# Continuation from a flat to a round Earth model in the coplanar orbit transfer problem

M. Cerf[1], T. Haberkorn, Emmanuel Trélat[1]

[1]EADS Astrium, les Mureaux
[2]MAPMO, Université d'Orléans

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# The coplanar orbit transfer problem

- Spherical Earth
- Central gravitational field $g(r) = \frac{\mu}{r^2}$

## System in cylindrical coordinates

$$
\begin{aligned}
\dot r(t) &= v(t)\sin\gamma(t) \\
\dot\varphi(t) &= \frac{v(t)}{r(t)}\cos\gamma(t) \\
\dot v(t) &= -g(r(t))\sin\gamma(t) + \frac{T_{\max}}{m(t)}u_1(t) \\
\dot\gamma(t) &= \left(\frac{v(t)}{r(t)} - \frac{g(r(t))}{v(t)}\right)\cos\gamma(t) + \frac{T_{\max}}{m(t)v(t)}u_2(t) \\
\dot m(t) &= -\beta T_{\max}\|u(t)\|
\end{aligned}
$$

- Thrust: $T(t) = u(t)T_{\max}$ ($T_{\max}$ large: strong thrust)
- Control: $u(t) = (u_1(t), u_2(t))$ satisfying $\|u(t)\| = \sqrt{u_1(t)^2 + u_2(t)^2} \leqslant 1$

# The coplanar orbit transfer problem

## Initial conditions

$$r(0) = r_0,\ \varphi(0) = \varphi_0,\ v(0) = v_0,\ \gamma(0) = \gamma_0,\ m(0) = m_0,$$

## Final conditions

- a point of a specified orbit: $r(t_f) = r_f,\ v(t_f) = v_f,\ \gamma(t_f) = \gamma_f,$
  or
- an elliptic orbit of energy $K_f < 0$ and eccentricity $e_f$:

$$\xi_{K_f} = \frac{v(t_f)^2}{2} - \frac{\mu}{r(t_f)} - K_f = 0,$$

$$\xi_{e_f} = \sin^2\gamma + \left(1 - \frac{r(t_f)v(t_f)^2}{\mu}\right)^2 \cos^2\gamma - e_f^2 = 0.$$

(orientation of the final orbit not prescribed: $\varphi(t_f)$ free; in other words: argument of the final perigee free)

## Optimization criterion

$\max m(t_f)$ (note that $t_f$ has to be fixed)

# Application of the Pontryagin Maximum Principle

## Hamiltonian

$$H(q,p,p^0,u) = p_r v \sin\gamma + p_\varphi \frac{v}{r}\cos\gamma + p_v\left(-g(r)\sin\gamma + \frac{T_{\max}}{m}u_1\right)$$
$$+ p_\gamma\left(\left(\frac{v}{r} - \frac{g(r)}{v}\right)\cos\gamma + \frac{T_{\max}}{mv}u_2\right) - p_m \beta T_{\max}\|u\|,$$

## Extremal equations

$$\dot q(t) = \frac{\partial H}{\partial p}(q(t),p(t),p^0,u(t)), \quad \dot p(t) = -\frac{\partial H}{\partial q}(q(t),p(t),p^0,u(t)),$$

## Maximization condition

$$H(q(t),p(t),p^0,u(t)) = \max_{\|w\|\leqslant 1} H(q(t),p(t),p^0,w)$$

# Application of the Pontryagin Maximum Principle

## Hamiltonian

$$H(q,p,p^0,u) = p_r v \sin\gamma + p_\varphi \frac{v}{r}\cos\gamma + p_v\left(-g(r)\sin\gamma + \frac{T_{\max}}{m}u_1\right) + p_\gamma\left(\left(\frac{v}{r} - \frac{g(r)}{v}\right)\cos\gamma + \frac{T_{\max}}{mv}u_2\right) - p_m \beta T_{\max}\|u\|,$$

## Maximization condition leads to

- $u(t) = (u_1(t), u_2(t)) = (0,0)$ whenever $\Phi(t) < 0$
- $u_1(t) = \dfrac{p_v(t)}{\sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}}}$, $u_2(t) = \dfrac{p_\gamma(t)}{v(t)\sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}}}$ whenever $\Phi(t) > 0$

where

$$\Phi(t) = \frac{1}{m(t)}\sqrt{p_v(t)^2 + \frac{p_\gamma(t)^2}{v(t)^2}} - \beta p_m(t) \quad \text{(switching function)}$$

# Application of the Pontryagin Maximum Principle

## Hamiltonian

$$H(q,p,p^0,u) = p_r v \sin\gamma + p_\varphi \frac{v}{r}\cos\gamma + p_v\left(-g(r)\sin\gamma + \frac{T_{\max}}{m}u_1\right) + p_\gamma\left(\left(\frac{v}{r} - \frac{g(r)}{v}\right)\cos\gamma + \frac{T_{\max}}{mv}u_2\right) - p_m\beta T_{\max}\|u\|,$$

## Transversality conditions

- case of a fixed point of a specified orbit: $p_\varphi(t_f) = 0,\ p_m(t_f) = -p^0$
- case of an orbit of given energy and eccentricity:

$$\partial_r\xi_{K_f}(p_\gamma\partial_v\xi_{e_f} - p_v\partial_\gamma\xi_{e_f}) + \partial_v\xi_{K_f}(p_r\partial_\gamma\xi_{e_f} - p_\gamma\partial_r\xi_{e_f}) = 0$$

## Remark

- $p^0 \neq 0$ (no abnormal) $\Rightarrow p^0 = -1$
- no singular arc (Bonnard - Caillau - Faubourg - Gergaud - Haberkorn - Noailles - Trélat)

# Shooting method

Given $(t_f, p_0)$, one can integrate the Hamiltonian flow from 0 to $t_f$ to have $(q(t_f), p(t_f))$.

**Find a zero of**

$$S(t_f, p_0) = \begin{pmatrix} r(t_f, p_0) - r_f \\ v(t_f, p_0) - v_f \\ \gamma(t_f, p_0) - \gamma_f \\ p_\varphi(t_f, p_0) \\ p_m(t_f, p_0) - 1 \end{pmatrix} \text{ or } \begin{pmatrix} \xi_{K_f}(p_0) \\ \xi_{e_f}(p_0) \\ * * * \\ p_\varphi(t_f, p_0) \\ p_m(t_f, p_0) - 1 \end{pmatrix},$$

A zero of $S(\cdot, \cdot)$ is an admissible trajectory satisfying the necessary conditions.

**Main problem: how to make the shooting method converge?**

- initialization of the shooting method
- discontinuities of the optimal control

# Shooting method

## Main problem: how to make the shooting method converge?

- initialization of the shooting method
- discontinuities of the optimal control

## Several methods:

- use first a direct method to provide a good initial guess, e.g. AMPL combined with IPOPT:

R. Fourer, D.M. Gay, B.W. Kernighan, *AMPL: A modeling language for mathematical programming*, Duxbury Press, Brooks-Cole Publishing Company (1993).

A. Wächter, L.T. Biegler *On the implementation of an interior-point Iter line- search algorithm for large-scale nonlinear programming*, Mathematical Programming **106** (2006), 25–57.

but usual flaws of direct methods (computationally demanding, lack of numerical precision).

# Shooting method

**Main problem: how to make the shooting method converge?**

- initialization of the shooting method
- discontinuities of the optimal control

**Several methods:**

- use the impulse transfer solution to provide a good initial guess:

  P. Augros, R. Delage, L. Perrot, *Computation of optimal coplanar orbit transfers*, AIAA 1999.

  but valid only for nearly circular initial and final orbits. See also:

  J. Gergaud, T. Haberkorn, *Orbital transfer: some links between the low-thrust and the impulse cases*, Acta Astronautica **60**, no. 6-9 (2007), 649–657.

  L.W. Neustadt, *A general theory of minimum-fuel space trajectories*, SIAM Journal on Control **3**, no. 2 (1965), 317–356.

# Shooting method

**Main problem: how to make the shooting method converge?**

- initialization of the shooting method
- discontinuities of the optimal control

**Several methods:**

- multiple shooting method parameterized by the number of thrust arcs:

  H. J. Oberle, K. Taubert, *Existence and multiple solutions of the minimum-fuel orbit transfer problem*, J. Optim. Theory Appl. **95** (1997), 243–262.

# Shooting method

**Main problem: how to make the shooting method converge?**

- initialization of the shooting method
- discontinuities of the optimal control

**Several methods:**

- differential or simplicial continuation method linking the minimization of the $L^2$-norm of the control to the minimization of the fuel consumption:

J. Gergaud, T. Haberkorn, P. Martinon, *Low thrust minimum fuel orbital transfer: an homotopic approach*, J. Guidance Cont. Dyn. **27**, 6 (2004), 1046–1060.

P. Martinon, J. Gergaud, *Using switching detection and variational equations for the shooting method*, Optimal Cont. Appl. Methods **28**, no. 2 (2007), 95–116.

but not adapted for high-thrust transfer.

# Flattening the Earth

**Observation:**

Solving the optimal control problem for a flat Earth model with constant gravity is simple and algorithmically very efficient.

In view of that:

Continuation from this simple model to the initial round Earth model.

# Simplified flat Earth model

**System**

$$\dot{x}(t) = v_x(t)$$
$$\dot{h}(t) = v_h(t)$$
$$\dot{v}_x(t) = \frac{T_{\max}}{m(t)} u_x(t)$$
$$\dot{v}_h(t) = \frac{T_{\max}}{m(t)} u_h(t) - g_0$$
$$\dot{m}(t) = -\beta T_{\max} \sqrt{u_x(t)^2 + u_h(t)^2}$$

max $m(t_f)$

$t_f$ free

**Control**

Control $(u_x(\cdot), u_h(\cdot))$ such that $u_x(\cdot)^2 + u_h(\cdot)^2 \leqslant 1$

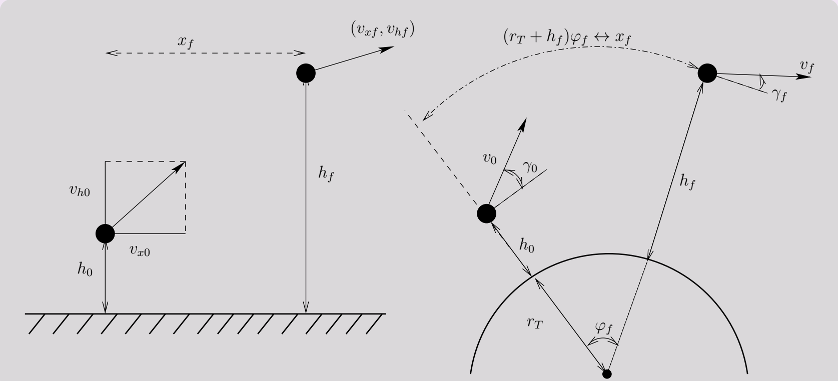
- initial conditions: $x(0) = x_0,\ h(0) = h_0,\ v_x(0) = v_{x0},\ v_h(0) = v_{h0},\ m(0) = m_0$
- final conditions: $h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_f) = 0$

# Modified flat Earth model

Idea: mapping circular orbits to horizontal trajectories

$$\begin{cases} x = r\varphi \\ h = r - r_T \\ v_x = v\cos\gamma \\ v_h = v\sin\gamma \end{cases} \iff \begin{cases} r = r_T + h \\ \varphi = \frac{x}{r_T+h} \\ v = \sqrt{v_x^2 + v_h^2} \\ \gamma = \arctan\frac{v_h}{v_x} \end{cases}$$

$$\begin{pmatrix} u_x \\ u_h \end{pmatrix} = \begin{pmatrix} \cos\gamma & -\sin\gamma \\ \sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$



# Modified flat Earth model

Plugging this change of coordinates into the initial round Earth model:

$$
\begin{aligned}
\dot{r}(t) &= v(t)\sin\gamma(t) \\
\dot{\varphi}(t) &= \frac{v(t)}{r(t)}\cos\gamma(t) \\
\dot{v}(t) &= -g(r_T)\sin\gamma(t) + \frac{T_{\max}}{m(t)}u_1(t) \\
\dot{\gamma}(t) &= \left(\frac{v(t)}{r(t)} - \frac{g(r_T)}{v(t)}\right)\cos\gamma(t) + \frac{T_{\max}}{m(t)v(t)}u_2(t) \\
\dot{m}(t) &= -\beta T_{\max}\|u(t)\|
\end{aligned}
$$

leads to...

# Modified flat Earth model

## Modified flat Earth model

$$\dot{x}(t) = v_x(t) + v_h(t)\frac{x(t)}{r_T + h(t)}$$

$$\dot{h}(t) = v_h(t)$$

$$\dot{v}_x(t) = \frac{T_{\max}}{m(t)}u_x(t) - \frac{v_x(t)v_h(t)}{r_T + h(t)}$$

$$\dot{v}_h(t) = \frac{T_{\max}}{m(t)}u_h(t) - g(r_T + h(t)) + \frac{v_x(t)^2}{r_T + h(t)}$$

$$\dot{m}(t) = -\beta T_{\max}\|u(t)\|$$

Differences with the simplified flat Earth model (with constant gravity):

- the term in green: variable (usual) gravity.
- the terms in red: "correcting terms" allowing the existence of horizontal (periodic up to translation in $x$) trajectories with no thrust.

# Continuation procedure

Simplified flat Earth model (with constant gravity) $\xrightarrow[\text{procedure}]{\text{continuation}}$ modified flat Earth model:

$$\dot{x}(t) = v_x(t) + \lambda_2 v_h(t) \frac{x(t)}{r_T + h(t)}$$
$$\dot{h}(t) = v_h(t)$$
$$\dot{v}_x(t) = \frac{T_{\max}}{m(t)} u_x(t) - \lambda_2 \frac{v_x(t) v_h(t)}{r_T + h(t)}$$
$$\dot{v}_h(t) = \frac{T_{\max}}{m(t)} u_h(t) - \frac{\mu}{(r_T + \lambda_1 h(t))^2} + \lambda_2 \frac{v_x(t)^2}{r_T + h(t)}$$
$$\dot{m}(t) = -\beta T_{\max} \sqrt{u_x(t)^2 + u_h(t)^2}$$

**2 parameters:**

$$0 \leqslant \lambda_1 \leqslant 1$$
$$0 \leqslant \lambda_2 \leqslant 1$$

- $\lambda_1 = \lambda_2 = 0$: simplified flat Earth model with constant gravity
- $\lambda_1 = 1$, $\lambda_2 = 0$: simplified flat Earth model with usual gravity
- $\lambda_1 = \lambda_2 = 1$: modified flat Earth model (equivalent to usual round Earth)

# Continuation procedure

Simplified flat Earth model (with constant gravity) $\xrightarrow[\text{procedure}]{\text{continuation}}$ modified flat Earth model:

$$
\begin{aligned}
\dot{x}(t) &= v_x(t) + \lambda_2 v_h(t)\frac{x(t)}{r_T + h(t)} \\
\dot{h}(t) &= v_h(t) \\
\dot{v}_x(t) &= \frac{T_{\max}}{m(t)} u_x(t) - \lambda_2 \frac{v_x(t)v_h(t)}{r_T + h(t)} \\
\dot{v}_h(t) &= \frac{T_{\max}}{m(t)} u_h(t) - \frac{\mu}{(r_T + \lambda_1 h(t))^2} + \lambda_2 \frac{v_x(t)^2}{r_T + h(t)} \\
\dot{m}(t) &= -\beta T_{\max}\sqrt{u_x(t)^2 + u_h(t)^2}
\end{aligned}
$$

**2 parameters:**

$0 \leqslant \lambda_1 \leqslant 1$

$0 \leqslant \lambda_2 \leqslant 1$

$\leadsto$ Two-parameters family of optimal control problems: $(\mathbf{OCP})_{\lambda_1,\lambda_2}$

# Continuation procedure

**$(\mathbf{OCP})_{0,0}$**

flat Earth model, constant gravity

(linear) continuation on $\lambda_1 \in [0, 1]$ →
final time $t_f$ **free**

**$(\mathbf{OCP})_{1,0}$**

flat Earth model, usual gravity

↓ (linear) continuation on $\lambda_2 \in [0, 1]$
final time $t_f$ **fixed**

**$(\mathbf{OCP})_{1,1}$**

modified flat Earth model (equivalent to round Earth model)

Application of the PMP to $(\mathbf{OCP})_{\lambda_1,\lambda_2}$ $\Rightarrow$ series of shooting problems.

# Change of coordinates

**Remark:** Once the continuation process has converged, we obtain the initial adjoint vector for (**OCP**)$_{1,1}$ in the modified coordinates.

To recover the adjoint vector in the usual cylindrical coordinates, we use the general fact:

**Lemma**

Change of coordinates $x_1 = \phi(x)$ and $u_1 = \psi(u)$
$\Rightarrow$ dynamics $f_1(x_1, u_1) = d\phi(x).f(\phi^{-1}(x_1), \psi^{-1}(u_1))$
and for the adjoint vectors:

$$p_1(\cdot) = {}^t d\phi(x(\cdot))^{-1} p(\cdot).$$

Here, this yields:

$$\begin{aligned} p_r &= \frac{x}{r_T + h} p_x + p_h \\ p_\varphi &= (r_T + h) p_x \\ p_v &= \cos\gamma\, p_{v_x} + \sin\gamma\, p_{v_h} \\ p_\gamma &= v(-\sin\gamma\, p_{v_x} + \cos\gamma\, p_{v_h}). \end{aligned}$$

# Analysis of the flat Earth model

**System**

$$\dot{x} = v_x$$
$$\dot{h} = v_h$$
$$\dot{v}_x = \frac{T_{\max}}{m} u_x$$
$$\dot{v}_h = \frac{T_{\max}}{m} u_h - g_0$$
$$\dot{m} = -\beta T_{\max} \sqrt{u_x^2 + u_h^2}$$

**Initial conditions**

$$x(0) = x_0$$
$$h(0) = h_0$$
$$v_x(0) = v_{x0}$$
$$v_h(0) = v_{h0}$$
$$m(0) = m_0$$

**Final conditions**

$$x(t_f) \text{ free}$$
$$h(t_f) = h_f$$
$$v_x(t_f) = v_{xf}$$
$$v_h(t_f) = 0$$
$$m(t_f) \text{ free}$$

$t_f$ free

max $m(t_f)$

**Theorem**

If $h_f > h_0 + \frac{v_{h0}^2}{2g_0}$, then the optimal trajectory is a succession of at most two arcs, and the thrust $\|u(\cdot)\| T_{\max}$ is

- either constant on $[0, t_f]$ and equal to $T_{\max}$,
- or of the type $T_{\max} - 0$,
- or of the type $0 - T_{\max}$.

# Analysis of the flat Earth model

Main ideas of the proof:

- Application of the PMP
- The switching function $\Phi = \frac{1}{m}\sqrt{p_{v_x}^2 + p_{v_h}^2} - \beta p_m$ satisfies:

$$\dot{\Phi} = \frac{-p_h p_{v_h}}{m\sqrt{p_{v_x}^2 + p_{v_h}^2}}$$

$$\ddot{\Phi} = \frac{\beta \|u\|}{m}\dot{\Phi} - \frac{m}{\sqrt{p_{v_x}^2 + p_{v_h}^2}}\dot{\Phi}^2 + \frac{p_h^2}{m\sqrt{p_{v_x}^2 + p_{v_h}^2}}$$

$\Rightarrow$ $\Phi$ has at most one minimum

$\Rightarrow$ strategies $T_{\max}$, $T_{\max} - 0$, $0 - T_{\max}$, or $T_{\max} - 0 - T_{\max}$

- The strategy $T_{\max} - 0 - T_{\max}$ cannot occur

# Shooting method in the flat Earth model

A priori, we have:

**5 unknowns**

$p_h, p_{v_x}, p_{v_h}(0), p_m(0)$, and $t_f$

**5 equations**

$h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_f) = 0,\ p_m(t_f) = 1,\ H(t_f) = 0$

but using several tricks and some system analysis, the shooting method can be simplified to:

**3 unknowns**

$p_{v_x}, p_{v_h}(0)$, and the first switching time $t_1$

**3 equations**

$h(t_f) = h_f,\ v_x(t_f) = v_{xf},\ v_h(t_1) + g_0 t_1 = g_0 p_{v_h}(0)$

$\Rightarrow$ very easy and efficient (instantaneous) algorithm

and the initialization of the shooting method is automatic (CV for any initial adjoint vector)

$\Rightarrow$ automatic tool for initializing the continuation procedure

# Numerical simulations

$T_{\max} = 180\ kN$

$Isp = 450\ s$

## Initial conditions

$\varphi_0 = 0 \quad (SSO)$

$h_0 = 200\ km$

$v_0 = 5.5\ km/s$

$\gamma_0 = 2\ deg$

$m_0 = 40000\ kg$

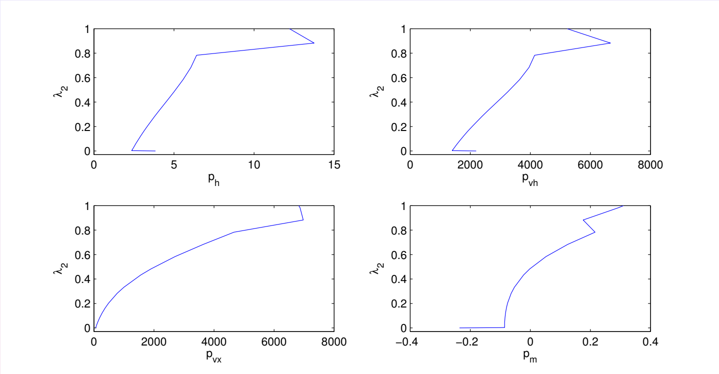
## Final conditions

$h_f = 800\ km$

$v_f = 7.5\ km/s$

$\gamma_f = 0\ deg$

(nearly circular final orbit)




Evolution of the shooting function unknowns $(p_h, p_{v_x}, p_{v_h}, p_m)$ (abscissa) with respect to homotopic parameter $\lambda_2$ (ordinate)

$\rightarrow$ continuous but not $C^1$ path: $\lambda_2 \approx 0.01$, $\lambda_2 \approx 0.8$, and $\lambda_2 \approx 0.82$:

- $0 \leqslant \lambda_2 \lesssim 0.01$: $T_{\max} - 0$
- $0.01 \lesssim \lambda_2 \lesssim 0.8$: $T_{\max} - 0 - T_{\max}$
- $0.8 \lesssim \lambda_2 \lesssim 0.82$: $T_{\max} - 0$
- $0.82 \lesssim \lambda_2 \leqslant 1$: $T_{\max} - 0 - T_{\max}$

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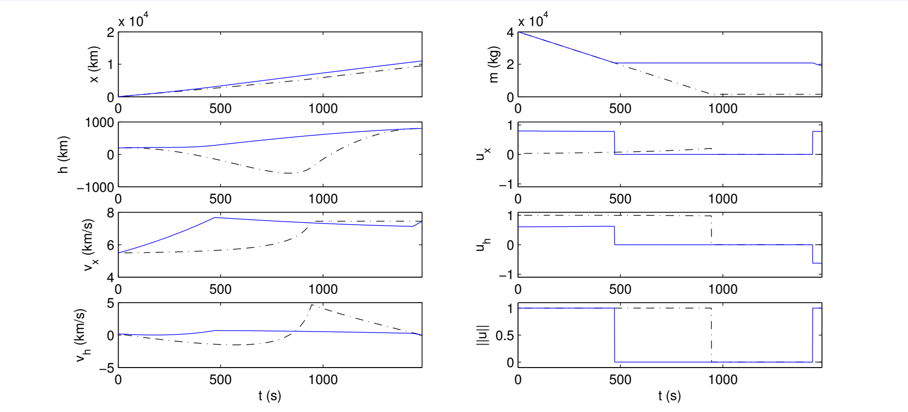
# Numerical simulations




Trajectory and control strategy of $(\mathbf{OCP})_{1,0}$ (dashed) and $(\mathbf{OCP})_{1,1}$ (plain). $t_f \simeq 1483s$

## Remark

In the case of a final orbit (no injecting point): additional continuation on transversality conditions.

# Numerical simulations

Comparison with a direct method:

- Heun (RK2) discretization with $N$ points
- combination of AMPL with IPOPT
- needs however a careful initial guess

## Continuation method

3 seconds:

- $(\mathbf{OCP})_{0,0}$: instantaneous
- from $(\mathbf{OCP})_{0,0}$ to $(\mathbf{OCP})_{1,0}$: 0.5 second
- from $(\mathbf{OCP})_{1,0}$ to $(\mathbf{OCP})_{1,1}$: 2.5 seconds

$\rightarrow$ Accuracy: $10^{-12}$

## Direct method

- $N = 100$: 5 seconds
- $N = 1000$: 165 seconds
  $\rightarrow$ Accuracy: $10^{-6}$

# Conclusion

- Algorithmic procedure to solve the problem of minimization of fuel consumption for the coplanar orbit transfer problem by shooting method approach
- Does not require any careful initial guess

## Open questions

- Is this procedure systematically efficient, for any possible coplanar orbit transfer?
- Extension to 3D