A Matrix-based Method for the N-dimensional Sine-Gordon Equation

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In [2], we have developed a numerical method for the N-dimensional sine-Gordon equation

$$u_{tt}(\bar{x},t) = \triangle u(\bar{x},t) - \sin(u(\bar{x},t)), \qquad u : [a_1,b_1] \times \cdots \times [a_N,b_N] \times \mathbb{R}^+ \subseteq \mathbb{R}^N \times \mathbb{R}^+ \to \mathbb{R}, \tag{1}$$

based on the confluence of two powerful ideas: The first one is the concept of a differentiation matrix, which has been proven to be a very useful tool in the numerical solutions of differential equations [3] [4]; and the second one is the solution of matrix differential equations [1, Chap. 10]. Indeed, by means of differentiation matrices and taking into account the boundary conditions, (1) can be discretized into

$$\begin{aligned}(U_{i_1 i_2 \ldots i_N})_{tt} = \sum_{j_1=1}^{n_1-1} A_{1\, i_1\, j_1} U_{j_1\, i_2 \ldots i_N} + \sum_{j_2=1}^{n_2-1} A_{2\, i_2\, j_2} U_{i_1\, j_2\, i_3 \ldots i_N} + \ldots + \sum_{j_N=1}^{n_N-1} A_{N\, i_N\, j_N} U_{i_1 \ldots i_{N-1}\, j_N} \\ + C_{i_1 i_2 \ldots i_N}(t) - \sin(U_{i_1 i_2 \ldots i_N}), \qquad 1 \le i_k \le n_k - 1, \quad k = 1, \ldots, N,\end{aligned} \tag{2}$$

for some time-independent matrices $\mathbf{A}_1, \mathbf{A}_2, \ldots, \mathbf{A}_N$. We have applied a fourth-order Runge-Kutta scheme with integrating factor to (2), **avoiding Kronecker tensor products**.
Our method is very intuitive and easy to express, and can be implemented without toil in any number of spatial dimensions. Although there is currently a vast literature on the numerical treatment of the one-dimensional sine-Gordon equation, the references for the two-dimensional case are much sparser, and virtually nonexistent for higher dimensions.
We have tested it with two-dimensional problems taken from the literature, showing that it largely outperforms the previously existing algorithms; while in three-dimensional problems, the results seem very promising.

Références

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