continued*)

- From **Sanov's theorem**, the empirical measures

$$\nu^n = \frac{1}{n}\sum_{i=1}^{n} \epsilon_{\mathcal{D}_i}$$

are $\mathbb{M}_1(S)$-valued random variables which satisfy and LDP with good and convex rate function $K(. \,||\, \phi)$

- **From the contraction principle, if the function $x \to \ln(f(x))$ from S to $I\!\!R$ is continuous and bounded, then the family of measures $\rho^n_{\mathcal{D}}$ on $I\!\!R$ satisfies an LDP with good and convex rate function**

$$\begin{aligned} I(u) &= \inf_{\tau\in\mathbb{M}_1(S):\ -\int_S \ln(f(x))\tau(dx)=u} K(\tau||\phi) \\ &= u - \sup_{\tau\in\mathbb{M}_1(S):\ K(\tau||\phi)+h(\tau)=u} h(\tau). \end{aligned}$$

POISSON LOWER BOUNDS ON SEN ERROR EXPONENTS (*continued*)

- **Example 3: LDP on empirical measures: Markov case**
- $S \subseteq \mathbb{R}^2$: **support of the measure on** $\mathbb{R}^2$ **with density** $a(x)p(y|x)$.
- **Under technical conditions, the empirical measures**

$$\frac{1}{n}\sum_{i=1}^{n} \epsilon_{\mathcal{D}_i,\mathcal{D}_{i+1}}$$

satisfy an LDP on $\mathbb{M}_1(S)$ **with good and convex rate function**

$$I(\tau) = \begin{cases} K(\tau || \tau_1 \otimes P) & \textbf{if } \tau \in \mathrm{Sym}\mathbb{M}_1(S) \\ \infty & \textbf{otherwise}. \end{cases}$$

– K: **Kullback-Leibler divergence of measures on** $I\!R^2$

– τ_1: **marginals of** τ; $\tau_1 \otimes P$: **the measure** $\tau_1(dx)P(x,y)dy$.

POISSON LOWER BOUNDS ON SEN ERROR EXPONENTS (*continued*)

- **If** $(x, y) \to \ln(p(y \mid x))$ **from** S **to** $I\!R$ **is continuous and bounded, Then** $\rho_{\mathcal{D}}^n$ **satisfies an LDP with good and convex rate function**

$$\begin{aligned} I(u) &= \inf_{\tau \in \mathrm{Sym}\mathbb{M}_1(S):\ \int_{\pi(x)p(y|x)>0} \ln(p(y|x))\tau(dxdy)=u} K(\tau || \tau_1 \otimes P) \\ &= u - \sup_{\tau \in \mathrm{Sym}\mathbb{M}_1(S):\ K(\tau||\tau_1 \otimes P)+h(\tau_2|\tau_1)=u} h(\tau_2|\tau_1), \end{aligned}$$

provided the last function is convex and admits an essentially smooth Fenchel-Legendre transform

POISSON LOWER BOUNDS ON SEN ERROR EXPONENTS (*continued*)

- **Lemma** [Volume Exponent] [AB 10]
  Assume that ρ^n satisfies a LDP with good rate function I. Then the stun level volumes verify:

$$\sup_{s<u}(s - I(s)) \leq \liminf_{n\to\infty} \frac{1}{n} \ln(V_{\mathcal{D}}^n(u)) \leq \limsup_{n\to\infty} \frac{1}{n} \ln(V_{\mathcal{D}}^n(u)) \leq \sup_{s\leq u}(s - I(s)).$$

  The function

$$J(u) = \sup_{s\leq u}(s - I(s)),$$

  the volume exponent, is upper semicontinuous.

POISSON LOWER BOUNDS ON SEN ERROR EXPONENTS (*continued*)

Theorem 3-sen-Poisson [Main Result], [AB 10]
Under the SEN assumption (LDP) on $\mathcal{D}$,

$$\underline{\pi}(\mathbf{Poisson}, \mathcal{L}(\mathcal{D}), \alpha, \mathcal{D}) \geq \inf_r \{F(r) + I(r)\} \quad ,$$

where

– $I(r)$ **is the rate function of the noise entropy spectrum** ρ^n

– $F(r)$ **is**

$$F(r) = (\ln(\alpha) + h(\mathcal{D}) - J(r))^+ \quad ,$$

with

$$J(r) = \sup_{s \leq r}(s - I(s))$$

the noise volume exponent.

IDEA OF PROOF

- **Use the Palm representation and the bound**

$$1 - e^{-\lambda_n V_f^n(r)} \leq \min(1, \lambda_n V_f^n(r))$$

to write

$$p_e(n) \leq \int_{r>0} e^{-n\phi_n(r)} \rho^n(dr),$$

with

$$\phi_n(r) = \left(\ln(\alpha) + h(\mathcal{X}) - \frac{1}{n} \ln(V_f^n(r)) \right)^+ .$$

- **Leverage the LDP on ρ^n:**
Use the Laplace–Varadhan integral lemma
(more precisely an extension of this lemma in Varadhan 84).

SYMMETRIC EXPONENTIAL WN EXAMPLE

- **LDP**

$$\mathbb{E}\left(f(X)^{-\theta}\right) = (\sqrt{2}\sigma)^{\theta}\mathbb{E}\left(\exp\left(\theta\frac{|X|\sqrt{2}}{\sigma}\right)\right) = (\sqrt{2}\sigma)^{\theta}\frac{1}{1-\theta}.$$

So, the LDP assumption holds with the good rate function:

$$I(x) = \sup_{\theta}\left(\theta x - \theta\ln(\sqrt{2}\sigma) + \ln(1-\theta)\right) = x - h(X) - \ln(x - \ln(\sqrt{2}\sigma))$$

- **Error exponent**

$$\underline{\pi}(\textbf{Poisson}, \mathcal{L}(\mathcal{D}), \alpha, \mathcal{D}) \geq \begin{cases} \alpha - 1 - \ln\alpha & \textbf{if } 1 \leq \alpha < \sqrt{2} \\ \sqrt{2} - 1 - \ln 2 + \ln\alpha & \textbf{if } \sqrt{2} \leq \alpha\textbf{.} \end{cases}$$

COLORED GAUSSIAN NOISE EXAMPLE

- **LDP**

From the Grenander–Szegö Theorem, the Gärtner-Ellis Theorem holds with

$$G(\theta) = \frac{\theta}{2}\ln(2\pi) - \frac{1}{2}\ln(1-\theta) + \frac{\theta}{2}\ln\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln(g(\beta))d\beta\right) ,$$

when $\theta < 1$ and $G(\theta) = \infty$ for $\theta > 1$. So, the LDP assumption hods with the good rate function

$$I(x) \ = \ x - h(\mathcal{D}) - 1/2\ln(2x - \ln(2\pi\sigma^2)) \ ,$$

- **Error exponent:**

$$\underline{\pi}(\alpha, \mathcal{D}) \geq \begin{cases} \frac{\alpha^2}{2} - \frac{1}{2} - \ln\alpha & \textbf{if } 1 \leq \alpha < \sqrt{2} \\ \frac{1}{2} - \ln 2 + \ln\alpha & \textbf{if } \sqrt{2} \leq \alpha < \infty \end{cases} .$$

OTHER EXAMPLES STUDIED

- **Uniform white noise**
- **Markov noise**

AWGN CAPACITY AND ERROR EXPONENTS WITH CONSTRAINTS

- M, n **positive integers.**
- $\mathcal{C}$ **an** (M, n) **code:**

$$\begin{bmatrix} t_1(1) & t_1(2) & \cdots & t_1(n) \\ \vdots & & & \vdots \\ t_M(1) & t_M(2) & \cdots & t_M(n) \end{bmatrix}$$

- **Rate of the code:** $\frac{1}{n}\ln(M)$
- **Power constraint:** $\frac{1}{n}\sum_{i=1}^{n} t_m(i)^2 \leq P$ **for all** m;
  i.e. all codewords belong to $B^n(0, \sqrt{nP})$.

AWGN CAPACITY AND ERROR EXPONENTS WITH CONSTRAINTS (*continued*)

- W **uniform on** $\{1,\ldots,M\}$.
- **Transmitter sends** $(T(1),\ldots,T(n)) = (t_W(1),\ldots,t_W(n))$
- **The channel adds an independent noise** $(D(1),\ldots,D(n))$ **where coordinates** $D(i)$ **are i.i.d.** $\mathcal{N}(0,\sigma^2)$.
- **The receiver gets** $(Z(1),\ldots,Z(n)))$ **with** $Z(i) = T(i) + D(i)$.
- **MLE decoding:**

$$\widehat{W} = \mathrm{argmin}_m \sum_{i=1}^{n} (Z(i) - t_m(i))^2$$

and

$$p_e(\mathcal{C}) = P(\widehat{W} \neq W)$$

AWGN CAPACITY AND ERROR EXPONENTS WITH CONSTRAINTS (*continued*)

- **Shannon capacity of the AWGN channel**

$$C = \frac{1}{2}\log\left(1 + \frac{P}{\sigma^2}\right)$$

– **If $R < C$, there exists a sequence of (e^{nR}, n) codes $\mathcal{C}_n$ s.t.**

$$p_e(\mathcal{C}_n) \to 0, \quad n \to \infty$$

– **If $R > C$, for all sequences of (e^{nR}, n) codes $\mathcal{C}_n$**

$$\liminf_n p_e(\mathcal{C}_n) = 1, \quad n \to \infty$$

Error exponents for the AWGN channel

– $P_{e,opt}(n, R, A)$: **infimum of** $P_e(\mathcal{C})$ **over all codebooks in** $\mathbb{R}^n$ **of rate at least** R **when the signal-to-noise ratio is** $A^2 = P/\sigma^2$**.**

$$\mathcal{E}(n, R, A) = -\frac{1}{n} \log P_{e,opt}(n, R, A) \ .$$

– **Error exponent:**

$$\begin{aligned} \bar{\mathcal{E}}(R, A) &= \limsup_n \mathcal{E}(n, R, A), \\ \underline{\mathcal{E}}(R, A) &= \liminf_n \mathcal{E}(n, R, A). \end{aligned}$$

ASEN CAPACITY WITH CONSTRAINTS

- **Same setting but with a stationary and ergodic noise $\mathcal{D}$**
- **Shannon capacity of the ASEN channel**

$$C_P(\mathcal{D}) = \lim_{n\to\infty} \frac{1}{n} \sup_{T^n,\ \mathbb{E}(\sum_{i=1}^n |T_i|^2)<nP} I(T^n, T^n + D^n),$$

where

– $I(X,Y)$ **is the mutual information of** X **and** Y**:**
$I(X,Y) = h(X) + h(Y) - h(X,Y)$

– **the supremum bears on all distribution functions for** $T^n \in \mathbb{R}^n$ **such that** $\mathbb{E}(\sum_{i=1}^n |T_i|^2) < nP$**.**

RELATION BETWEEN CAPACITY WITH AND WITHOUT CONSTRAINTS

- **Additive stationary and ergodic noise $\mathcal{D}$ with**
  - **Differential entropy rate:** $h(\mathcal{D})$
  - **Variance:** $\sigma^2 = \mathbb{E}(\mathcal{D}(0)^2)$
  - **Capacity:** $c(\mathcal{D}) = -h(\mathcal{D})$
- **Lemma** **[Shannon]**

Under the above assumptions

$$\frac{1}{2}\ln(2\pi e P) + c(\mathcal{D}) \leq C_P(\mathcal{D}) \leq \frac{1}{2}\ln(2\pi e(P+\sigma^2)) + c(\mathcal{D})$$

and

$$C_P(\mathcal{D}) = \frac{1}{2}\ln(2\pi e P) + c(\mathcal{D}) + O(1/P)\,, \quad P \to \infty.$$

ASEN ERROR EXPONENTS WITH CONSTRAINTS

- **As above:**

$$\mathcal{E}(n, R, P, \mathcal{D}) = -\frac{1}{n} \log P_{e,opt}(n, R, P, \mathcal{D}) \ .$$

- **Error exponent:**

$$\begin{aligned} \bar{\mathcal{E}}(R, P, \mathcal{D}) &= \limsup_n \mathcal{E}(n, R, P, \mathcal{D}), \\ \underline{\mathcal{E}}(R, P, \mathcal{D}) &= \liminf_n \mathcal{E}(n, R, P, \mathcal{D}). \end{aligned}$$

RELATION BETWEEN ERROR EXPONENTS WITH AND WITHOUT CONSTRAINTS

– $\underline{\mathcal{E}}(R, P, \mathcal{D})$: **e-e for rate R, power constraint P, noise $\mathcal{D}$.**

– $\underline{\pi}(\mu, \mathcal{L}(\mathcal{D}), \alpha, \mathcal{D})$: **e-e for μ with intensity e^{nR},**
$R = -h(\mathcal{D}) - \ln(\alpha)$, **noise $\mathcal{D}$ and MLE.**

- **Theorem [AB 10]**
  Under the above assumptions,
  for all μ and $\mathcal{C}$ with $\alpha > 1$, for all $P > 0$,

$$\underline{\mathcal{E}}\left(\frac{1}{2}\ln(2\pi e P) - h(\mathcal{D}) - \ln(\alpha), P, \mathcal{D}\right) \geq \underline{\pi}(\mu, \mathcal{L}(\mathcal{D}), \alpha-, \mathcal{D}).$$

- **Matches Shannon's random error exponent and expurgated exponent in AWGN**

IDEA OF PROOF

- **Consider as codebook the restriction of the p.p. μ^n of intensity $e^{n(-h(\mathcal{D})-\ln(\alpha))}$ in the ball of radius $\sqrt{nP}$.**
- **The mean number of codewords is $e^{nR+o(1)}$ with**

$$R = -h(\mathcal{D}) - \ln(\alpha) + \frac{1}{2}\ln(2\pi e P).$$

- **Compare the error in this codebook and in the stationary point process.**

IDEA OF PROOF (*continued*)

- **For all n,**

$$p_e^{pp}(n,\mu^n,\mathcal{C}^n,\alpha,\mathcal{D}) = \frac{\mathbb{E}^n\left(\sum_{k \textbf{ s.t. } T_k^n \in B^n(0,\sqrt{nP})} p_{e,k}\right)}{e^{-nh(\mathcal{D})}e^{-n\ln(\alpha)}V_B^n(\sqrt{nP})}\ ,$$

with $p_{e,k}$ the probability that $T_k^n + D_k^n \notin \mathcal{C}_k^n$ given $\{T_l^n, \mathcal{C}_l^n\}_l$.

IDEA OF PROOF (*continued*)

- **Hence, for all $\gamma > 0$,**

$$p_e^{pp}(n, \mu^n, \mathcal{C}^n, \alpha, \mathcal{D})$$

$$\geq \frac{\mathbb{E}^n \sum_{k \text{ s.t. } T_k^n \in B^n(0,\sqrt{nP})} p_{e,k} 1_{\mu^n(B^n(0,\sqrt{nP})) \geq (2\pi eP)^{\frac{n}{2}} e^{-nh(\mathcal{D})} e^{-n\ln(\alpha+\gamma)}}}{e^{-nh(\mathcal{D})} e^{-n\ln(\alpha)} V_B^n(\sqrt{nP})}$$

$$\geq \mathbb{P}^n \left(\mu^n(B^n(0,\sqrt{nP})) \geq (2\pi eP)^{\frac{n}{2}} e^{-nh(\mathcal{D})} e^{-n\ln(\alpha+\gamma)} \right)$$

$$p_{e,opt}(n, \frac{1}{2}\ln(2\pi eP) - h(\mathcal{D}) - \ln(\alpha+\gamma), P, \mathcal{D}) e^{-n\ln(\alpha+\gamma)} e^{n\ln(\alpha)} \frac{(2\pi eP)^{\frac{n}{2}}}{V_B^n(\sqrt{nP})}.$$

IDEA OF PROOF (*continued*)

Hence

$$
\begin{aligned}
& -\frac{1}{n} \ln \left(p_e^{pp}(n, \mu^n, \mathcal{C}^n, \alpha, \mathcal{D}) \right) \\
\leq \; & -\frac{1}{n} \ln \left(p_{e,opt}(n, \frac{1}{2} \ln(2\pi e P) - h(\mathcal{D}) - \ln(\alpha + \gamma), P, \mathcal{D}) \right) \\
& -\frac{1}{n} \ln \left(\mathbb{P}^n \left(\mu^n(B^n(0, \sqrt{nP})) \geq (2\pi e P)^{\frac{n}{2}} e^{-nh(\mathcal{D})} e^{-n(\alpha+\gamma)} \right) \right) \\
& - \ln(\alpha) + \ln(\alpha + \gamma) - \frac{1}{n} \ln \left(\frac{(2\pi e P)^{\frac{n}{2}}}{V_B^n(\sqrt{nP})} \right) \quad .
\end{aligned}
$$

Proof concluded when taking

– **a liminf in** n**;**

– **a lim in** γ**.**

MATERN POINT PROCESSES FOR WGN

- **Built from a Poisson processes** μ_n **of rate** $\lambda_n = e^{nR}$ **where** $R = \frac{1}{2}\ln\frac{1}{2\pi e\alpha^2\sigma^2}$
- **Exclusion radius** $(\alpha - \epsilon)\sigma\sqrt{n}$
- **The intensity of this Matérn p.p. is**

$$\widetilde{\lambda}_n = \lambda_n e^{-\lambda_n V_B^n((\alpha-\epsilon)\sigma\sqrt{n})}$$

- **We have**

$$\frac{\widetilde{\lambda}_n}{\lambda_n} \to_{n\to\infty} 1.$$

MATERN LOWER BOUNDS ON WGN ERROR EXPONENTS

$$\underline{\eta}(\alpha, \mathcal{D}) \geq \underline{\pi}(\textbf{Poisson}, \mathcal{L}(\mathcal{D}), \alpha, \mathcal{D}) \quad \textbf{(random coding e-e)}$$
$$\underline{\eta}(\alpha, \mathcal{D}) \geq \underline{\pi}(\textbf{Matern}, \mathcal{L}(\mathcal{D}), \alpha, \mathcal{D}) \quad \textbf{(expurgated e-e)}$$

Theorem 3-wgn-Matérn [AB 08] **In the $\mathcal{D}$=AWGN case**

3 For $\alpha > 2$: $\underline{\pi}(\textbf{Matérn}, \mathcal{L}(\mathcal{D}), \alpha, \mathcal{D}) \geq \frac{\alpha^2}{8}$

- **Matches Poltyrev's expurgated error exponent** Poltyrev [94]

MATERN POINT PROCESSES FOR SEN

- **Assume for simplicity that $f^n(x^n) = f^n(-x^n)$.**
- **If two points S and T of the Poisson point process μ^n are such that $\mathbb{S}(T, S) < \xi$,, then T is discarded.**
- **The surviving points form the Matérn-$\mathcal{D}$-ξ point process $\widehat{\mu}^n$.**
- **Lemma The probability of error for the Matérn-$\mathcal{D}$-ξ point process satisfies the bound**

$$p_e(n) \le \int_{x^n \in \mathbb{R}^n} \min \left(1, \lambda_n \int_{y^n \in \mathbb{R}^n} 1_{\mathbb{S}(y^n,0) \ge \xi} \, 1_{\mathbb{S}(x^n,y^n) \le \mathbb{S}(x^n,0)} \; dy^n \right) f^n(x^n) dx^n.$$

SYMMETRIC EXPONENTIAL EXAMPLE

- **For $\mathcal{D}$ symmetric exponential**

$$\underline{\pi}(\mathbf{Mat}, \mathcal{L}(\mathcal{D}), \alpha, \mathcal{D}) \geq \begin{cases} \alpha - \ln(\alpha) - 1 & \textbf{for } \alpha \leq 2 \\ \ln(\alpha) + 1 - 2\ln(2) & \textbf{for } 2 \leq \alpha \leq 4 \\ \frac{\alpha}{2} - \ln(\alpha) - 1 + 2\ln(2) & \textbf{for } \alpha \geq 4. \end{cases}$$

CONCLUSION, FUTURE WORK

Bridge between
Information Theory and Stochastic Geometry

New viewpoint on error exponents
New stochastic geometry problems in high dimension

Other point processes to be investigated:
Matérn, Gibbs, Determinantal, etc.

VARADHAN'S LEMMA

- Theorem 2.3 in Varadhan 84
- P_ϵ satisfies a LDP with good rate function $I(.)$.
- F_ϵ non negative and such that

$$\liminf_{\epsilon \to 0, y \to x} F_\epsilon(y) \geq F(x), \quad \forall x$$

with F lower semicontinuous.

- Then

$$\liminf_{\epsilon \to 0} -\epsilon \ln \left(\int e^{-\frac{F_\epsilon(x)}{\epsilon}} P_\epsilon(dx) \right) \geq \inf\{F(x) + I(x)\}.$$

Szego's Theorem

- **Under technical conditions, the eigenvalues** $\tau(i, n)$, $i = 0, \ldots, n-1$ **of the covariance matrix** Γ^n **are such that**

$$\lim \frac{1}{n} \sum_{i=0}^{n} F(\tau(i, n)) = \frac{1}{2\pi} \int_{0}^{2\pi} F(f(\beta)) d\beta.$$

- **Key relations**

$$r_k = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-ik\beta} g(\beta) d\beta.$$

$$g(\beta) = \sum_{k=-\infty}^{\infty} t_k e^{ik\beta}, \quad \beta \in [0, 2\pi].$$