# Localized spherical deconvolution

T. M. Pham Ngoc
joint work with
G. Kerkyacharian and D. Picard

Laboratoire de Probabilités et Modèles Aléatoires

# Statistical model

We observe

$$Z_i = \varepsilon_i X_i \quad i = 1...N$$

$Z_i$, $X_i$ i.i.d random elements of $\mathbb{S}^2$, the unit sphere of $\mathbb{R}^3$, $\varepsilon_i \in SO(3)$ are i.i.d., $X_i$ and $\varepsilon_i$ are supposed to be independent.

- Distributions of $X$, $Z$ , $\varepsilon$ are absolutely continuous with respect to the uniform probability measure on $\mathbb{S}^2$, $\mathbb{S}^2$ and the Haar measure on $SO(3)$ with densities $f$, $f_Z$ et $f_\varepsilon$.

# Convolution product

We have the following formula :

$$f_Z = f_\varepsilon * f$$

For $f_\varepsilon \in \mathbb{L}_2(SO(3)),\ f \in \mathbb{L}_2(\mathbb{S}^2)$, we define as follows the convolution product :

$$f_\varepsilon * f(\omega) = \int_{SO(3)} f_\varepsilon(u) f(u^{-1}\omega) du$$

# Bibliography


- A. van Rooij et F. Ruymgaart (1991). *Regularized deconvolution on the circle and the sphere*
- D. Healy, J. Hendriks et P. Kim (1998). *Spherical deconvolution.*
- P. Kim et J. Y. Koo (2002). *Optimal spherical deconvolution.*

# Motivations

- Astrophysics :
  study of the origins of UHECR i.e Ultra High Energy cosmic rays, extreme kinetic energy $10^{20}$ electronvolts.
- Identify their sources :
  Supermassive black holes at the AGN centers (active galactic nuclei), Hypernovae, relic particles from the Big Bang.
- UHECR arrive with a probability law that we aim at estimating. We observe the cosmic ray incident points on the Earth. They might be deviated by several phenomenons.

# Fourier Analysis on $SO(3)$ and $\mathbb{S}^2$

## Definition

We define the rotational Fourier transform on $SO(3)$

$$f^{\star l}_{mn} = \int_{SO(3)} f(g) D^l_{mn}(g) dg, \quad l = 0, 1, 2..., \quad -l \leq m, n \leq l$$

where the $D^l_{mn}$ are the rotational harmonics which form an orthonormal basis of $L_2(SO(3))$

- $f^{\star l} = [f^{\star l}_{m,n}]$ is a matrix of dimension $(2l+1) \times (2l+1)$ with $l = 0, 1, 2, \dots$ et $-l \leq m, n \leq l$ .

# Fourier Analysis on $SO(3)$ et $\mathbb{S}^2$

**Definition**

The Fourier transform on $\mathbb{S}^2$ is defined as

$$f_m^{\star l} = \int_{\mathbb{S}^2} f(g)\overline{Y_m^l}(g)dg, \quad l = 0, 1, 2..., \quad -l \leq m \leq l$$

where the $Y_m^l$ are the spherical harmonics which form an orthonormal basis of $L_2(\mathbb{S}^2)$

- $f^{\star l} = [f_m^{\star l}]$ is an array of size $2l + 1$ with $l = 0, 1, 2, \ldots$ et $-l \leq m \leq l$ .

# Classical approach of inverse problems

$$f_\varepsilon * f(\omega) = \int_{SO(3)} f_\varepsilon(u) f(u^{-1}\omega) du$$

## Lemma

*We have for all* $-l \leq m \leq l$, $l = 0, 1, \ldots,$ *:*

$$(f_\varepsilon * f)^{\star l}_m = \sum_{n=-l}^{l} f^{\star l}_{\varepsilon, mn} f^{\star l}_n := (f^{\star l}_\varepsilon f^{\star l})_m. \quad (1)$$

- We inverse the convolution operator thanks to the Fourier Transform.

# Classical approach of inverse problems

- By considering the vectors $f^{\star l}$, $f_Z^{\star l}$ and the matrix $f_\varepsilon^{\star l}$, for all $l \geq 0$, using (1), we get :

$$
\begin{aligned}
f^{\star l} &= (f_\varepsilon^{\star l})^{-1} f_Z^{\star l} \\
f_m^{\star l} &= \sum_{n=-l}^{l} f_{\varepsilon^{-1},mn}^{\star l} f_{Z,n}^{\star l}
\end{aligned}
$$

where $f_{\varepsilon^{-1},mn}^{\star l} := (f_\varepsilon^{\star l})_{mn}^{-1}$

- We consider the empirical Fourier transform $\hat{f}_Z^{\star l}$ of $f_Z^{\star l}$

$$
\hat{f}_{Z,n}^{\star l} = 1/N \sum_{j=1}^{N} \overline{Y_n^l(Z_j)}
$$

- We deduce the following estimator $\hat{f}_m^{\star l}$

$$
\hat{f}_m^{\star l} := \frac{1}{N} \sum_{j=1}^{N} \sum_{n=-l}^{l} f_{\varepsilon^{-1},mn}^{\star l} \overline{Y_n^l(Z_j)}
$$

# Classical approach of inverse problems

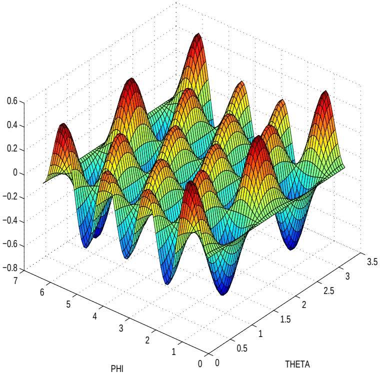
- We get by the inversion formula an estimator of the distribution $f$

$$\hat{f}(\omega) = \sum_{l=0}^{\tilde{N}} \sum_{m=-l}^{l} \hat{f}_m^{\star l} Y_m^l(\omega),$$

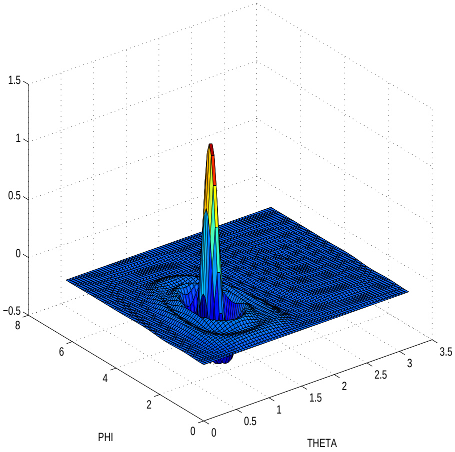
  with $\tilde{N}$ a parameter depending on the number of observations.

- Drawbacks of this method :
  The spherical harmonics are not localized on the sphere.
  This method may be unable to detect irregularities of the target function $f$.

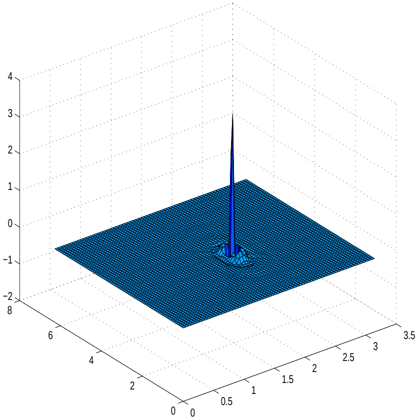
# Spherical harmonic $l = 8$ $m = 2$



# Needlet $j = 3$ $\eta = 250$



# Needlet $j = 5$ $\eta = 5000$



# Bibliography about Needlets


- F. Narcowich, P. Petrushev and J. Ward. Local tight frames on spheres. *SIAM J. Math. Anal.*, 2006.
- G. Kerkyacharian, P. Petrushev, D. Picard and T. Willer. Needlet algorithms for estimation in inverse problems. *Electron. J. Stat.* 2007.
- P. Baldi, G. Kerkyacharian, D. Marinucci and D. Picard. Adaptive density estimation for directional data using needlets. *Ann. Statist.*, 2009.
- P. Baldi, G. Kerkyacharian, D. Marinucci and D. Picard. Subsampling needlet coefficients on the sphere. *Bernoulli*, 2009.
- G. Kerkyacharian, G. Kyriazis, E. Le Pennec, P. Petrushev and D. Picard. Inversion of noisy radon transform by svd based needlets. *ACHA*, 2009.

# Bibliography


- Delabrouille, Cardoso, Le Jeune, Betoule, Faÿ, Guilloux. A full sky, low foreground, high resolution CMB map from WMAP *Astronomy and Astrophysics*, 2009.
- Faÿ, Guilloux, Betoule, Cardoso, Delabrouille, Le Jeune. CMB power spectrum estimation using wavelets *Physical Review D*, 2008.
- Guilloux, Faÿ, Cardoso. Practical wavelet design on the sphere. *Applied and Computational Harmonic Analysis*, 2009.

# Localization result

$$\psi_{j\eta}(x) = \sqrt{\lambda_{j\eta}} \sum_{l=2^{j-1}}^{2^{j+1}} b(l/2^j) \sum_{m=-l}^{l} \overline{Y_m^l(\xi_{j\eta})} Y_m^l(x).$$

For all $k \in \mathbb{N}$ there exists a constant $c_k$ such that for all $\xi \in \mathbb{S}^2$ :

$$|\psi_{j,\eta}(\xi)| \leq \frac{c_k 2^j}{(1 + 2^j d(\eta, \xi))^k} \; .$$

# Thresholding estimation procedure

- $$f = \sum_j \sum_{\eta \in \mathscr{Z}_j} (f, \psi_{j\eta})_{\mathbb{L}_2(\mathbb{S}^2)} \psi_{j\eta}.$$
- By Parseval equality $\beta_{j\eta} = (f, \psi_{j\eta})_{\mathbb{L}_2(\mathbb{S}^2)} = \sum_{lm} f_m^{\star l} \psi_{j\eta,m}^{\star l}$ but we already had

  $$\hat{f}_m^{\star l} := \frac{1}{N} \sum_{j=1}^{N} \sum_{n=-l}^{l} f_{\varepsilon^{-1},mn}^{\star l} \overline{Y_n^l(Z_j)}$$

  hence an unbiased estimator of $\beta_{j\eta}$

  $$\hat{\beta}_{j\eta} = \sum_{lm} \hat{f}_m^{\star l} \psi_{j\eta,m}^{\star l}. \tag{2}$$

  Finally, an estimator of $f$ is

  $$\hat{f} = \sum_{j=-1}^{J} \sum_{\eta \in \mathscr{Z}_j} t(\hat{\beta}_{j\eta}) \psi_{j\eta}.$$

# Thresholding estimation procedure

where $t$ is a thresholding procedure defined as follows :

$$t(\hat{\beta}_{j\eta}) = \hat{\beta}_{j\eta} I\{|\hat{\beta}_{j\eta}| \geq \kappa t_N |\sigma_j|\} \quad \text{with}$$

$$t_N = \sqrt{\frac{\log N}{N}},$$

$$\sigma_j^2 = A \sum_{ln} | \sum_m \psi^{\star l}_{j\eta,m} f^{\star l}_{\varepsilon^{-1} mn}|^2,$$

with $\|f_Z\|_\infty \leq A$.

## Theorem

*Let* $1 \leq p < \infty$, $\nu > 0$, *we suppose that*

$$\sigma_j^2 := A \sum_{ln} | \sum_{m} \psi^{\star l}_{j\eta,m} f^{\star l}_{\varepsilon^{-1} mn}|^2 \leq C 2^{2j\nu}, \ \forall\, j \geq 0. \qquad (3)$$

*Take* $\kappa^2 \geq \sqrt{3\pi A}$, $\sqrt{3\pi A}\kappa > \max 8p, 2p+1$ $2^J = d[t_N]^{\frac{-1}{(\nu+1)}}$ *with* $t_N = \sqrt{\frac{\log N}{N}}$ *et* $d > 0$. *Then if* $\pi \geq 1$, $s > 2/\pi$, $r \geq 1$ *(with the restriction* $r \leq \pi$ *if* $s = (\nu+1)(\frac{p}{\pi} - 1)$*), there exists a constant* $C$ *such that :*

$$\sup_{f \in B^s_{\pi,r}(M)} \mathbb{E}\|\hat{f} - f\|_p^p \leq C(\log(N))^{p-1}[N^{-1/2}\sqrt{\log(N)}]^{\mu p}, \qquad (4)$$

*where*

$$\mu = \frac{s}{s+\nu+1}, \quad \text{if } s \geq (\nu+1)(\frac{p}{\pi} - 1)$$
$$\mu = \frac{s - 2/\pi + 2/p}{s+\nu-2/\pi+1}, \quad \text{if } \frac{2}{\pi} < s < (\nu+1)(\frac{p}{\pi} - 1).$$

# The case of an unknown noise

$$\hat{\beta}_{j\eta} = \frac{1}{N}\sqrt{\lambda_{j\eta}} \sum_{l=2^{j-1}}^{2^{j+1}} b(l/2^j) \sum_{m=-l}^{l} \overline{Y_m^l(\xi_{j\eta})} \sum_{n=-l}^{l} f^{\star l}_{\varepsilon^{-1},mn} \sum_{u=1}^{N} Y_n^l(Z_u).$$

- We replace the rotational Fourier transform $(f^{\star l}_{\varepsilon})_{mn} := \hat{f}^{\star l}_{\varepsilon,mn}$ by its empirical version.
- $f^{\star l}_{\varepsilon^{-1},mn}$ denotes the $(m, n)$ element of the matrix $(f^{\star l}_{\varepsilon})^{-1} := f^{\star l}_{\varepsilon^{-1}}$ which is the inverse of the $(2l+1) \times (2l+1)$ matrix $(f^{\star l}_{\varepsilon})$.
- To get the empirical version $\hat{f}^{\star l}_{\varepsilon^{-1},mn}$ of $f^{\star l}_{\varepsilon^{-1},mn}$

  Compute the empirical matrix $(\hat{f}^{\star l}_{\varepsilon})$ then inverse it to get the matrix $(\hat{f}^{\star l}_{\varepsilon})^{-1} := \hat{f}^{\star l}_{\varepsilon^{-1}}$. The $(m, n)$ entry of the matrix $(\hat{f}^{\star l}_{\varepsilon})$ is given by the formula :

$$\hat{f}^{\star l}_{\varepsilon,mn} = \frac{1}{N}\sum_{j=1}^{N} D^l_{m,n}(\varepsilon_j),$$

# Simulations : Estimation of the uniform density probability $f = \frac{1}{4\pi}\mathbf{1}_{\mathbb{S}^2}$

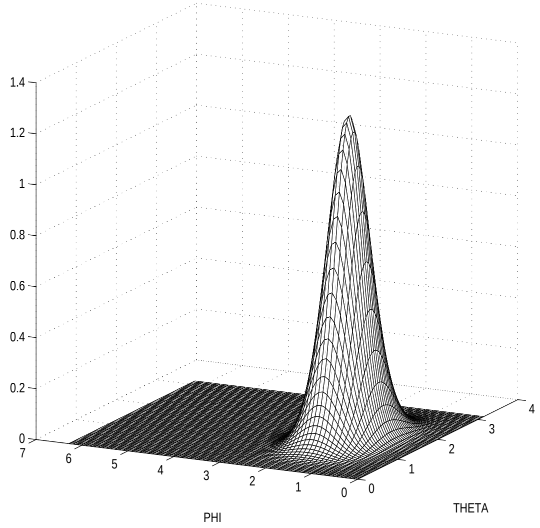
| | $j = 0$ | $j = 1$ | $j = 2$ | $j = 3$ |
|---|---|---|---|---|
| $\kappa = 0.2$ | 0 | 7 | 30 | 110 |
| $\kappa = 0.3$ | 0 | 0 | 2 | 6 |
| $\kappa = 0.4$ | 0 | 0 | 0 | 3 |

TABLE: Number of non zero coefficients surviving thresholding $\phi \sim U[0, \pi/8]$

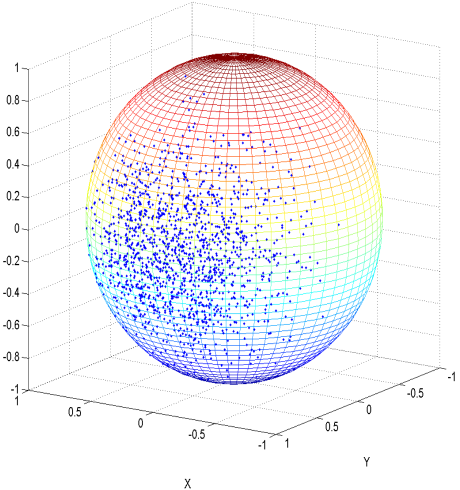
| | $j = 0$ | $j = 1$ | $j = 2$ | $j = 3$ |
|---|---|---|---|---|
| $\kappa = 0.2$ | 2 | 3 | 77 | 350 |
| $\kappa = 0.3$ | 0 | 0 | 4 | 10 |
| $\kappa = 0.4$ | 0 | 0 | 0 | 6 |

TABLE: Number of nonzero coefficients surviving thresholding $\phi \sim U[0, \pi]$

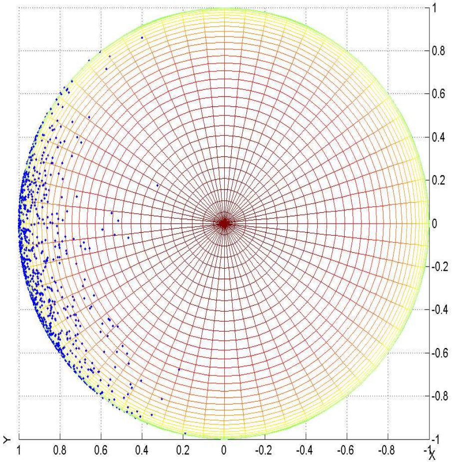
# Case of an unimodal density probability $f = Ce^{-4|\omega-\omega_1|^2}\mathbf{1}_{\mathbb{S}^2}$ with $\omega_1 = (0, 1, 0)$, $\omega_1 = (\frac{\pi}{2}, \frac{\pi}{2})$



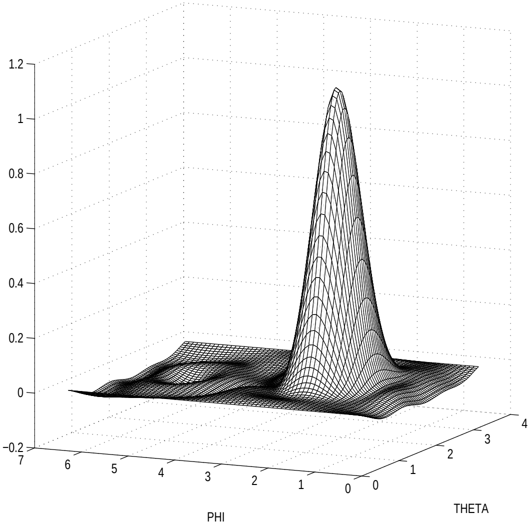
# Observations $\phi \sim U[0, \pi/8]$



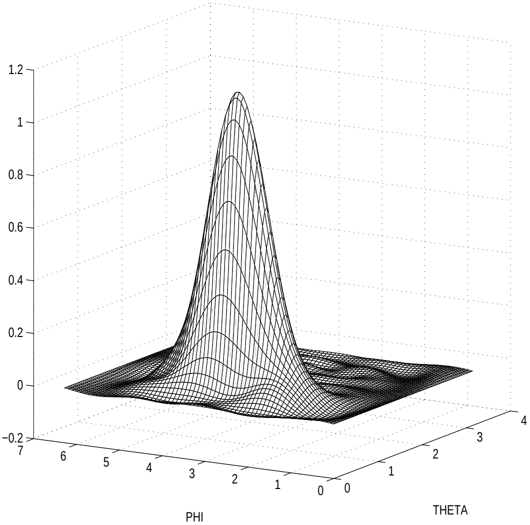
# Observations $\phi \sim U[0, \pi/8]$

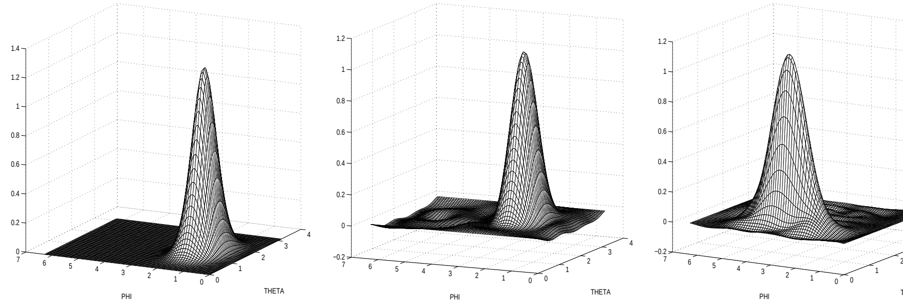


# Estimated density $\kappa = 0.5$



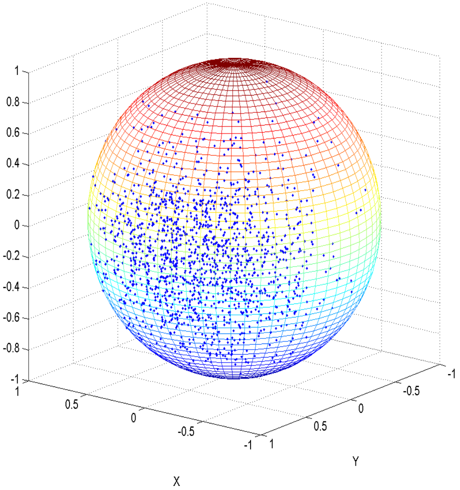
# Estimated density by the first method



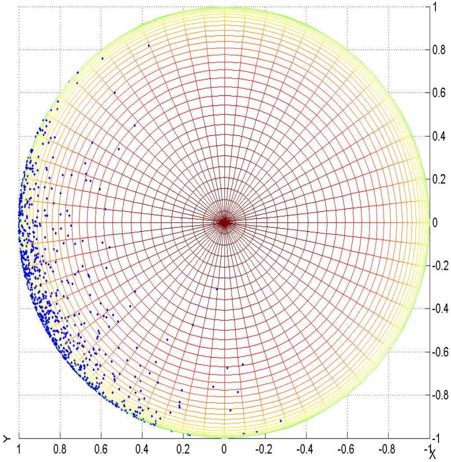


PHI
THETA
PHI
THETA
PHI

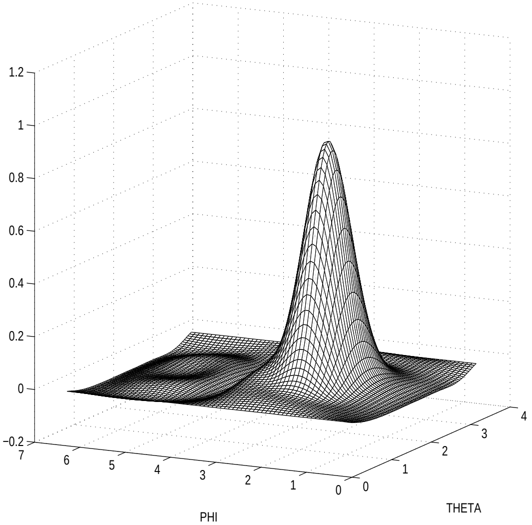
# Observations $\phi \sim U[0, \pi/4]$



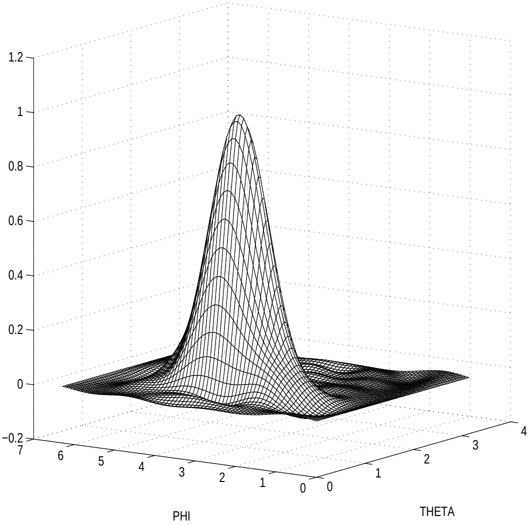
# Observations $\phi \sim U[0, \pi/4]$

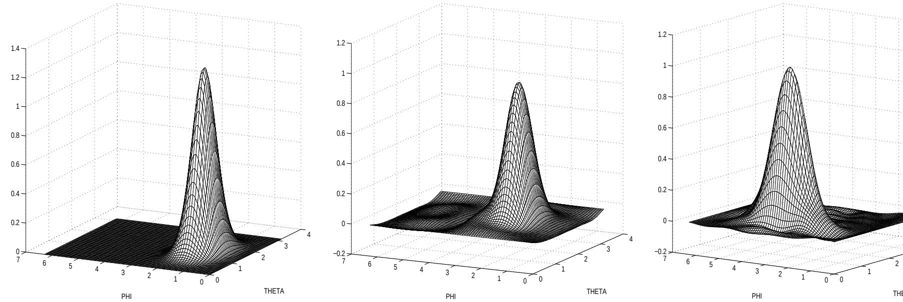


# Estimated density $\kappa = 0.5$



# Estimated density by the first method




PHI
THETA
PHI
THETA
PHI
THE

Kerkyacharian, Picard and Pham Ngoc. (2010). Localized spherical deconvolution. *In minor revision for the Annals of Statistics.*