# A Posteriori Error Estimation for the Nonlinear Diffusion Equations

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# A posteriori error estimation

- A posteriori error estimation for finite volume discretization
  - Morley-type interpolation of the original piecewise constant finite volume approximation for Laplace Equation (S. Nacaise)
  - The discrete duality finite volume (DDFV) discretization for Laplace equation (P. Omnes et al.)
  - Postprocessed approximation based on the flux through the segment of primal cells (M. Vohralík)
- A posteriori error estimation for the nonlinear equations
  - Mixed finite element discretization for nonlinear diffusion equations ( D. Kim et al.)
  - Finite element method for $p$-Laplace (L. El Alaoui et al.).

# Nonlinear diffusion equations

Let $\Omega$ be an open bounded polygonal subset of $\mathbb{R}^2$, $\Gamma = \partial\Omega$, $f$ be a given function from $\Omega$ to $\mathbb{R}$ and $H$ be a given function on $\mathbb{R}$. We consider the following nonlinear equation:

$$\begin{cases} -\mathrm{div}(H(\hat{u})\nabla\hat{u})(x) & = f(x), \quad x \in \Omega \\ \hat{u}(x) & = 0, \qquad x \in \Gamma \end{cases} \tag{1}$$

Where H is global Lipschitz and bounded with the positive constants, $f \in L^2(\Omega)$.

# The Scheme

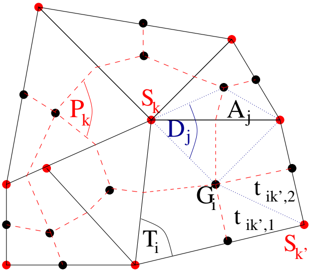
We will use the fixed point method to linearize and the DDFV discretization to discretize the equation.

- The unknowns of the sheme were located at the centers and the vertices of mesh.
- Give $u_h^{m,*}$, find the $u_h^{m+1}$:

$$-(\nabla_h^T \cdot H(u_h^{m,*})(\nabla_h^D u_h^{m+1}))_i = f_i^T$$

$$-(\nabla_h^P \cdot H(u_h^{m,*})(\nabla_h^D u_h^{m+1}))_k = f_k^P,$$

$$u_{h,j}^{m,*} = \frac{(u^{m,*})_{h,k_2(j)}^P + (u_h^{m,*})_{h,k_1(j)}^P}{2}.$$



# Estimate

We reconstruct the postprocessed approximation M base on the flux through of the edges in the scheme. There holds

$$\|H(\hat{u})\nabla\hat{u} - \nabla M\|_{L^2(\Omega)} \leq \eta_D + \eta_L.$$

- the estimation includes two terms: discretization and linearization estimators
- When the number of iteration is large enough, the linearization estimator is much smaller than the discretization one. Then, the estimation is almost equal to the discretization estimator.

# Numerical experiment

We show here some numerical results obtained on a domain $\Omega = ]-1;1[\times]-1;1[\backslash[0;1]\times[-1;0]$. The exact solution is $\hat{u}(r,\theta) = r^{2/3} sin(2\theta/3)$. Our tests have two parts.

- Discuss our stopping criterion and classical stopping one on some different functions $H$.
- Compare the estimated and actual errors, the effectivity indices for a uniform and an adaptive refinement.

# Compare stopping criterion

- Classical stopping criterion

$$\frac{\left(\sum_{i\in[1,I]}|T_i|\left[\sum_{j\in\partial T_i}\frac{|A_j|}{T_i}F_{i,j}(u_h^{n+1},u_h^{n+1})-f_i^T\right]^2\right)^{1/2}}{\left(\sum_{i\in[1,I]}|T_i||f_i^T|^2\right)^{1/2}}\leq 10^{-8}.$$

- Our stopping criterion

$$\eta_{\mathrm{L}}\leq\gamma_{\mathrm{D}}\eta_{\mathrm{D}},$$

where we will choose parameter $\gamma_{\mathrm{D}}=0.01$

# Compare stopping criterion

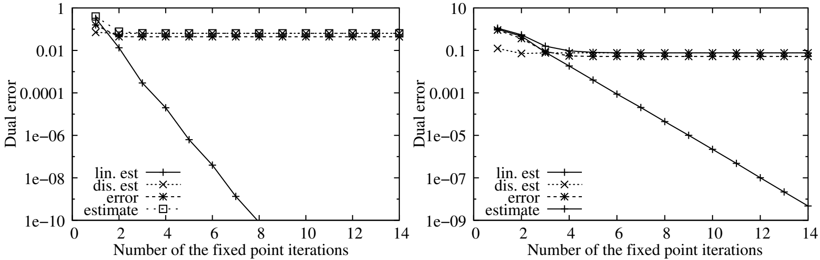
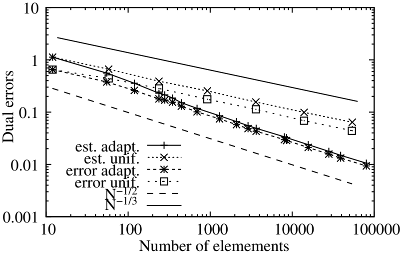



Figure: Total error, total estimator, discretization and linearization estimators as a function of iteration for $H(x) = 1 + 1/(1 + x^2)$ (left) and $H(x) = 2 + \sin(10x)$ (right)

# Compare the uniform and the adaptive refinement



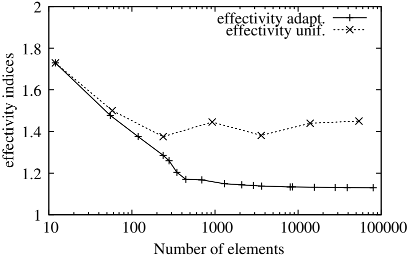




Figure: Estimated and actual errors for uniform and an adaptive refinement (left) and effectivity indices for uniform and an adaptive refinement (right) for $H = 1 + 1/(1 + x^2)$

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- Conclusion
  - Get the scheme for the approximate solution of the nonlinear diffusion equation by two process: discretization and linearization.
  - Derive an a posteriori error estimation: discretization and linearization estimators.
  - Get our stopping criterion which leads to computional saving.
- Future work
  - A posteriroi error estimation for the nonlinear parabolic equations.
  - A posteriroi error estimation for more general nonlinear diffusion equations.