RANDOM WALKS AND RANDOM INTERLACEMENTS

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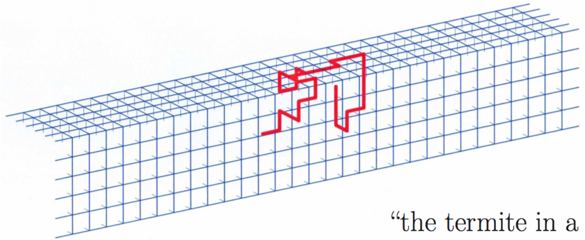
DISCONNECTION OF DISC. CYLINDERS

$E_N = (\mathbb{Z}/N\mathbb{Z})^d \times \mathbb{Z}$, $d \geq 1$, N large



"the termite in a wooden beam"

$X_0, X_1, \ldots, X_n, \ldots$ random walk on E_N

$$\begin{aligned} T_N &= \text{disconnection time} \\ &= \inf\{n \geq 0;\ \{X_0, X_1, \ldots, X_n\} \text{ disconnects } E_N\}. \end{aligned}$$

Question of H.J. Hilhorst: How large is T_N?
Where is X_{T_N}?
How does $E_N \setminus \{X_0, \ldots, X_{T_N}\}$ look?

Disconnection of Cylinders

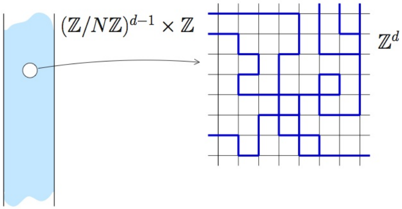
T_N = disconnection time

C_N = cover time of $(\mathbb{Z}/N\mathbb{Z})^d \times \{0\}$ by X

THEOR. *(Dembo-Sznitman 06)* $d \geq 1$

$$(*) \qquad \lim_N \frac{\log T_N}{\log N} = \lim \frac{\log C_N}{\log N} = 2d, \quad P_0\text{-prob.}$$

To improve on (∗)



have a better understanding of the microscopic structure

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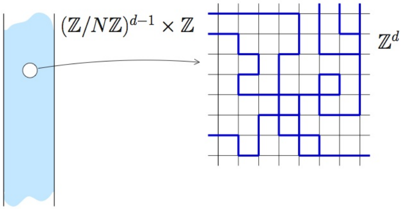

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The Random Interlacements

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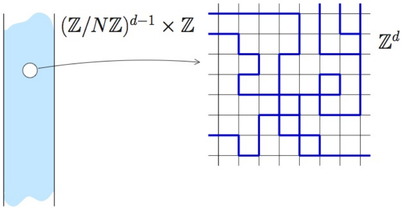

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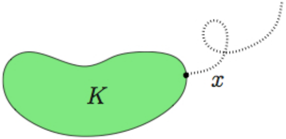
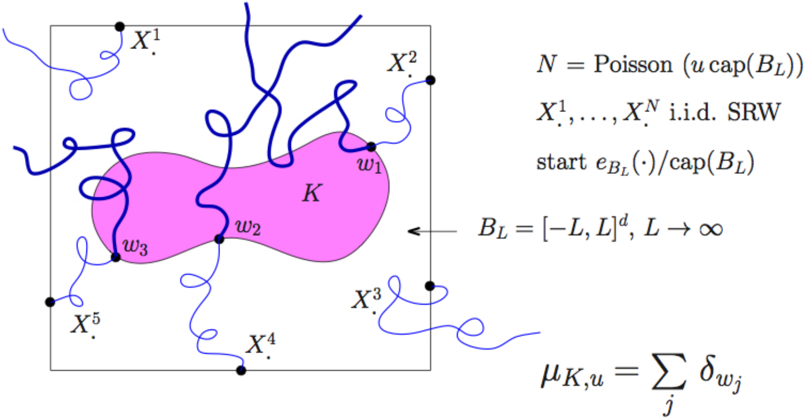
The Random Interlacements

$\mathbb{Z}^d$, $d \geq 3$ (more generally: transient weighted graph)
$K \subset \mathbb{Z}^d$, finite



$e_K(x) = P_x[\widetilde{H}_K = \infty],\ x \in K$, equilibrium meas. of K

$\mathrm{cap}(K) = \sum_{x \in K} e_K(x)$: capacity of K



$\mu_{K,u}$ Poisson point meas. intensity uP_{e_K}

As K varies compatibility:

For $K \supset K'$ $\quad \mu_{K',u}$ obtained by “sweeping” $\mu_{K,u}$ on K'.

$\mathcal{I}^u(\omega) \subseteq \mathbb{Z}^d$: Random Interlacement at level u,

$\mathcal{I}^u(\omega) \cap K =$ trace on K of traj. in support $\mu_{K',u}$, $K' \supset K$

$\mathcal{V}^u(\omega) = \mathbb{Z}^d \backslash \mathcal{I}^u(\omega)$: Vacant set at level u.

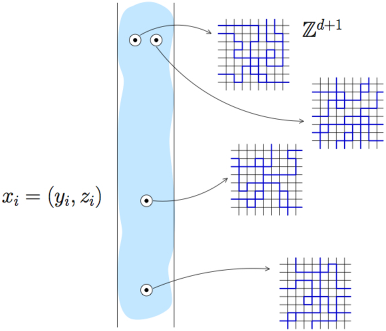
THEOR. *(Sznitman 07)*

- $\mathcal{I}^u$ infinite conn. subset of $\mathbb{Z}^d$, ergodic under transl.
- $\mathbb{P}[\mathcal{I}^u \cap K = \phi] = e^{-u\,\mathrm{cap}(K)}$, so

$$\mathbb{P}[0 \in \mathcal{I}^u] = 1 - e^{-\frac{u}{g(0)}},$$

$$\mathrm{cov}(x \in \mathcal{I}^u, y \in \mathcal{I}^u) \sim \frac{c(u)}{|x-y|^{d-2}}, \ |x-y| \to \infty.$$

RI and SRW on the cylinder (I): the local picture



π can proj. $\mathbb{Z}^{d+1} \to E_N$

$X_. = (Y_., Z_.)$ SRW on E_N
unif. start at height $z = 0$

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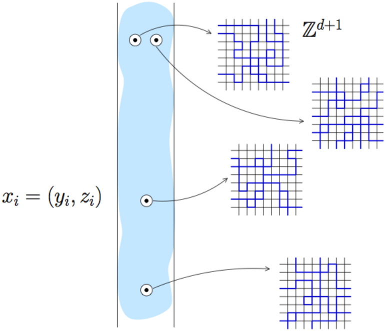

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$L_n^z = \sum_{0 \le m < n} 1\{Z_m = z\}$ local time of $Z_.$ at z

$\varphi_{x,n} =$ ind. funct. $\pi^{-1}(X_{[0,n]} - x)(\in \{0,1\}^{\mathbb{Z}^{d+1}})$,
local conf. by time n centered at x.

$L(v,t)$, $v \in \mathbb{R}, t \ge 0$, Brownian local time.

THEOR: *(Sznitman 08)* $M \geq 1$, assume:

$$\lim_N \inf_{i \neq j} |x_i - x_j| = \infty, \qquad \lim_N \frac{z_i}{N^d} = v_i \in \mathbb{R}, \ 1 \leq i \leq M,$$

and

$$\tau_N \geq 0, \text{ R.V. s.t. } \quad \tau_N / N^{2d} \xrightarrow{\text{Prob.}} \alpha > 0,$$

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then

$$(\varphi_{x_1,\tau_N}, \ldots, \varphi_{x_M,\tau_N}, L^{z_1}_{\tau_N}/N^d, \ldots, L^{z_M}_{\tau_N}/N^d) \overset{\text{law}}{\longrightarrow}$$

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and given $(\mathcal{U}_1, \dots, \mathcal{U}_M)$, $\varphi_1, \dots, \varphi_M$ are indep. with resp. dist. $\mathcal{I}^u$ under $\mathbb{P}$, for $u = \mathcal{U}_1, \dots, \mathcal{U}_M$.

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- case SRW on $(\mathbb{Z}/N\mathbb{Z})^d$, $d \geq 3$ *(Windisch 08, 10)*
- SRW on cylinders $G_N \times \mathbb{Z}$

Percolation and RI:

can $\mathcal{I}^u$ be a "rainproof fabric"?

$\eta(u) = \mathbb{P}\,[0 \stackrel{\mathcal{V}^u}{\longleftrightarrow} \infty]$,

$\eta(u) > 0 \iff \mathcal{V}^u$ percolates (0-1 law).

$u_* = \inf\{u \geq 0;\ \eta(u) = 0\} \in [0, \infty]$, critical value.

Is u_* non-degenerate?

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THEOR: *(Sznitman 07, Sidoravicius-Sznitman 08)*

i) $u_* < \infty$, for $d \geq 3$

ii) $u_* > 0$, for $d \geq 3$, and for small u, $\mathcal{V}^u$ percolates in planes.

(So, u_* always non-degenerate!)

Proofs: renormalization, sprinkling, path surgery.

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Connect. bounds: u small, $d \geq 5$ *(Teixeira 09)*

$u > u_{**}$, $d \geq 3$ *(Sidoravicius-Sznitman 09)*

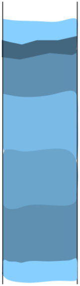
Back to the disconnection time T_N

$$u_{**} = \inf\Big\{u \geq 0;\, \exists\, \alpha > 0,\, L^{\alpha}\, \mathbb{P}\big[B_L \overset{\mathcal{V}^u}{\longleftrightarrow} \partial B_{2L}\big] \underset{L}{\to} 0\Big\}.$$

Proof $u_* < \infty$ also shows $u_* \leq u_{**} < \infty$.

also $\exists u_- > 0,\; L^{6d}\, \mathbb{P}\big[0 \overset{*-\mathcal{I}^{u_-}\cap\mathbb{Z}^2}{\longleftrightarrow} \partial B_L\big] \underset{L}{\to} 0.$

$$\zeta(u) = \inf\Big\{\alpha \geq 0;\; \sup_{v\in\mathbb{R}} (d+1)\, L\Big(v, \frac{\alpha}{d+1}\Big) \geq u\Big\}.$$



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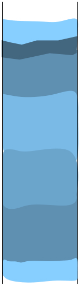

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THEOR. *(Sznitman 08, 09)* $d \ge 2$, for $\gamma \ge 0$,

$\limsup_N\; P\big[T_N/N^{2d} \ge \gamma\big] \le W[\zeta(u_{**}) \ge \gamma]$, (W Wiener meas.)

$\liminf_N\; P\big[T_N/N^{2d} > \gamma\big] \ge W[\zeta(u_-) > \gamma]$

(so T_N/N^{2d} and N^{2d}/T_N tight).

Questions:

- $u_* = u_{**}$?

- Is there "strong percolation" for $u < u_*$?

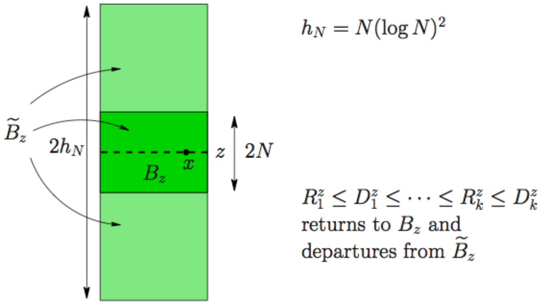
- Does $T_N/N^{2d} \xrightarrow{\text{law}} \zeta(u_*)$? What about X_{T_N} ?
  Self-induced criticality ?

- What is the local picture viewed from disc. point X_{T_N}?

- What is the dimension of the infinite vacant components near disc. point X_{T_N}?

- Universality?

RI and SRW on the cylinder (II): coupling



THEOR. *(Sznitman 08, 09)*

For $0 < \varepsilon < 1$, $u > (d+1)\alpha$, $N \geq c$, $x = (y, z) \in E_N$
$\exists Q_x$ coupling SRW with $\mathcal{I}^u$ s.t.
$Q_x[(X_{[0,D_K^z]} - x) \cap B \subseteq \mathcal{I}^u \cap B] \geq 1 - N^{-3d}, \quad K = \alpha\, N^d / h_N$
and $B = B(0, N^{1-\varepsilon})$

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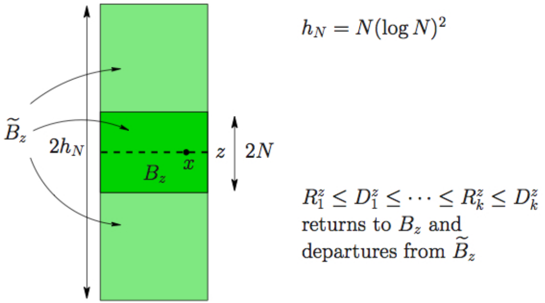

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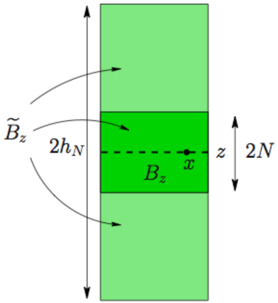

$h_N = N(\log N)^2$

$R_1^z \leq D_1^z \leq \cdots \leq R_k^z \leq D_k^z$
returns to B_z and
departures from $\widetilde{B}_z$

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For torus Teixeira-Windisch 10.