An optimal variance estimate in stochastic homogenization

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We consider a discrete elliptic equation with random coefficients $A$, which (to fix ideas) are identically distributed and independent from grid point to grid point $x \in \mathbb{Z}^d$. On scales large w. r. t. the grid size (i. e. unity), the solution operator is known to behave like the solution operator of a (continuous) elliptic equation with constant deterministic coefficients. These symmetric “homogenized” coefficients $A_{hom}$ are characterized by

$$\xi \cdot A_{hom}\xi = \langle ((\xi + \nabla \phi) \cdot A(\xi + \nabla \phi))(0) \rangle, \quad \xi \in \mathbb{R}^d,$$

where the random field $\phi$ is the unique stationary solution of the “corrector problem”

$$-\nabla \cdot A(\xi + \nabla \phi) = 0$$

and $\langle \cdot \rangle$ denotes the ensemble average.

It is known (“by ergodicity”) that the above ensemble average of the energy density $e = (\xi + \nabla \phi) \cdot A(\xi + \nabla \phi)$, which is a stationary random field, can be recovered by a system average. We quantify this by proving that the variance of a spatial average of $e$ on length scales $L$ is estimated as follows:

$$\text{var} \left[ \sum_{x \in \mathbb{Z}^d} \eta_L(x) e(x) \right] \lesssim L^{-d},$$

where the averaging function (i. e. $\sum_{x \in \mathbb{Z}^d} \eta_L(x) = 1$, supp$\eta_L \subset [-L, L]^d$) has to be smooth in the sense that $|\nabla \eta_L| \lesssim L^{-1}$. In two space dimensions (i. e. $d = 2$), there is a logarithmic correction.

In other words, smooth averages of the energy density $e$ behave like as if $e$ would be independent from grid point to grid point (which it is not for $d > 1$). This result is of practical significance, since it allows to estimate the error when numerically computing $A_{hom}$.

Références