An extension of Davis and Lo’s contagion model

Areski Cousin, Diana Dorobantu and Didier Rullière
ISFA, University of Lyon

Journées MAS

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Das et al. (2007) : Conditional independence assumption with no contagion effect is rejected by historical default data. The conditional independence assumption is not enough to capture historical default dependency.

Boissay (2006), Jorion and Zhang (2007, 2009) analyze the mechanism of default propagation and provide financial evidence of chain reactions or dominos effects.

We present a multi-period extension of Davis and Lo’s contagion model.
In the spirit of Davis and Lo’s contagion model:

- First models: Davis and Lo (2001) and Jarrow and Yu (2001)

Other contagion models in the credit risk field:

- Copula: Schönbucher and Schubert (2001)
Modeling of credit contagion for a pool of defaultable entities

- One-period model
- Credit references may default either directly or as a consequence of a contagion effect

Example: Portfolio with 3 credit references
End of the period: direct default
End of the period: default by contagion (one possibility)
One-period contagion model

- $n$ : number of credit references,
- $X_i$ : direct default indicator of name $i$ (i.e. $X_i = 1$ if $i$ defaults directly, $X_i = 0$ otherwise),
- $C_i$ : indirect default indicator of name $i$,
- $Z_i$ : default indicator (direct or indirect) such that :

$$Z_i = X_i + (1 - X_i)C_i$$

where :

$$C_i = \mathbb{1}_{\text{at least one } x_jy_{ji}=1, \ j=1,...,n}$$

- $Y_{ji} = 1$ if the contagion link is activated from name $j$ to name $i$, $Y_{ji} = 0$ otherwise.
Davis and Lo’s contagion model

\[ N = \sum_{i=1}^{n} Z_i : \text{total number of defaults} \]

Distribution of total number of defaults (Davis and Lo)

\[ P[N = k] = \binom{n}{k} \sum_{i=1}^{k} \binom{k}{i} p^i (1 - p)^{n-i} (1 - (1 - q)^i)^{k-i} (1 - q)^{i(n-k)}. \]

Under the assumptions:

- \( X_i, i = 1, \ldots, n \): iid Bernoulli with parameter \( p \)
- \( Y_{ij}, i, j = 1, \ldots, n \): iid Bernoulli with parameter \( q \)
- One default alone may trigger a contamination effect
- A name that has been infected cannot contaminate other names (no chain-reaction effect)
Extension of Davis and Lo’s contagion model

Dominos Effect

Two defaults required to trigger a contagion effect
Multi-period contagion model: \( t = 0, 1, 2, \ldots, T \), in period \([t, t+1]\) :

- \( n \): number of credit references,
- \( X_i^t \): direct default indicator of name \( i \),
- \( C_i^t \): indirect default indicator of name \( i \),
- \( Z_i^t \): default indicator (direct or indirect) such that:

\[
Z_i^t = Z_i^{t-1} + (1 - Z_i^{t-1})[X_i^t + (1 - X_i^t)C_i^t]
\]

where

\[
C_i^t = f\left(\sum_{j\in F_t} Y_{ji}^t\right)
\]

- \( Y_{ji}^t, i, j = 1, \ldots, n \) are Bernoulli random variables such that \( Y_{ji}^t = 1 \) if name \( j \) may infect name \( i \) between \( t \) and \( t+1 \)
- \( F_t \) is the set of names that are likely to infect other names between \( t \) and \( t+1 \)
- \( f \) is a contamination trigger function, for example \( f = \mathbb{1}_{x \geq 1} \) (Davis and Lo) or \( f = \mathbb{1}_{x \geq 2} \)
Extension of Davis and Lo’s contagion model

\[ N_t = \sum_{i=1}^{n} Z_t^i \]: total number of defaults at time \( t \)

**Main result**

\[
P[N_t = r] = \sum_{k=0}^{r} P[N_{t-1} = k] C_{n-k}^{r-k} \sum_{\gamma=0}^{n-k-\gamma} C_{\alpha}^{\gamma} (-1)^{j+\alpha} \xi_j + r - k - \gamma, t(\gamma).
\]

**Under the assumptions:**

- \( X_t^i, i = 1, \ldots, n \) are **conditionally independent** Bernoulli r.v. with the same marginal distribution \( \mu_{k,t} \) and \( X_t = (X_t^1, \ldots, X_t^n), t = 1, \ldots, T \) are independent vectors.

- \( Y_t^{ij}, i, j = 1, \ldots, n \) are **conditionally independent** Bernoulli r.v. with the same marginal distribution and \( Y_t = (Y_t^{ij})_{1 \leq i, j \leq n}, t = 1, \ldots, T \) are independent vectors.

- \( (X_t)_{t=1,\ldots,T} \) and \( (Y_t)_{t=1,\ldots,T} \) are independent.
Cash-flows of CDO tranches driven by the aggregate loss process

\[ L_t = \sum_{i=1}^{n} (1 - R_i) E_i Z_t^i \]

where \( R_i \) is the recovery rate associated with name \( i \) and \( E_i \) is the nominal of \( i \).
We restrict ourselves to the case where for all $t$:

- $X^i_t \sim \text{Bernoulli}(\Theta)$ where $\Theta \sim \text{Beta}$, $E[\Theta] = p$ and $\text{Var}(\Theta) = \sigma^2$, $i = 1, \ldots, n$
- $Y^{ij}_t$ are iid $Y^{ij}_t \sim \text{Bernoulli}(q)$, $i, j = 1, \ldots, n$
- Only one default is required to trigger a default by contagion

Moreover

- $n = 125$, $r = 3\%$ (short-term interest rate)
- $R_i = R = 40\%$, $E_i = E = 1$ for any $i = 1, \ldots, n$

$$L_t = (1 - R)N_t$$

- Computation of CDO tranche price only requires marginal loss distributions at several time horizons
Least square calibration procedure: Find $\alpha^* = (p^*, \sigma^*, q^*)$ which minimizes:

$$RMSE(\alpha) = \sqrt{\frac{1}{6} \sum_{i=1}^{6} \left( \frac{\tilde{s}_i - s_i(\alpha)}{\tilde{s}_i} \right)^2}.$$  

where

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<th>6%-9%</th>
<th>9%-12%</th>
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Calibration on 5-years iTraxx tranche quotes

Four calibration procedures:

- **Calibration 1**: All available market spreads are included in the fitting
- **Calibration 2**: The equity [0%-3%] tranche spread is excluded
- **Calibration 3**: Both equity [0%-3%] tranche and CDS index spreads are excluded
- **Calibration 4**: All tranche spreads are excluded except equity tranche and CDS index spreads.

Two calibration dates before and during the credit crisis:

- 31 August 2005
- 31 March 2008
Calibration on 5-years iTraxx tranche quotes

31 August 2005

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Annual scaled optimal parameters

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Calibration on 5-years iTraxx tranche quotes

31 March 2008

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We propose a multi-period extension of Davis and Lo's contagion model that accounts for:

- possibly dominos or chain reaction effect
- flexible contagion mechanism (ex: more than one default required to trigger a contamination)

We provide a recursive formula for the distribution of the number of defaults at different time horizons:

- When direct defaults and contagion events are conditionally independent

The multi-period setting is required to price synthetic CDO tranches:

- The contagion parameter has a significant impact on the model ability to fit CDO tranche quotes.