3D-2D analysis for the optimal elastic compliance problem

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The simplest way for designing elastic plates is to use plane layers with constant thickness and made of a single material. The resistance-weight ratio can be improved by allowing for a varying thickness. In this paper we only assume that the structure is made of a single material lying in some subset (of fixed volume) of a thin layer and we do not add any extra geometrical constraint. As we look for an optimal design and study its limit when the thickness of the layer tends to zero, our study is at the junction of two research directions: the so-called 3D-2D asymptotic analysis and shape optimization.

Let us recall that the compliance of an elastic material occupying a domain $\Omega \subset \mathbb{R}^3$, characterized by a strain potential $j$, and subject to a given system of loads $F$, is the opposite of the total energy at equilibrium (the higher is this compliance, the smaller is the resistance of the material to the load). It is given by

$$C_{j,F}(\Omega) := \sup \left\{ \langle F, u \rangle - \int_{\Omega} j(e(u)) \, dx \, : \, u \in C^\infty(\mathbb{R}^3; \mathbb{R}^3) \right\}$$

where $2\epsilon(u) := \nabla u + (\nabla u)^T$. The asymptotic study of the compliance when the domain $\Omega$ is a cylinder $Q_\delta := \overline{D} \times [-\delta/2, +\delta/2]$ of infinitesimal height $\delta$, plays an important role in mechanical engineering. In the standard case, with a suitable scaling for the load, it turns out that the limit can be written in terms of descriptors depending only on the 2D transverse spatial variables $x'$. This is due to the special structure of the limit displacement: a Kirchoff-Love displacement. This limit behavior may not hold when the body is non homogeneous with a high contrast or in presence of voids, that is when the domain $\Omega$ is a proper subset of the thin cylinder.

When performing shape optimization engineers look for domains included in some design region and which, for a given volume, minimize the compliance. This problem is generally ill-posed. During the minimization process the characteristic functions of $\Omega_\alpha$ converge to a function $\theta$ with values in $[0,1]$, and no optimal shape exists. Relaxation theory teach us that intermediate values have to be accepted and $j$ has to be replaced by an integrand $j^\text{eff}_\theta$. The explicit computation of which using homogenization theory is a challenging problem.

In this paper, we study the asymptotics of the optimal shape problem when the design region is the flattening cylinder $Q_\delta$, with $\delta \to 0$. Obviously, the load as well as the volume constraint have to be adapted to the thin design region. The volume constraint is replaced by a volume penalization through a Lagrange multiplier $k$. The problem reads

$$\phi_{j,F}^\delta(k) := \inf \left\{ C_{j,F^\delta}(\Omega) + \frac{k}{\delta} |\Omega| \, : \, \Omega \subset Q_\delta \right\}.$$  \hspace{1cm} (2)

We prove that the limit as $\delta \to 0$ of $\phi_{j,F}^\delta(k)$ is given by

$$\phi(k) := \inf_{\theta \in L^\infty(Q;[0,1])} \sup \left\{ \langle \overline{F}, v \rangle - \int_D \int_{-1/2}^{1/2} [\tilde{j}(e(v_1, v_2) - x_3 \nabla^2 v_3) - k] \, \theta \, dx \, : \, v_\alpha \in H^1(D), \, v_3 \in H^2(D) \right\}$$

where $\overline{F}$ is a suitable 2D-average load and $\tilde{j}$ is the 2D energy density in the plane stress case. This is due to the special structure of the limit displacement: a Kirchoff-Love displacement. This limit behavior may not hold when the body is non homogeneous with a high contrast or in presence of voids, that is when the domain $\Omega$ is a proper subset of the thin cylinder.

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