Wavelets and differential operators: 
from fractals to Marr’s primal sketch

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Invariance is an attractive principle for specifying image processing algorithms. In this presentation, we promote affine invariance—more precisely, invariance with respect to translation, scaling and rotation. As starting point, we identify the corresponding class of invariant 2D operators: these are combinations of the (fractional) Laplacian and the complex gradient (or Wirtinger operator). We then specify some corresponding differential equation and show that the solution in the real-valued case is either a fractional Brownian field (Mandelbrot and Van Ness, 1968) or a polyharmonic spline (Duchon, 1976), depending on the nature of the system input (driving term): stochastic (white noise) or deterministic (stream of Dirac impulses). The affine invariance of the operator has two important consequences: (1) the statistical self-similarity of the fractional Brownian field, and (2) the fact that the polyharmonic splines specify a multiresolution analysis of $L_2(\mathbb{R}^2)$ and lend themselves to the construction of wavelet bases. The other fundamental implication is that the corresponding wavelets behave like multi-scale versions of the operator from which they are derived; this makes them ideally suited for the analysis of multidimensional signals with fractal characteristics (whitening property of the fractional Laplacian).

The complex extension of the approach yields a new complex wavelet basis that replicates the behavior of the Laplace-gradient operator and is therefore adapted to edge detection. We introduce the Marr wavelet pyramid which corresponds to a slightly redundant version of this transform with a Gaussian-like smoothing kernel that has been optimized for better steerability. We demonstrate that this multiresolution representation is well suited for a variety of image-processing tasks. In particular, we use it to derive a primal wavelet sketch—a compact description of the image by a multiscale, subsampled edge map—and provide a corresponding iterative reconstruction algorithm.