Partial integro-differential equations for option prices in semimartingale models

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In Markovian stochastic models used in finance, option prices are characterized in terms of solutions to a backward PDE, whose variables are time (to maturity) and the value of the underlying asset. This characterization is very useful for computation of option prices. Dupire [3] derived a *forward* PDE for call options, which holds in a more general context than the backward PDE, when the (risk-neutral) dynamics of the underlying asset is not necessarily Markovian, but described by a continuous Brownian martingale $S_t = S_0 + \int_0^t \delta_u dW_u$, the call option price verifies a forward PDE where the diffusion coefficient is given by the local (or effective) volatility function $\sigma(t,S)$ given by $\sigma(t,S) = \sqrt{E[\delta_t^2|S_t = S]}$. This method, also known as "Markovian projection", is linked to the construction of a Markov process which mimicks the marginal distributions of a martingale [2, 4]. We extend the result of Dupire to the setting of semimartingales with jumps and derive a partial integro-differential equation (PIDE) which extends the Dupire equation in this general setting.

Consider a price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by a stochastic volatility model with jumps $S_T = S_0 + \int_0^T r(t)S_{t^-}dt + \int_0^T S_{t^-}\delta_t dW_t + \int_0^T \int_{-\infty}^{+\infty} S_{t^-}(e^y - 1)\tilde{M}(dt\,dy)$, where $r(t) > 0$ represents a (deterministic) bounded discount rate, δ_t the (random) volatility process and M is an integer-valued random measure with compensator $\mu(dt\,dy;\omega) = m(t, dy, \omega)\,dt$, representing jumps in the log-price and $\tilde{M} = M - \mu$ is the compensated random measure associated to M. Define the call option price by $C_t(T,K) = E[\exp(-\int_t^T r(u)du)(S_T - K)_+|\mathcal{F}_t]$.

Theorem 1 (Forward PIDE for call options)**.** *If* $\forall T > 0$, $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2\,dt + \int_0^T dt \int_{\mathbb{R}}(e^y - 1)^2 m(t,dy)\right)\right] < \infty$*, then the call option price* $(T,K) \mapsto C_{t_0}(T,K)$*, as a function of maturity and strike, is a solution of the partial integro-differential equation:*

$$\frac{\partial C_{t_0}}{\partial T} = -r(T)K\frac{\partial C_{t_0}}{\partial K} + \frac{K^2\sigma(T,K)^2}{2}\,\frac{\partial^2 C_{t_0}}{\partial K^2} + \int_0^{+\infty} y\frac{\partial^2 C_{t_0}}{\partial K^2}(T,dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \tag{1}$$

on $[t_0,\infty[\times]0,\infty[$ *with the initial condition:* $\forall K > 0 \quad C_{t_0}(t_0,K) = (S_{t_0} - K)_+$*, where* $\sigma(t,z) = \sqrt{\mathbb{E}\left[\delta_t^2|S_{t^-} = z\right]}$ *and* $\chi_{t,y}(z) = \mathbb{E}\left[\psi_t\left(z\right)|S_{t-} = y\right]$ *with* ψ_t *is the exponential double tail of the compensator* $m(t,dy)$

Our derivation does not require ellipticity or non-degeneracy of the diffusion coefficient. We give various examples of applications to stochastic volatility models with jumps, pure jump models and point process models used in equity and credit risk modeling. The case of index options in a multivariate jump-diffusion model illustrates how the forward equation may be used to project a high dimensional pricing problem into a one-dimensional state equation.

Références

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