Homologies et réseaux de capteurs

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MAT4NET
- Eye tremor
- Shoulder ligament strains
- Spinal ligament strains
- Elbow ligament strains
- Wireless EMG and EKG
- Wrist ligament strains
- Knee ligament strains
- Ankle ligament strains
- Wireless smart insoles measure force
- Depth of corneal implant orientation sensor for improved tooth crown prep
- Wireless vertebral bone strains
- 3DM-G measures orientation and motion
- Hip replacement - sensors for measuring micromotion
- Smart wireless sensor measures implant subsidence
- Smart total knee replacement
- Achilles tendon strains
- Arch support strains
Fig. 1. A collection of sensor nodes generates a convex workspace. The Rips complex of the network is an abstract simplicial complex which has no localization or coordinate data. In the example illustrated, the Rips complex encodes the communication network as one closed 3-simplex, even closed 2-simplices, and even closed 1-simplices connected as shown. The ‘holes’ in this Rips complex reflect the holes in the sensor cover, below.

Fig. 2. [left] The Rips complex has the property that all 2-simplices determine triangles in the domain which lie within the radius $r_c$ cover. However, the Rips complex does not capture the topology of the cover. A contractible union of $r_c$ balls can have Rips complex with nontrivial homology in dimension one [center], in which $R$ is a quadrilateral, two [right, in which $R$ is the boundary of a solid octahedron, or higher.

Fig. 3. In a sensor network with a sufficiently large hole in cover [left], the communication graph [center] has a cycle that cannot be ‘filled in’ by triangles. The filled in Rips complex [right] ‘sees’ this hole, even as an abstract complex devoid of sensor node location data.
Complexe simplicial

- Généralise la notion de graphes
- Constitué d’arêtes, de triangles, de tétraèdres, ...
Fig. 1. A collection of sensor nodes generates a coverage in the workspace. The Rips complex of the network is an abstract simplicial complex which has no localization or coordinate data. In the example illustrated, the Rips complex encodes the communication network as one closed 3-simplex, even closed 2-simplices, and even closed 1-simplices connected as shown. The ‘holes’ in this Rips complex reflect the holes in the sensor coverage, below.

Fig. 2. The Rips complex has the property that all 2-simplices determine triangles in the domain which lie within the radius $r_c$ coverage. However, the Rips complex does not capture the topology of the coverage. A contractible union of $r_c$ balls can have Rips complex with non-trivial homology in dimension one, in which $R$ is a quadrilateral, two, in which $R$ is the boundary of a solid octahedron, or higher.

Fig. 3. In a sensor network with a sufficiently large hole in coverage, the communication graph has a cycle that cannot be ‘filled in’ by triangles. The filled in Rips complex ‘sees’ this hole, even as an abstract complex devoid of sensor node location data.
Complexe de Cech
Sommets : \{ a, b, c, d, e \} = C_0
Arêtes : \{ ab, bc, ca, be, ec, ed \} = C_1
Triangles : \{ bec \} = C_2
Tétrahèdre : \emptyset = C_3
Exemple plus compliqué

**Figure 3** A sensors’ network and its associated Čech complex.

**Definition 9 (Vietoris-Rips complex)**
Given \((X, d)\) a metric space, a finite set of points in \(X\), and \(\varepsilon\) a real positive number. The Vietoris-Rips complex of parameter \(\varepsilon\) of \(X\), denoted \(R_\varepsilon(X)\), is the abstract simplicial complex whose \(k\)-simplices correspond to unordered \((k+1)\)-tuples of vertices in \(X\) which are pairwise within distance less than \(\varepsilon\) of each other.

In general, unlike the Čech one, Vietoris-Rips complexes are not topologically equivalent to the coverage of an area. However, the following gives us the relation between coverage and Vietoris-Rips complexes:

**Lemma 1**
Given \((X, d)\) a metric space, \(X\) a finite set of points in \(X\), and \(\varepsilon\) a real positive number, \(R_{\varepsilon+1/2}(X)\) \(\cong\) \(C_{\varepsilon}(X)\) \(\cong\) \(R_{2\varepsilon}(X)\).

In the Erdös-Rényi model, which is a random graph model, there is no geometric considerations, we extend the model to the homology:

**Definition 10 (Erdös-Rényi complex)**
Given \(n\) an integer and \(p\) a real number in \([0, 1]\), the Erdös-Rényi complex of parameters \(n\) and \(p\), denoted \(G(n, p)\), is an abstract simplicial complex with \(n\) vertices which are connected randomly. Each edge is included in the complex with probability \(p\) independent from every other edge. Then a \(k\)-simplex, for \(k \geq 2\), is included in the complex if and only if all its faces already are.

Only graph description is required to build a Vietoris-Rips or a Erdös-Rényi complex. That is why here we will give examples only on these two complexes.

### 3 Moments of random variables of an abstract simplicial complexe

By means of Malliavin calculus, we have computed explicitly the \(n\)-th order moment of the number of \(k\)-simplices. The computation of these moments are not detailed here, only are given the main theorems.
Opérateur de bord

Définition

\[ \partial_k : C_k \longrightarrow C_{k-1} \]

\[ [v_0, \cdots, v_k] \longmapsto \sum_{j=0}^{k} (-1)^j [v_0, \cdots, \hat{v_j}, \cdots] \]

Exemple

\[ \partial(bec) = ec - bc + be \]
\[ \partial^2(bec) = c - e - (c - b) + e - b = 0 \]
Théorème

\[ \partial_k \partial_{k+1} = 0 \]

Conséquence

\[ \text{Im} \ \partial_{k+1} \subset \ker \partial_k \]

Définition

\[ \beta_k = \dim \ker \partial_k - \text{range} \ \partial_{k+1} \]
Interprétation

- $\beta_0$ : nb de composantes connexes
- $\beta_1$ : nb de trous
- $\beta_2$ : nb de « vides »
- ...

Exemple

Rappel :

\[ C_0 = \{a, b, c, d, e\}, \quad C_1 = \{ab, bc, ca, be, ec, ed\} \]

\[ \partial_0 \equiv 0, \quad \partial_1 = \begin{pmatrix}
-1 & 0 & 1 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & -1 & 0
\end{pmatrix} \]

Nb de composantes connexes

dim ker \( \partial_0 \) = 5, range \( \partial_1 \) = 4 donc \( \beta_0 = 1 \)
Nombre de trous

Rappel :

\[ C_1 = \{ ab, bc, ca, be, ec, ed \}, \quad C_2 = \{ bec \} \]

\[
\partial_2 = \begin{pmatrix}
0 \\ -1 \\ 0 \\ 1 \\ 1 \\ 0
\end{pmatrix}
\]

dim ker\( \partial_1 \) = 2, range \( \partial_2 \) = 1 donc \( \beta_1 = 1 \)
Caractéristique d’Euler

Définition

\[ \chi = \sum_{j=0}^{d} (-1)^j \beta_j = \sum_{j=0}^{\infty} (-1)^j |C_k| \]

Inégalité de Morse

\[ -|C_{k-1}| + |C_k| - |C_{k+1}| \leq \beta_k \leq |C_k| \]
• Algorithme centralisé
• Nécessite de connaître les positions exactes

Complexe de Rips

\[ [x_0, \cdots , x_k] \in \mathcal{R}_k(\epsilon) \iff |x_i - x_j| \leq \epsilon \]
• Si distance $= l^\infty$, $C_k(\epsilon) = R_k(\epsilon)$
• Pour la distance euclidienne

$$R_k(\epsilon \sqrt{\frac{d + 1}{2d}}) \subset C_k(\epsilon) \subset R_k(2\epsilon)$$
Quelques résultats (D-Ferraz-Randriam-Vergne)

$n$ points, uniformément répartis sur un $d$-tore d’arête $a$

$k$ simplexes

$$E[|C_k(n)|] = \binom{n}{k+1}(k+1)^d \left(\frac{2\epsilon}{a}\right)^{dk}$$

Caractéristique d’Euler

$$E[\chi(n)] = \sum_{k=0}^{n} \binom{n}{k+1}(-1)^k(k+1)^d \left(\frac{2\epsilon}{a}\right)^{dk}$$
Dimension 5

Domination regions of $\beta_k$ when $d = 5$ in function of $\lambda$

Dominating $\beta_k$
- $\beta_0$
- $\beta_0$ and $\beta_1$
- $\beta_1$
- $\beta_1$ and $\beta_2$
- $\beta_2$
- $\beta_2$ and $\beta_3$
- $\beta_3$
- $\beta_3$ and $\beta_4$
- No dominance
Mise en œuvre

- Calculs algébriques classiques
- Base « minimale » des e.v. quotients donne les bords des trous
Green networking
Eteindre des capteurs en maintenant la couverture

Hauteur d’une arête
Ordre du plus grand simplexe auquel elle appartient

Indice d’un sommet
Minimum des hauteurs des arêtes adjacentes
We can see in Figure 5 the realisation of the coverage algorithm on a Vietoris-Rips complex of parameter $\varepsilon = 1$ based on a Poisson point process of intensity $\lambda = 4$ on a square of side length 2, with a fixed boundary of vertices on the square perimeter. The boundary vertices are circled in red.

For this configuration, on average on 200 runs, the algorithm removed $69.22\%$ of the non-boundary vertices, and computed in 206.01 seconds.

We can see in Figure 6 the realisation of the connectivity algorithm on an Erdös-Rényi complex of parameter $n = 15$ and $p = 0.3$, with random active vertices. We chose a small number of vertices for the figure to be readable. A vertex is active with probability $p_a = 0.5$ independently from every other vertices. The graph key is the same as before.

Figure 5: A Vietoris-Rips complex before and after the coverage reduction algorithm.

Figure 6: A Erdös-Rényi complex before and after the connectivity reduction algorithm.
Complexité (D.-Vergne)

Régime sous-critique

Si

\[ k \frac{1 + \eta - d}{k-1} < \theta := \left( \frac{\epsilon_n}{a} \right)^d < k \frac{1 + \eta + d}{k+1} \]

alors la hauteur tend vers \( k \) quand \( n \) tend vers l’infini.

Régime critique

Si \( n\theta_n \to 1 \) alors

\[ (\ln n)^{1-\eta} < \text{hauteur} < \ln n, \quad \forall \eta > 0. \]

Régime sur-critique

Si \( n\theta_n \to \infty \) alors hauteur \( \sim n\theta_n \).
Rips-Cech (D-Feng-Martins)

- Norme euclidienne
- Rayon de couverture $R_S$
- Rayon de communication $R_C$

![Graph showing intensity $\lambda$ vs. $p_{2d}(\lambda)(\%)$ with various lines indicating simulations and lower bounds for different values of $\gamma$.]
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