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PERCOLATION PRESQUE-CRITIQUE EN DEUX DIMENSIONS, ET QUELQUES MODÈLES LIÉS

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NEAR-CRITICAL PERCOLATION IN TWO DIMENSIONS, AND RELATED MODELS

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PhD Thesis supervised by Wendelin WERNER

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Contents

Introduction (français) **1**

1 Contexte . . . 1

1.1 Percolation par sites . . . 1

1.2 Invariance conforme de la percolation critique et exposants critiques . . . 3

1.3 Modèles de formes aléatoires . . . 4

2 Aperçu de nos résultats . . . 5

2.1 Percolation presque-critique en deux dimensions . . . 5

2.2 Propriétés à grande échelle de l'IIIC en deux dimensions . . . 7

2.3 Asymétrie des interfaces en régime presque-critique . . . 7

2.4 Exposants critiques pour la percolation en gradient . . . 8

2.5 Limites d'échelle pour le front de la percolation en gradient . . . 9

2.6 Géométrie des fronts de diffusion . . . 9

Introduction **11**

1 Background . . . 11

1.1 Site percolation . . . 11

1.2 Conformal invariance of critical percolation and critical exponents . . . 12

1.3 Random shape models . . . 14

2 Overview of our results . . . 14

2.1 Near-critical percolation in two dimensions . . . 15

2.2 Large scale properties of the IIIC for 2D percolation . . . 17

2.3 Asymmetry of near-critical percolation interfaces . . . 17

2.4 Critical exponents of planar gradient percolation . . . 18

2.5 Scaling limits for the gradient percolation front . . . 19

2.6 Geometry of diffusion fronts . . . 19

1 Near-critical percolation in two dimensions **21**

1.1 Introduction . . . 22

1.2 Percolation background . . . 23

1.2.1 Notations . . . 23
1.2.2 General properties . . . 25
1.2.3 Some technical tools . . . 26
1.3 Near-critical percolation overview . . . 29
1.3.1 Characteristic length . . . 29
1.3.2 Russo-Seymour-Welsh type estimates . . . 30
1.3.3 Outline of the paper . . . 31
1.4 Arm separation . . . 31
1.4.1 Arm events . . . 31
1.4.2 Well-separateness . . . 32
1.4.3 Statement of the results . . . 35
1.4.4 Proof of the main result . . . 37
1.4.5 Some consequences . . . 41
1.4.6 Arms in the half-plane . . . 44
1.5 Description of critical percolation . . . 44
1.5.1 Arm exponents for critical percolation . . . 45
1.5.2 Universal exponents . . . 47
1.6 Arm events near criticality . . . 50
1.6.1 Statement of the theorem . . . 50
1.6.2 Proof of the theorem . . . 51
1.6.3 Some complements . . . 59
1.7 Consequences for the characteristic functions . . . 60
1.7.1 Different characteristic lengths . . . 60
1.7.2 Main critical exponents . . . 62
1.7.3 Critical exponent for L . . . 62
1.7.4 Uniform exponential decay, critical exponent for θ . . . 66
1.7.5 Further estimates, critical exponents for χ and ξ . . . 69
1.8 Concluding remarks . . . 75
1.8.1 Other lattices . . . 75
1.8.2 Some related issues . . . 76

2 Large scale properties of the IIIC for 2D percolation 77
2.1 Introduction . . . 78
2.2 Setup and statement of theorems . . . 78
2.2.1 Setting . . . 78
2.2.2 Correlation lengths . . . 80
2.2.3 Description of the model . . . 81

2.3 Proofs . . . 83
2.4 A sharp transition . . . 88
2.5 Appendix . . . 89

3 Asymmetry of near-critical percolation interfaces 91
3.1 Introduction . . . 92
3.2 Preliminaries . . . 97
3.3 Tightness . . . 100
3.4 Length and dimension of near-critical interfaces . . . 101
3.4.1 Length of discrete interfaces . . . 101
3.4.2 Dimension of scaling limits . . . 103
3.5 The alternative . . . 104
3.6 Asymmetry of near-critical interfaces . . . 107
3.6.1 Discrete asymmetry . . . 107
3.6.2 Continuous asymmetry . . . 108
3.7 Informal discussion . . . 111

4 Critical exponents of planar gradient percolation 113
4.1 Introduction . . . 114
4.2 Homogeneous percolation preliminaries . . . 115
4.2.1 Setting . . . 115
4.2.2 Arm exponents . . . 116
4.2.3 Behavior near criticality . . . 117
4.3 Localization of the front . . . 119
4.3.1 Framework of gradient percolation . . . 119
4.3.2 Localization . . . 120
4.3.3 Uniqueness of the front . . . 123
4.4 Length of the front . . . 126
4.4.1 Two-arm estimates . . . 126
4.4.2 Expected length of the front . . . 131
4.4.3 Convergence in L^2 . . . 133
4.5 Outer boundary . . . 135
4.6 Open questions . . . 136

5 Scaling limits for the gradient percolation front 137
5.1 Introduction . . . 138
5.2 Characteristic length for gradient percolation . . . 139
5.2.1 Percolation background . . . 139

5.2.2 Regularity and asymptotic behavior of L_ϵ 140
5.2.3 Gradient percolation: setup 141
5.2.4 Characteristic length for gradient percolation 141
5.2.5 Macroscopic properties of the front 142
5.3 Existence of non-trivial scaling limits 146
5.4 Properties of the scaling limits 148
5.4.1 Similarities with critical percolation 148
5.4.2 Local asymmetry 148

6 Geometry of diffusion fronts **149**
6.1 Introduction 150
6.2 Setup and statement of the results 152
6.2.1 Description of the model 152
6.2.2 Main ingredients 153
6.2.3 Statement of the results 155
6.3 Proofs 156
6.4 Model with a source 160

Bibliography **167**

Introduction (français)

Cette thèse consiste en six chapitres étudiant différents aspects de la percolation près de sa transition de phase en deux dimensions. Cette introduction est divisée en deux parties : nous rappelons d'abord brièvement les résultats préexistants reliés à notre travail, et nous présentons ensuite chacun des six chapitres.

1 Contexte

1.1 Percolation par sites

La théorie mathématique de la percolation a été initiée par Broadbent et Hammersley en 1957 [15]. Nous rappelons d'abord rapidement ses principales propriétés, et nous renvoyons le lecteur aux références classiques [39, 35] pour plus de détails.

La percolation par sites sur un réseau $\mathbb{L}$ ($\mathbb{Z}^d$ par exemple) peut être décrite comme suit. Pour chaque site x de $\mathbb{L}$, on lance une pièce (biaisée) pour décider s'il est occupé (ouvert, noir) ou vacant (fermé, blanc) : pour un certain paramètre p entre 0 et 1, x est occupé avec probabilité p et vacant avec probabilité $1-p$, indépendamment des autres sites.

On s'intéresse ensuite aux propriétés de connexité de l'ensemble des sites occupés : deux sites occupés x et y sont dits connectés, ce que l'on note $x \rightsquigarrow y$, s'il existe un chemin de x à y sur le réseau composé uniquement de sites occupés. Les sites occupés peuvent ainsi être regroupés en composantes connexes (maximales), que l'on appelle "clusters".

On dit qu'un site x est connecté à ∞ s'il appartient à un cluster infini, et on définit

$$\theta(p) = \mathbb{P}_p(0 \rightsquigarrow \infty).$$

Notons que l'invariance par translation du modèle (si on considère un réseau transitif) entraîne que pour n'importe quel site x, $\theta(p) = \mathbb{P}_p(x \rightsquigarrow \infty)$. Il est en réalité possible et classique de montrer (en combinant l'invariance par translation et des idées d'isopérimétrie) que pour un réseau d-dimensionnel, lorsque $\theta(p) > 0$ il existe presque sûrement une unique composante connexe infinie.

Une caractéristique importante de la percolation est l'existence d'une transition de phase. Si on introduit le paramètre critique

$$p_c(\mathbb{L}) = \sup\{p \text{ t.q. } \theta(p) = 0\},$$

on peut aisément vérifier que $0 < p_c(\mathbb{L}) < 1$ pour les réseaux usuels tels que $\mathbb{Z}^d$ ou le réseau triangulaire $\mathbb{T}$ en deux dimensions (voir Figure 1). Ainsi,

- Lorsque $p < p_c$, il n'y a p.s. aucun cluster infini (régime sous-critique).
- Lorsque $p > p_c$, il y a p.s. un unique cluster infini (régime sur-critique).

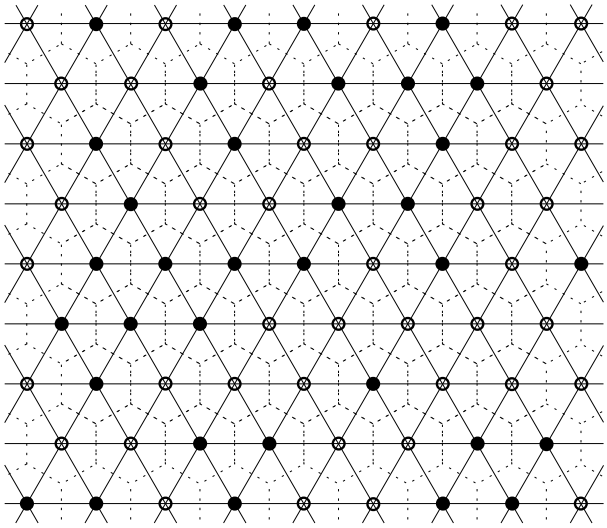


FIG. 1 – Percolation par sites sur le réseau triangulaire.

Le cas $p = p_c$ est appelé régime critique. Déterminer s'il y a ou non un cluster infini dans ce cas peut s'avérer être une question très difficile : pour $\mathbb{Z}^3$, la réponse n'est en fait pas connue pour le moment. Mentionnons également que le régime sous-critique présente la propriété de décroissance exponentielle suivante : la probabilité que 0 soit connecté à un site à distance n décroît exponentiellement en n (plus précisément, cette probabilité est bornée par $C_1(p)e^{-C_2(p)n}$ pour deux constantes $C_1, C_2 > 0$ dépendant de p) – un résultat similaire peut être formulé pour le régime sur-critique.

À partir de maintenant, nous nous restreignons aux réseaux deux-dimensionnels. Nous nous concentrerons principalement sur le réseau triangulaire $\mathbb{T}$, car c'est actuellement le réseau pour lequel la description la plus fine est connue, comme nous l'expliquerons dans la section suivante. La percolation par sites sur ce réseau est souvent représentée comme un coloriage aléatoire des faces du réseau hexagonal dual, et elle possède une propriété d'auto-dualité (self-matching) qui amène naturellement à conjecturer que

$$p_c(\mathbb{T}) = \frac{1}{2},$$

ce qui est un résultat célèbre dû à Kesten [38]. Nous savons de plus dans ce cas qu'il n'y a aucun cluster infini au point critique, autrement dit que $\theta(1/2) = 0$.

La décroissance exponentielle des probabilités de connexion rend les régimes sous- et sur-critiques relativement « prévisibles » sur des grandes échelles. En régime sous-critique par exemple, dans une portion du réseau de diamètre N, on n'observe typiquement que des composantes occupées très petites, de taille au plus $\log N$, perdues dans une « mer » de sites vacants. En un certain sens, le régime critique est bien plus riche : les probabilités de connexion ne décroissent plus exponentiellement vite, et dans une région de diamètre N, on observe à la fois des structures occupées et des structures vacantes de diamètres comparables à N, et des objets fractals apparaissent lorsque N devient très grand.

Nous examinerons ici différents aspects du comportement de la percolation près du point critique, i.e. de la transition du régime sous-critique au régime sur-critique en passant par le régime critique.

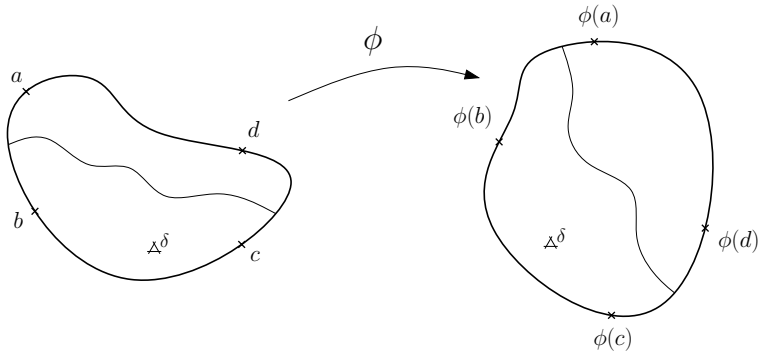


FIG. 2 – Invariance conforme de la probabilité de croisement.

1.2 Invariance conforme de la percolation critique et exposants critiques

La compréhension mathématique de la percolation critique en deux dimensions s'est considérablement accrue lors de la dernière décennie, grâce à la preuve par Smirnov de l'invariance conforme sur le réseau triangulaire, et à la définition des courbes SLE par Schramm qui décrivent les limites d'échelle conformément invariantes possibles.

Intéressons-nous plus en détails au résultat de Smirnov, et pour cela prenons n'importe quel domaine de Jordan D (lisse), et deux points a et b sur ∂D. Pour tout $\delta > 0$, on peut considérer le réseau de pas δ, et une approximation $(D^\delta, a^\delta, b^\delta)$ de (D, a, b) sur ce réseau. Les deux points a^δ et b^δ divisent ∂D^δ en deux arcs : on décide de colorier l'un d'eux en noir, et l'autre en blanc. On regarde ensuite l'interface entre le cluster de sites noirs connectés à l'arc colorié en noir et le cluster de sites blancs connectés à l'arc blanc. Cette interface γ^δ peut être explorée de manière dynamique par un « processus d'exploration » de a à b. Le théorème de Smirnov [69, 70, 78, 17] affirme alors que lorsque le pas $\delta \searrow 0$, γ^δ converge en loi vers un SLE(6) « chordal », qui est une courbe continue aléatoire de a à b (cette convergence en loi a lieu par rapport à la distance uniforme sur les courbes continues « à reparamétrisation monotone près »).

Une étape clé pour prouver cette convergence consiste à établir la formule de Cardy : dans un domaine D avec quatre points a, b, c, d sur ∂D (dans cet ordre), cette formule donne la probabilité d'existence d'un croisement entre les arcs (ab) et (cd) lorsque $\delta \searrow 0$. Cette formule affirme en particulier que la probabilité de croisement est conformément invariante : si ϕ est une bijection conforme de D vers un autre domaine D', alors la probabilité d'existence d'un croisement de (ab) à (cd) dans D est égale à la probabilité d'existence dans D', de $(\phi(a)\phi(b))$ à $(\phi(c)\phi(d))$. Cette invariance conforme est cruciale pour relier la percolation aux processus SLE.

Pour décrire la percolation critique et presque-critique, certains événements exceptionnels sont très utiles : les *événements de bras*, faisant référence à l'existence dans des anneaux d'un certain nombre de croisements (« bras »), la couleur de chacun de ces croisements (noir ou blanc) étant prescrite. Pour deux entiers strictement positifs $n \leq N$, on définit l'événement

$$A_{j,\sigma}(n, N) = \{\partial S_n \rightsquigarrow_{j,\sigma} \partial S_N\}$$

qu'il existe j bras monochromatiques disjoints dans l'anneau $S_{n,N}$ de rayons n et N (centré sur l'origine), dont les couleurs dans le sens inverse des aiguilles d'une montre sont celles prescrites par la suite de couleurs σ. Ces événements de bras peuvent être exprimés en termes de processus d'exploration, si bien que la limite

$$\lim_{\delta \searrow 0} \mathbb{P}_{1/2}(A_{j,\sigma}(\delta n, \delta N))$$

peut être calculée en utilisant le SLE(6). On peut ensuite prouver que pour tout n_0 fixé (suffisamment grand),

$$\mathbb{P}_{1/2}(A_{j,\sigma}(n_0, N)) = N^{-\alpha_j + o(1)}$$

lorsque $N \to \infty$, avec

- $\alpha_1 = 5/48$,
- et pour $j \geq 2$, $\alpha_j = (j^2 - 1)/12$ (si σ n'est pas constante).

Ces exposants α_j sont appelés « exposants de bras ». Nous renvoyons le lecteur à [46, 78] pour une preuve complète de ce résultat.

Ayant à notre disposition ces exposants de bras pour la percolation critique, nous pouvons ensuite utiliser les relations de « scaling » dues à Kesten [43] pour décrire la percolation près du point critique. Nous n'entrerons pas dans les détails ici comme c'est le sujet de notre premier chapitre, mais mentionnons seulement que l'on peut prouver de cette manière un paradigme de la physique statistique venant de la théorie de la renormalisation, à savoir que les fonctions caractéristiques usuelles se comportent comme des puissances de $|p - p_c|$ quand $p \to p_c$. Pour θ, on a par exemple :

$$\theta(p) = (p - 1/2)^{5/36 + o(1)}$$

lorsque $p \to 1/2^+$.

Dans notre travail, nous utiliserons en réalité les processus SLE indirectement. En effet, nous nous baserons sur les résultats pour les exposants de bras qui ont été obtenus par des calculs sur les SLE, mais nous n'utiliserons jamais vraiment leur définition via l'équation de Loewner, nous ne la rappelons donc pas dans cette introduction.

1.3 Modèles de formes aléatoires

Notre étude est motivée par des modèles et des travaux provenant de la physique de la matière condensée. Dans [63], les physiciens Gouyet, Rosso et Sapoval ont commencé à étudier les propriétés géométriques des fronts de diffusion. Ils considèrent dans cet article un processus d'exclusion simple avec source, menant à une densité de particules $p(z)$ qui est une fonction décroissante de l'ordonnée y. Ils proposent alors l'approximation que les différents sites sont indépendamment occupés ou vacants, ce qui les amène à considérer un processus de percolation inhomogène de paramètre $p(z)$, un modèle qu'ils appelleront par la suite « percolation en gradient ». Cette approximation leur permet de réaliser des simulations précises et de montrer numériquement que les fronts de diffusion sont des objets fractals. Ils mesurent ainsi la dimension $D_f = 1.76 \pm 0.02 \simeq 7/4$, une valeur qui avait déjà été observée pour les interfaces de percolation en régime critique. Se basant aussi sur l'idée que les fronts restent là où $p(z)$ est proche de la valeur critique p_c, ils conjecturent donc que les fronts de diffusion ressemblent à des bords de clusters de percolation.

La percolation en gradient est maintenant devenue un outil classique pour modéliser des milieux aléatoires (voir par exemple [64, 29, 34]). Mentionnons aussi un modèle de corrosion ou d'érosion, connu sous le nom de percolation en gradient « de corrosion » ([65, 62, 31, 66]). Dans ce modèle, qui est apparu d'abord dans des expériences [5], des résistances aléatoires sont associées à chaque site d'un matériau initialement lisse, que l'on place en contact avec une solution corrosive. Lorqu'un site « faible » (de petite résistance) est atteint par le liquide, il est dissout, et le pouvoir corrosif du liquide décroît. On atteint alors en temps fini une interface finale, qui semble être reliée à la percolation en gradient : en particulier sa dimension mesurée est très proche de 7/4.

La percolation en gradient étant plutôt facile à simuler, elle fournit aussi une manière efficace d'estimer les paramètres critiques, en calculant la hauteur moyenne de l'interface. Cette idée a été

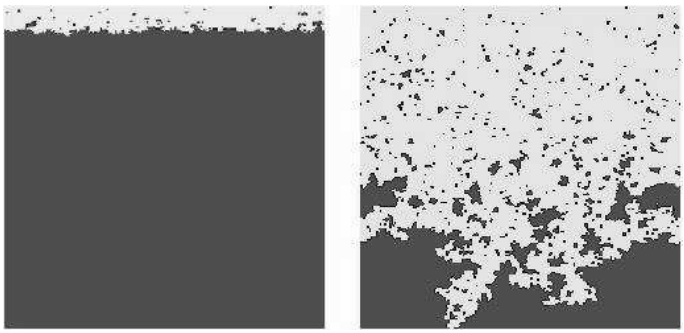


FIG. 3 – Côtes fractales (Figure B. Sapoval).

utilisée pour la première fois dans [59] pour $\mathbb{Z}^2$, où la valeur $p_c = 0.59280 \pm 10^{-5}$ est obtenue, et utilisée plus tard dans [79, 60, 80, 75] pour divers autres réseaux et pour la percolation continue.

2 Aperçu de nos résultats

Ce travail consiste en six chapitres. Dans le Chapitre 1, nous présentons en détails les techniques et les résultats dûs à Kesten qui permettent de décrire la percolation presque-critique sur le réseau triangulaire, et nous obtenons aussi quelques nouvelles conséquences. Dans les chapitres suivants, nous appliquons ensuite ces idées dans diverses situations : nous étudions successivement le modèle d' « Inhomogeneous Incipient Infinite Cluster » (Chapitre 2), les propriétés géométriques des interfaces de percolation en régime presque-critique (Chapitre 3), le modèle de percolation en gradient (Chapitres 4 et 5), et finalement (Chapitre 6) un certain modèle de diffusion dans l'esprit de l'article original de Gouyet, Rosso et Sapoval [63]. Notons que les quatre premiers chapitres correspondent aux articles [55], [28], [56] et [54] respectivement.

2.1 Percolation presque-critique en deux dimensions

Dans ce premier chapitre, nous présentons et nous développons les idées et les techniques permettant de décrire la percolation critique et presque-critique en deux dimensions, nous basant sur les relations de « scaling » dues à Kesten, et sur la description de la percolation critique obtenue par Lawler, Schramm, Smirnov et Werner [71, 50].

Un outil important dans nos considérations est la longueur caractéristique dite de « finite-size scaling » L_ϵ, introduite dans [23], que nous définissons maintenant. Pour cela, considérons l'événement $\mathcal{C}_H$ qu'il existe un croisement horizontal dans le losange $[0,n] \times [0,n]$, de côté n : à $p = 1/2$, $\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) = 1/2$, et cette probabilité tend vers 0 ou 1 quand n tend vers l'infini, selon que $p < 1/2$ ou $p > 1/2$. La longueur caractéristique L_ϵ mesure alors l'échelle jusqu'à laquelle cette probabilité de croisement reste éloignée de 0 et de 1 : pour chaque $\epsilon \in (0, 1/2)$, on définit

$$L_\epsilon(p) = \begin{cases} \min\{n \text{ t.q. } \mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \leq \epsilon\} & \text{si } p < 1/2, \\ \min\{n \text{ t.q. } \mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \geq 1 - \epsilon\} & \text{si } p > 1/2. \end{cases}$$

Comme évoqué précédemment, les événements de bras jouent un rôle central. Ces événements possèdent une propriété importante de « séparation » : si nous prescrivons aussi l'endroit où les bras arrivent sur les bords (intérieur et extérieur) de $S_{n,N}$, ce qui donne lieu à sous-événement

$\tilde{A}_{j,\sigma}(n,N)$ de $A_{j,\sigma}(n,N)$, alors la probabilité reste du même ordre de grandeur. Plus précisément, nous avons[1]

$$\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}(n,N)) \asymp \hat{\mathbb{P}}(A_{j,\sigma}(n,N))$$

uniformément en p, $\hat{\mathbb{P}}$ entre $\mathbb{P}_p$ et $\mathbb{P}_{1-p}$ (i.e. avec des paramètres compris entre p et $1-p$) et $n \leq N \leq L_\epsilon(p)$. Cette propriété de séparation a été utilisée dans [71] pour obtenir les exposants de bras pour la percolation critique.

Les exposants de bras pour la percolation critique peuvent ensuite être utilisés pour décrire la percolation presque-critique. Pour cela, nous aurons besoin de comparer les événements de bras au point critique et près du point critique. En étendant certains résultats de Kesten, nous prouvons que pour tout j, tout σ,

$$\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}(n,N)) \asymp \mathbb{P}_{1/2}(A_{j,\sigma}(n,N))$$

uniformément en p, $\hat{\mathbb{P}}$ entre $\mathbb{P}_p$ et $\mathbb{P}_{1-p}$ et $n \leq N \leq L_\epsilon(p)$.

Cela permet d'établir l'existence et les valeurs des exposants critiques associés aux principales fonctions caractéristiques du modèle. Tout d'abord, nous prouvons que pour tout $\epsilon \in (0,1/2)$,

$$|p-1/2|\big(L_\epsilon(p)\big)^2\mathbb{P}_{1/2}(0 \rightsquigarrow_4 \partial S_{L_\epsilon(p)}) \asymp 1$$

quand $p \to 1/2$, ce qui combiné avec l'exposant à 4 bras fournit

$$L_\epsilon(p) = |p-1/2|^{-4/3+o(1)}.$$

Cette relation implique aussi que

$$L_\epsilon(p) \asymp L_{\epsilon'}(p)$$

pour n'importe quels $\epsilon, \epsilon' \in (0,1/2)$ fixés.

Nous utilisons ensuite un dernier ingrédient, une propriété de décroissance exponentielle uniforme en p : pour n'importe quel $k \geq 1$ fixé, pour tout $p < 1/2$ et tout n,

$$\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,kn])) \leq C_1 e^{-C_2 n/L_\epsilon(p)},$$

où C_1 et C_2 sont des constantes ne dépendant que de k et ϵ. On obtient l'exposant critique pour θ : lorsque $p \to 1/2^+$,

$$\theta(p) \asymp \mathbb{P}_p(0 \rightsquigarrow \partial S_{L_\epsilon(p)}) = (L_\epsilon(p))^{-5/48+o(1)} = (p-1/2)^{5/36+o(1)}.$$

En ce qui concerne les deux autres fonctions habituelles

$$\chi(p) = \mathbb{E}_p\big[|C(0)|; |C(0)| < \infty\big]$$

(la taille moyenne d'un cluster fini) et

$$\xi(p) = \left[\frac{1}{\mathbb{E}_p\big[|C(0)|; |C(0)| < \infty\big]} \sum_x \|x\|_\infty^2 \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big)\right]^{1/2}$$

(le rayon moyen d'un cluster fini), il a été prouvé dans [43] que les sites à distance inférieure à $L_\epsilon(p)$ produisent une fraction positive de la somme totale : la contribution des sites à distance $\gg L_\epsilon(p)$ est négligeable. Ainsi, quand $p \to 1/2$,

$$\chi(p) \asymp (L_\epsilon(p))^2\mathbb{P}_p(0 \rightsquigarrow \partial S_{L_\epsilon(p)})^2 = |p-1/2|^{-43/18+o(1)}$$

et

$$\xi(p) \asymp L_\epsilon(p) = |p-1/2|^{-4/3+o(1)}.$$

[1] Nous utilisons le symbole "$\asymp$" pour indiquer que deux quantités sont du même ordre de grandeur.

2.2 Propriétés à grande échelle de l'IIIC en deux dimensions

Dans ce chapitre, nous étudions le modèle d'Inhomogeneous Incipient Infinite Cluster (IIIC), introduit dans [25]. Ce modèle consiste en un processus de percolation indépendante inhomogène, avec un paramètre légèrement supérieur à p_c, et décroissant vers p_c comme une puissance de la distance à l'origine : nous choisissons comme paramètre d'occupation du site z

$$p(z) = p_c + \|z\|^{-\lambda}.$$

Il a été prouvé dans [25] qu'une transition a lieu pour $\lambda_c = 1/\nu$, où $\nu = 4/3$ est l'exposant critique pour la longueur caractéristique mentionné précédemment :

- Quand $\lambda > \lambda_c$, il n'y a p.s. aucun cluster infini.
- Quand $\lambda < \lambda_c$, il y a p.s. un unique cluster infini $\mathcal{C}_\infty$.

On otient donc dans le deuxième cas un « incipient infinite cluster », qui peut être décrit assez précisément en utilisant des techniques de percolation presque-critique. Nous montrons d'abord que dans S_N,

$$\mathbb{E}[|\mathcal{C}_\infty \cap S_N|] = N^{2-\lambda\beta+o(1)}$$

lorsque $N \to \infty$. Nous prouvons aussi que

$$\mathrm{Var}(|\mathcal{C}_\infty \cap S_N|) = o(\mathbb{E}[|\mathcal{C}_\infty \cap S_N|]^2),$$

de telle sorte que $|\mathcal{C}_\infty \cap S_N|$ est proche de son espérance avec forte probabilité.

Nous étudions aussi plus en détail la transition à $\lambda = \lambda_c$. Cette transition se révèle être très abrupte : de manière informelle (le résultat précis est moins explicite car des problèmes techniques apparaissent), si l'on prend

$$p(z) = p_c + C\|z\|^{-1/\nu}(\log\log\|z\|)^{1/\nu},$$

alors la valeur de la constante C détermine la présence ou l'absence d'un cluster infini.

2.3 Asymétrie des interfaces en régime presque-critique

Dans ce chapitre, nous étudions les propriétés géométriques des interfaces apparaissant en régime presque-critique. Comprendre ce régime est important pour décrire les modèles où la transition de phase de la percolation apparaît spontanément, comme la percolation en gradient introduite dans le prochain chapitre. En effet, souvent ce que l'on observe est relié à la percolation *près du point critique* plutôt qu'*au point critique* exactement.

Pour fixer les idées, considérons le cadre suivant : le triangle de côté N, avec le côté inférieur centré sur l'origine. On colorie ensuite en noir le bord gauche et en blanc le bord droit, comme montré sur la Figure 4, et on regarde l'interface partant de l'origine entre le cluster noir connecté au côté gauche, et le cluster blanc connecté au côté droit.

Considérons maintenant deux suites $N_k \to \infty$ et $p_k \geq 1/2$. Si nous envoyons (en la réduisant) cette interface dans le triangle T, la suite γ_k des courbes obtenues possède des limites d'échelle, d'après des arguments de tension dûs à Aizenman et Burchard [2]. Nous montrons que trois régimes principaux apparaissent, suivant le comportement de la quantité $N_k^2\mathbb{P}_{1/2}(0 \rightsquigarrow_4 \partial S_{L(p_k)})$:

- Si celle-ci tend vers 0, alors γ converge (en loi) vers un SLE(6) « chordal », de l'origine vers le sommet supérieur de T (cas critique).
- Si elle tend vers ∞, alors γ converge vers le bord droit de T (cas sur-critique).
- Si elle reste éloignée de 0 et de ∞, on obtient un régime intermédiaire, appelé régime « off-critique ».

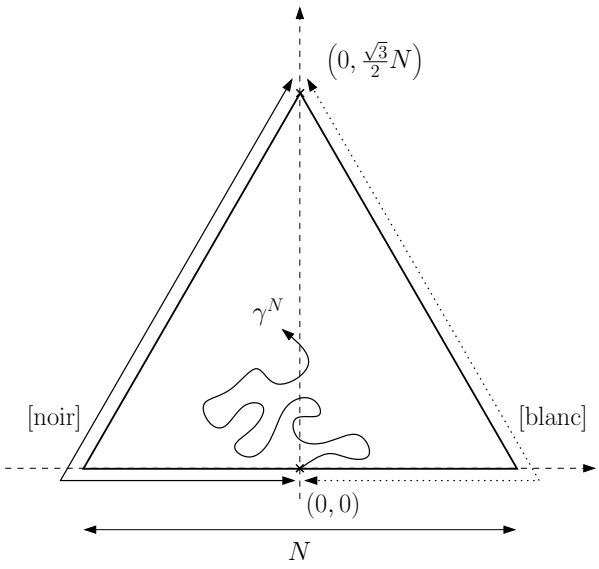


FIG. 4 – On s'intéresse à l'interface entre le cluster noir relié au côté gauche, et le cluster blanc relié au côté droit.

Concentrons-nous maintenant sur ce régime intermédiaire. Tout d'abord, nous prouvons qu'il partage de nombreuses propriétés avec le régime critique. Comme les probabilités des événements de bras restent du même ordre de grandeur, les propriétés macroscopiques telles que la dimension de Hausdorff (7/4) restent les mêmes. D'un autre côté, nous mettons en évidence une propriété d'asymétrie qui distingue ce régime du régime critique : la courbe a une légère tendance à tourner plus vers la droite que vers la gauche. Cette propriété est facile à obtenir au niveau discret, et nous prouvons qu'elle reste visible à la limite d'échelle, sur n'importe quelle portion de l'interface : un « arrière-goût » de percolation sur-critique reste donc présent. Pour cela, nous sub-divisons T en petits triangles de taille η, et nous comptons dans combien d'entre eux l'interface sort du côté droit plutôt que du côté gauche.

2.4 Exposants critiques pour la percolation en gradient

Nous étudions dans ce chapitre le modèle de percolation en gradient, introduit dans [63] par les physiciens Gouyet, Rosso et Sapoval pour simuler des fronts de diffusion. Le cadre est le suivant : dans une bande de largeur $2N$ et de longueur ℓ_N, nous réalisons un processus de percolation indépendante inhomogène, avec un paramètre $p(z)$ au site z qui décroît linéairement de 1 (pour $y = -N$) à 0 (pour $y = N$) suivant l'ordonnée y :

$$p(z) = 1/2 - y/2N.$$

En bas de la bande, le paramètre est proche de 1 et on observe un grand cluster de sites noirs, et de manière similaire on observe en haut un grand cluster de sites blancs. Le phénomène caractéristique de ce modèle est l'apparition d'un unique « front », une interface touchant simultanément ces deux clusters, comme le montre la Figure 5.

Supposons que la longueur de la bande satisfait $N^{4/7+\delta} \leq \ell_N \leq N^{\gamma}$ pour deux constantes $\delta, \gamma > 0$. Alors pour un certain $\epsilon > 0$, le front $\mathcal{F}_N$ est bien défini avec probabilité au moins $1 - C_1 e^{-C_2 N^{\epsilon}}$. Dans ce cas, une arête e est sur $\mathcal{F}_N$ si et seulement si elle est reliée par deux bras aux bords de la bande, un bras noir vers le bas et un bras blanc vers le haut.

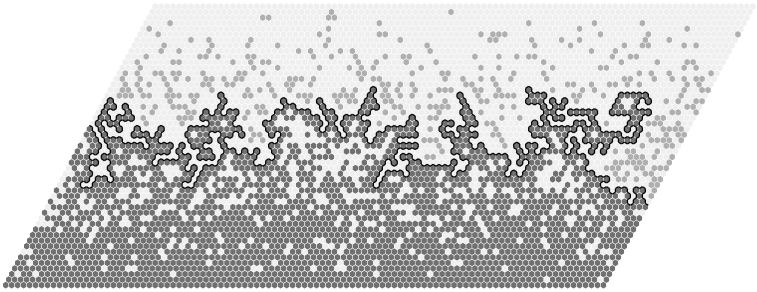


FIG. 5 – L'interface apparaissant dans le modèle de percolation en gradient.

Nous prouvons aussi que ce front reste localisé dans une bande autour de la droite $y = 0$ où $p = p_c$, avec des fluctuations d'ordre $N^{4/7}$. Plus précisément, nous prouvons que pour n'importe quel $\eta > 0$, avec probabilité au moins $1 - C_1' e^{-C_2' N^{\epsilon'}}$, $\mathcal{F}_N$ reste dans la bande de largeur $2N^{4/7+\eta}$ autour de $y = 0$, et sort de la bande de largeur $2N^{4/7-\eta}$.

En utilisant la caractérisation de $\mathcal{F}_N$ comme l'ensemble des arêtes possédant deux bras, on peut estimer diverses quantités le décrivant via l'exposant à 2 bras. Nous obtenons entre autres que sa longueur T_N vérifie :

$$\mathbb{E}[T_N] = \ell_N N^{3/7+o(1)}.$$

De plus, une propriété de décorrélation horizontale peut être démontrée, si bien que

$$\mathrm{Var}(T_N) = o(\mathbb{E}[T_N]^2).$$

Ainsi, T_N est proche de son espérance avec forte probabilité.

2.5 Limites d'échelle pour le front de la percolation en gradient

Ce court chapitre est une suite du précédent. Nous raffinons les résultats obtenus pour la percolation en gradient dans une bande en introduisant la quantité

$$\sigma_N^\epsilon = \sup\big\{\sigma \text{ t.q. } L_\epsilon\big(p(\sigma)\big) \geq \sigma\big\},$$

qui mesure les fluctuations verticales du front. Cette quantité s'avère être une longueur caractéristique – à la fois horizontale et verticale – pour le modèle. Nous montrons que $\sigma_N^\epsilon \asymp \sigma_N^{\epsilon'}$ pour n'importe quels $\epsilon, \epsilon' \in (0, 1/2)$, et que σ_N^ϵ fournit la seule manière (à constantes multiplicatives près) de renormaliser le front pour obtenir des limites non triviales. Nous construisons ensuite des limites d'échelle en utilisant les arguments de tension de Aizenman et Burchard. Ces limites d'échelle se comportent localement comme des interfaces en régime off-critique.

2.6 Géométrie des fronts de diffusion

Dans ce dernier chapitre, nous étudions certains modèles de fronts de diffusion : ce sont des modèles de ce type qui amenèrent Sapoval, Rosso et Gouyet [63] à introduire la percolation en gradient. Notre modèle principal est le suivant : sur un réseau, on dépose un certain nombre fixé n de particules à l'origine, et on les laisse réaliser des marches aléatoires simples indépendantes pendant une durée t. Un gradient de concentration apparaît (voir Figure 6), et une approximation Poissonienne, valide localement, permet de décrire l'interface en utilisant notre étude de la percolation en gradient.

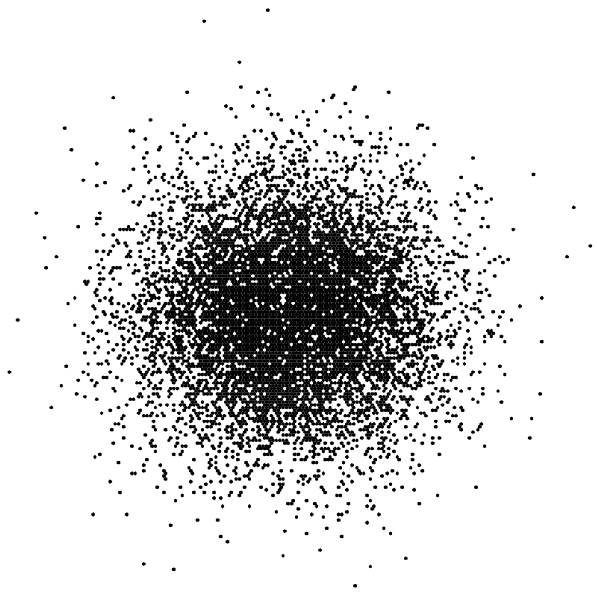


FIG. 6 – Image obtenue lorsqu'on laisse n particules diffuser indépendamment.

Pour n très grand, deux régimes principaux apparaissent selon la valeur de t : il existe une constante universelle λ_c telle que à $t = \lambda n$,

- Si $\lambda < \lambda_c$, il y a un cluster macroscopique autour de l'origine (*phase dense*). Son rayon est d'ordre $\sqrt{t}$, avec des fluctuations d'ordre $(\sqrt{t})^{4/7} = t^{2/7}$. Nous prouvons aussi que son rayon croît jusqu'à une certaine valeur $\lambda = \lambda_{\max} < \lambda_c$, puis décroît jusqu'à $\lambda = \lambda_c$, valeur pour laquelle il s'annule.
- Si $\lambda > \lambda_c$, l'image globale est similaire à la percolation sous-critique : il n'y a que des clusters très petits (*phase diluée*).

Nous considérons également un modèle avec une source constante de particules à l'origine, qui pourrait être utilisé par exemple pour modéliser la croissance d'une tache d'encre : le réseau est vide au départ, et à chaque instant t, de nouvelles particules arrivent à l'origine avec un taux $\mu > 0$. Nous montrons que ce modèle est similaire au précédent modèle en phase dense (par exemple le rayon et les fluctuations sont décrits par les mêmes exposants), pour n'importe quel $\mu > 0$.

Introduction

This thesis consists of six chapters that study aspects of two-dimensional percolation near its phase transition. This introduction is divided into two parts: we first briefly recall the existing results that are directly related to our work, and we then survey each of our six chapters.

1 Background

1.1 Site percolation

The mathematical theory of percolation was initiated by Broadbent and Hammersley in 1957 [15]. We first briefly recall its main properties, and we refer the reader to the classical references [39, 35] for more details.

Site percolation on a lattice $\mathbb{L}$ ($\mathbb{Z}^d$ for instance) can be described as follows. For each site x of $\mathbb{L}$, we toss a (biased) coin to decide whether it is occupied (open, black) or vacant (closed, white): for some parameter p between 0 and 1, x is occupied with probability p and vacant with probability $1-p$, independently of the other sites.

We are then interested in the connectivity properties of the set of occupied sites: two occupied sites x and y are said to be connected, which we denote by $x \rightsquigarrow y$, if there exists a nearest-neighbor path from x to y on the lattice that consists only of occupied sites. The occupied sites can thus be grouped into (maximal) connected components, that are called "clusters".

A site x is said to be connected to ∞ if it belongs to an infinite cluster, and one defines

$$\theta(p) = \mathbb{P}_p(0 \rightsquigarrow \infty).$$

Note that $\theta(p) = \mathbb{P}_p(x \rightsquigarrow \infty)$ for any site x because of translation invariance of the model (if one considers a transitive lattice). In fact, it is possible and classical to show (combining translation-invariance with isoperimetric ideas) that for d-dimensional lattices, when $\theta(p) > 0$ there almost surely exists exactly one infinite connected component.

An important feature of percolation is its phase transition. Let us introduce the percolation threshold

$$p_c(\mathbb{L}) = \sup\{p \text{ s.t. } \theta(p) = 0\}.$$

Then, one can easily check that $0 < p_c(\mathbb{L}) < 1$ for common lattices like $\mathbb{Z}^d$ or the triangular lattice $\mathbb{T}$ in two dimensions (see Figure 1). Hence,

- When $p < p_c$, there is a.s. no infinite cluster (sub-critical regime).
- When $p > p_c$, there is a.s. a unique infinite cluster (super-critical regime).

The case where $p = p_c$ is called critical. Deciding whether there is an infinite cluster or not in this case can be quite difficult: for $\mathbb{Z}^3$, the answer is in fact not known. Let us also mention that

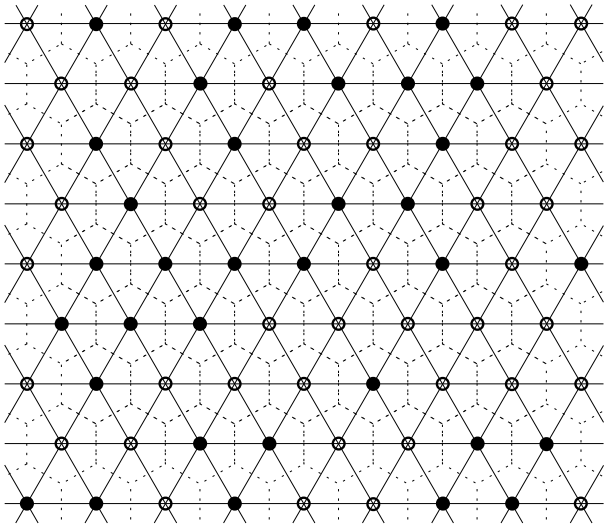


Figure 1: Site percolation on the triangular lattice.

the sub-critical regime presents the following exponential decay property: the probability that 0 is connected to a site at distance n decays exponentially in n (more precisely, this probability is bounded by $C_1(p)e^{-C_2(p)n}$ for two constants $C_1, C_2 > 0$ depending on p) – a similar statement can be made for the super-critical regime.

From now on, we restrict ourselves to two-dimensional lattices. We will focus mainly on the triangular lattice $\mathbb{T}$, since at present this is the lattice for which the finest description is known, as we will explain in the next section. Site percolation on this lattice is often represented as a random coloring of the faces of the dual hexagonal lattice, and it presents a very convenient self-matching property that naturally leads to the conjecture that

$$p_c(\mathbb{T}) = \frac{1}{2},$$

which is actually a celebrated result of Kesten [38]. We know furthermore in this case that there is no infinite cluster at criticality, in other words that $\theta(1/2) = 0$.

The exponential decay of connection probabilities makes the sub- and super-critical regimes quite "predictible" on large scales. In the sub-critical regime for instance, in a portion of the lattice with diameter N, one typically observes only very small occupied components, of size at most $\log N$, lost in a "sea" of vacant sites. In some sense, the critical regime is much richer: connection probabilities do not decay exponentially fast any more, and in a piece of diameter N, one observes both occupied and vacant structures of diameter comparable to N, and fractal objects arise as N gets very large.

Here we will investigate aspects of the behavior of percolation near criticality, i.e. of the transition from sub- to super-critical regime through the critical regime.

1.2 Conformal invariance of critical percolation and critical exponents

The mathematical understanding of critical two-dimensional percolation has tremendously improved over the last decade, thanks to Smirnov's proof of conformal invariance for percolation on the triangular lattice, and to the definition of SLE curves by Schramm that describes the possible conformally invariant scaling limits.

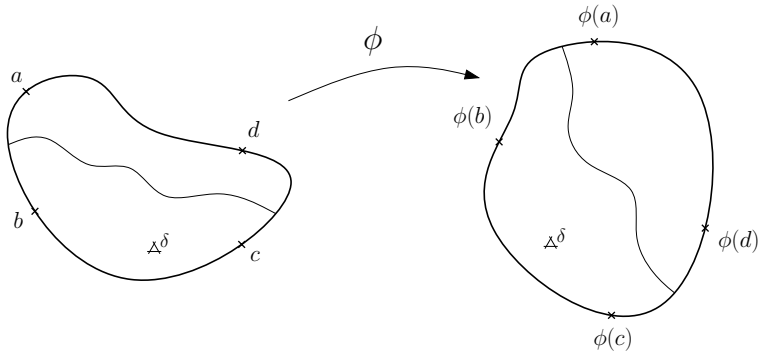


Figure 2: Conformal invariance of the crossing probability.

Let us describe in more detail Smirnov's result. Let us take D any (smooth) Jordan domain, and two marked points a and b on ∂D. For any $\delta > 0$, we can consider the lattice of mesh size δ, and an approximation $(D^\delta, a^\delta, b^\delta)$ of (D, a, b) on this lattice. The two points a^δ and b^δ divide ∂D^δ into two arcs: we decide to color one of them in black, and the other one in white. We then look at the interface between the cluster of black sites connected to the arc colored in black and the cluster of white sites connected to the white arc. This interface γ^δ can be explored dynamically by an "exploration process" from a to b. Then Smirnov's theorem [69, 70, 78, 17] states that as the mesh size $\delta \searrow 0$, γ^δ converges in distribution toward Schramm's chordal SLE(6) which is a random continuous curve from a to b (this convergence in law takes place with respect to the uniform distance on continuous curves "up to monotone reparametrization").

A key step to prove this convergence is to derive Cardy's formula: in a domain D with four marked points a, b, c, d on ∂D (in this order), this formula gives the probability of existence of a crossing between the arcs (ab) and (cd) as $\delta \searrow 0$. This formula states in particular that the probability of crossing is conformally invariant: if ϕ is a conformal bijection from D to another domain D', then the probability of existence of a crossing from (ab) to (cd) in D is equal to the probability of existence in D', from $(\phi(a)\phi(b))$ to $(\phi(c)\phi(d))$. This conformal invariance is the key to the relation to SLE processes.

To describe critical and near-critical percolation, certain exceptional events are instrumental: the *arm events*, referring to the existence in annuli of some number of crossings ("arms"), the color of each crossing (black or white) being prescribed. For any two positive integers $n \leq N$, we define the event

$$A_{j,\sigma}(n, N) = \{\partial S_n \rightsquigarrow_{j,\sigma} \partial S_N\}$$

that there exist j disjoint monochromatic arms in the annulus $S_{n,N}$ of radii n and N (centered on the origin), whose colors in counterclockwise order are those prescribed by the sequence of colors σ. These arm events can be expressed in terms of exploration process, so that

$$\lim_{\delta \searrow 0} \mathbb{P}_{1/2}(A_{j,\sigma}(\delta n, \delta N))$$

can be computed using SLE(6). We can then prove that for any fixed n_0 (large enough),

$$\mathbb{P}_{1/2}(A_{j,\sigma}(n_0, N)) = N^{-\alpha_j + o(1)}$$

as $N \to \infty$, with

- $\alpha_1 = 5/48$,

- and for $j \geq 2$, $\alpha_j = (j^2 - 1)/12$ (if σ is non-constant).

These exponents α_j are known as "arm exponents". We refer the reader to [46, 78] for a detailed account of this derivation.

Having these arm exponents for critical percolation at our disposal, we can then use Kesten's scaling relations [43] to describe percolation near criticality. We do not enter into details since this is the subject of our first chapter, but let us just mention that we can prove in this way a paradigm of statistical physics coming from renormalization theory, that the common characteristic functions behave like powers of $|p - p_c|$ as $p \to p_c$. For θ, one has for instance:

$$\theta(p) = (p - 1/2)^{5/36+o(1)}$$

as $p \to 1/2^+$.

In fact, in our work, we will use SLE processes indirectly i.e. we will build on these results for arm exponents that were derived via SLE computations. But we will never really use its definition via Loewner's equation, so we do not recall it in this introduction.

1.3 Random shape models

Our studies are motivated by models and works from condensed matter physics. In [63], the physicists Gouyet, Rosso and Sapoval started to study the geometric properties of diffusion fronts. They considered a simple exclusion process with a source, leading to a density of particles $p(z)$ that is a decreasing function of the y-coordinate. They then proposed the approximation that different sites are independently occupied or vacant. This led them to consider an inhomogeneous percolation process with parameter $p(z)$, a model that they later called "gradient percolation". This approximation enabled them to carry on precise simulations and to show numerically that the diffusion fronts are fractal objects. They measured the dimension $D_f = 1.76 \pm 0.02 \simeq 7/4$, a value that had already been observed for critical percolation interfaces. They thus conjectured, based also on the idea that the fronts remain where $p(z)$ is close to the critical value p_c, that diffusion fronts are related to percolation cluster boundaries.

Gradient percolation has now become a classical tool to model inhomogeneous media (see e.g. [64, 29, 34]). Let us mention a model for chemical etching or erosion, known as "etching gradient percolation" ([65, 62, 31, 66]). In this model, that first appeared in experiences [5], random resistances are assigned to each site of a flat material, that we put in contact with a corroding solution. When a "weak" site (of small resistance) is reached by the liquid, it is dissolved, and the corroding power of the liquid decreases. We reach some final interface in finite time, that seems to be closely related to gradient percolation: in particular its dimension has been measured to be very close to 7/4.

Since gradient percolation is rather easy to simulate, it is also an efficient way to estimate percolation thresholds, by computing the mean height of the interface. This idea was first used in [59] for $\mathbb{Z}^2$, where the value $p_c = 0.59280 \pm 10^{-5}$ was obtained, and later used in [79, 60, 80, 75] for various other lattices and continuum percolation.

2 Overview of our results

This work consists of six chapters. In Chapter 1, we present in detail Kesten's results and techniques that allow to describe near-critical percolation on the triangular lattice, and we also derive some new consequences. In the next chapters, we then apply these ideas to various situations: the Inhomogeneous Incipient Infinite Cluster (Chapter 2), the geometric properties

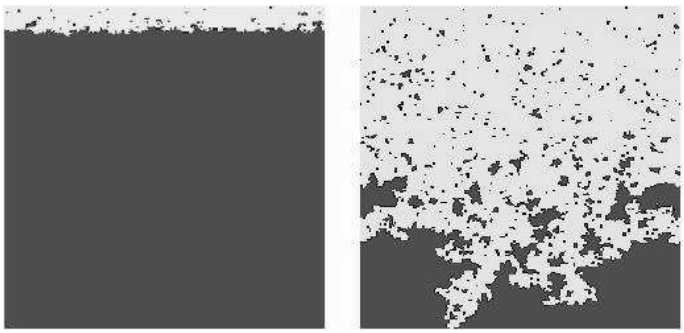


Figure 3: Fractal sea coasts (Figure B. Sapoval).

of near-critical percolation interfaces (Chapter 3), the gradient percolation model (Chapters 4 and 5), and finally (Chapter 6) to some diffusion model in the spirit of the original paper of Gouyet, Rosso and Sapoval [63]. Note that each chapter corresponds to a paper (we will often refer to our chapters as papers): even if we tried to use consistent notation, we decided to keep the introductions and preliminary sections of each paper, as it helps seeing what is used and where. The four first chapters have already been submitted for publication, and correspond to the papers [55], [28], [56] and [54] respectively.

2.1 Near-critical percolation in two dimensions

In this first chapter, we present and develop ideas and techniques that allow to describe critical and near-critical percolation in two dimensions, based on Kesten's scaling relations [43] and the description of critical percolation obtained by Lawler, Schramm, Smirnov and Werner [71, 50].

An important tool in our considerations is the *finite-size scaling* characteristic length L_ϵ, introduced in [23], that we now define. For that, consider the event $\mathcal{C}_H$ that there exists a horizontal crossing in the rhombus $[0,n]\times[0,n]$, of side length n: at $p=1/2$, $\mathbb{P}_p(\mathcal{C}_H([0,n]\times[0,n]))=1/2$, and this probability tends to 0 or 1 when n goes to infinity, according to $p<1/2$ or $p>1/2$. The characteristic length L_ϵ then measures the scale up to which this crossing probability remains bounded away from 0 and 1: for each $\epsilon\in(0,1/2)$, we define

$$L_\epsilon(p)=\begin{cases}\min\{n \text{ s.t. } \mathbb{P}_p(\mathcal{C}_H([0,n]\times[0,n]))\le\epsilon\} & \text{when } p<1/2,\\ \min\{n \text{ s.t. } \mathbb{P}_p(\mathcal{C}_H([0,n]\times[0,n]))\ge 1-\epsilon\} & \text{when } p>1/2.\end{cases}$$

As mentioned earlier, the arm events play a central role. These events possess an important "separation" property: if we also prescribe where the arms land on the (inner and outer) boundaries of $S_{n,N}$, which is a sub-event $\tilde{A}_{j,\sigma}(n,N)$ of $A_{j,\sigma}(n,N)$, then the probability remains of the same order of magnitude. More precisely, we have[2]

$$\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}(n,N))\asymp\hat{\mathbb{P}}(A_{j,\sigma}(n,N))$$

uniformly for p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$ (i.e. with parameters between p and $1-p$) and $n\le N\le L_\epsilon(p)$. This separation property was used in [71] to derive the arm exponents for critical percolation.

[2]We use the symbol "$\asymp$" to indicate that two quantities are of the same order of magnitude.

The arm exponents for critical percolation can then be used to describe near-critical percolation. For that purpose, we need to compare arm events at criticality and near criticality. Extending previous results of Kesten, we show that for all j and σ,

$$\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}(n,N)) \asymp \mathbb{P}_{1/2}(A_{j,\sigma}(n,N))$$

uniformly for p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$ and $n \leq N \leq L_\epsilon(p)$.

This allows to establish the existence and the values of the critical exponents associated to the main characteristic functions of the model. First of all, we prove that for any $\epsilon \in (0,1/2)$,

$$|p-1/2|\big(L_\epsilon(p)\big)^2 \mathbb{P}_{1/2}(0 \rightsquigarrow_4 \partial S_{L_\epsilon(p)}) \asymp 1$$

as $p \to 1/2$, which combined with the 4-arm exponent gives

$$L_\epsilon(p) = |p-1/2|^{-4/3+o(1)}.$$

This relation also implies that

$$L_\epsilon(p) \asymp L_{\epsilon'}(p)$$

for any two $\epsilon, \epsilon' \in (0,1/2)$.

We then use a last ingredient, a property of exponential decay uniform in p: for any fixed $k \geq 1$, for all $p < 1/2$ and all n,

$$\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,kn])) \leq C_1 e^{-C_2 n/L_\epsilon(p)},$$

where C_1 and C_2 are constants depending only on k and ϵ. We get the critical exponent for θ: as $p \to 1/2^+$,

$$\theta(p) \asymp \mathbb{P}_p(0 \rightsquigarrow \partial S_{L_\epsilon(p)}) = (L_\epsilon(p))^{-5/48+o(1)} = (p-1/2)^{5/36+o(1)}.$$

As for the two other classical functions

$$\chi(p) = \mathbb{E}_p\big[|C(0)|; |C(0)| < \infty\big]$$

(the average size of a finite cluster) and

$$\xi(p) = \left[\frac{1}{\mathbb{E}_p\big[|C(0)|; |C(0)| < \infty\big]} \sum_x \|x\|_\infty^2 \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big)\right]^{1/2}$$

(the mean radius of a finite cluster), it had been shown in [43] that the sites at distance less than $L_\epsilon(p)$ produce a positive fraction of the whole sum: the contribution of the sites at distance $\gg L_\epsilon(p)$ is negligible. Hence, as $p \to 1/2$,

$$\chi(p) \asymp (L_\epsilon(p))^2 \mathbb{P}_p(0 \rightsquigarrow \partial S_{L_\epsilon(p)})^2 = |p-1/2|^{-43/18+o(1)}$$

and

$$\xi(p) \asymp L_\epsilon(p) = |p-1/2|^{-4/3+o(1)}.$$

2.2 Large scale properties of the IIIC for 2D percolation

In this chapter, we discuss the Inhomogeneous Incipient Infinite Cluster (IIIC) model, introduced in [25]. This model consists in an independent inhomogeneous percolation process, with a parameter slightly larger than p_c, decaying to p_c as a power of the distance to the origin: the occupation parameter of a site z is taken to be

$$p(z) = p_c + \|z\|^{-\lambda}.$$

In was shown in [25] that a transition occurs for $\lambda_c = 1/\nu$, where $\nu = 4/3$ is the previously mentioned critical exponent for the characteristic length:

- When $\lambda > \lambda_c$, there is a.s. no infinite cluster.
- When $\lambda < \lambda_c$, there is a.s. a unique infinite cluster $\mathcal{C}_\infty$.

We thus get in the second case an incipient infinite cluster, which can be described rather precisely using near-critical techniques. First, we show that in S_N,

$$\mathbb{E}[|\mathcal{C}_\infty \cap S_N|] = N^{2-\lambda\beta+o(1)}$$

as $N \to \infty$. We also prove that

$$\mathrm{Var}(|\mathcal{C}_\infty \cap S_N|) = o(\mathbb{E}[|\mathcal{C}_\infty \cap S_N|]^2),$$

so that $|\mathcal{C}_\infty \cap S_N|$ is close to its expectation with high probability.

We also study in more detail the transition at $\lambda = \lambda_c$. This transition reveals to be very sharp: roughly speaking (actually, the precise statement is less explicit since technical issues arise), if one takes

$$p(z) = p_c + C\|z\|^{-1/\nu}(\log\log\|z\|)^{1/\nu},$$

then the value of the constant C determines the presence or the absence of an infinite cluster.

2.3 Asymmetry of near-critical percolation interfaces

In this chapter, we study the geometric properties of the interfaces arising in near-critical regime. Understanding this regime is important to describe the models where the phase transition of percolation spontaneously appears, like gradient percolation introduced in the next chapter. Indeed, in many cases what we observe is percolation *near criticality* rather than percolation *at criticality* exactly.

To fix ideas, let us consider the following setting: the triangle of side length N, with bottom edge centered on the origin. We then color the left boundary in black and the right boundary in white, as depicted on Figure 4, and we look at the interface starting at the origin between the black cluster connected to the left side, and the white cluster connected to the right side.

Let us now consider two sequences $N_k \to \infty$ and $p_k \geq 1/2$. If we scale this interface to the unit triangle T, the sequence γ_k of curves so obtained possesses scaling limits, using tightness arguments due to Aizenman and Burchard [2]. We show that three main regimes arise, according to the behavior of $N_k^2\mathbb{P}_{1/2}(0 \rightsquigarrow_4 \partial S_{L(p_k)})$:

- If it tends to 0, then γ converges (in distribution) to a chordal SLE(6), from the origin to the top corner of T (critical case).
- If it tends to ∞, then γ tends to the right boundary of T (super-critical case).

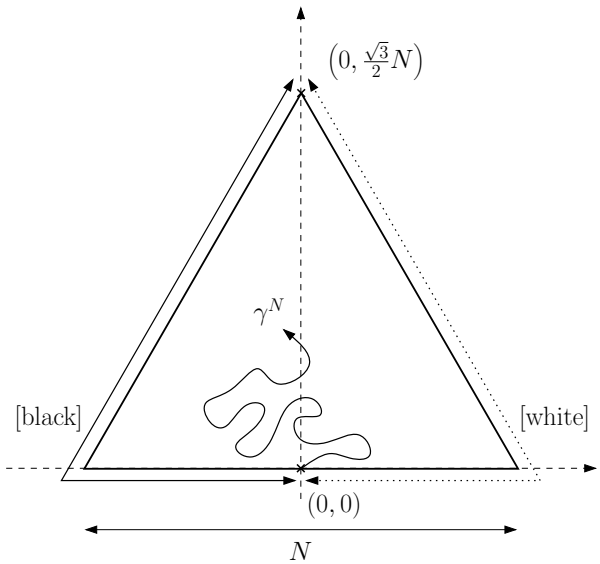


Figure 4: We look at the interface between the black cluster connected to the left side, and the white cluster connected to the right side.

- If it remains bounded away from 0 and ∞, we get an intermediate regime, called *off-critical regime*.

Let us now focus on this intermediate regime. First of all, we show that it shares many important properties with the critical regime. Since the probabilities of arm events remain of the same order of magnitude, macroscopic properties like the Hausdorff dimension (7/4) remain the same. On the other hand, we point out an asymmetry property that distinguishes this regime from the critical regime: the curve has a slight tendancy to turn more to the right than to the left. This property is easy to obtain on the discrete level, and we show that it remains visible in the scaling limit, on any portion of the interface: a flavor of super-critical percolation thus remains. For that purpose, we sub-divide T into small triangles of size η, and count in how many of them the interface exits on the right side rather than on the left side.

2.4 Critical exponents of planar gradient percolation

We study in this chapter the gradient percolation model, introduced in [63] by the physicists Gouyet, Rosso and Sapoval to simulate diffusion fronts. The setting is the following: in a strip of width $2N$ and of length ℓ_N, we perform an independent inhomogeneous percolation process, with a parameter $p(z)$ at site z that decays linearly from 1 (for $y = -N$) to 0 (for $y = N$) with the y-coordinate:

$$p(z) = 1/2 - y/2N.$$

At the bottom of the strip, the parameter is close to 1 and we observe a large cluster of black sites, and similarly we observe at the top a large cluster of white sites. The characteristic phenomenon in this model is the appearance of a unique "front", an interface touching simultaneously these two clusters, as depicted on Figure 5.

Let us assume that the length of the strip satisfies $N^{4/7+\delta} \leq \ell_N \leq N^{\gamma}$ for two constants $\delta, \gamma > 0$. Then for some $\epsilon > 0$, the front $\mathcal{F}_N$ is well-defined with probability at least $1 - C_1 e^{-C_2 N^{\epsilon}}$.

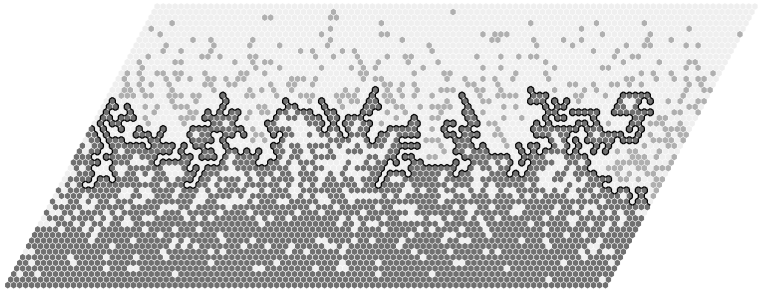


Figure 5: The interface appearing in the gradient percolation model.

In this case, an edge e is on $\mathcal{F}_N$ if and only if it connected by two arms to the boundaries of the strip, a black arm to the bottom and a white arm to the top.

We also show that this front remains localized in a strip around the line $y = 0$ where $p = p_c$, with fluctuations of order $N^{4/7}$. More precisely, we prove that for any $\eta > 0$, with probability at least $1 - C_1' e^{-C_2' N^{\epsilon'}}$, $\mathcal{F}_N$ stays in the strip of width $2N^{4/7+\eta}$ around $y = 0$, and exits the strip of width $2N^{4/7-\eta}$.

Using the characterization of $\mathcal{F}_N$ as the set of edges possessing two arms, we can estimate various quantities related to it via the 2-arm exponent. For instance, we get that its length T_N satisfies:

$$\mathbb{E}[T_N] = \ell_N N^{3/7+o(1)}.$$

Moreover, a horizontal decorrelation property holds, so that for instance

$$\mathrm{Var}(T_N) = o(\mathbb{E}[T_N]^2).$$

Hence, T_N is close to its expectation with high probability.

2.5 Scaling limits for the gradient percolation front

This short chapter is a continuation of the previous one. We tighten the results obtained for gradient percolation in a strip by introducing a quantity

$$\sigma_N^\epsilon = \sup\big\{\sigma \text{ s.t. } L_\epsilon\big(p(\sigma)\big) \geq \sigma\big\},$$

measuring the vertical fluctuations of the front. This quantity turns out to be a – both horizontal and vertical – characteristic length for the model. We show that $\sigma_N^\epsilon \asymp \sigma_N^{\epsilon'}$ for any two $\epsilon, \epsilon' \in (0, 1/2)$, and that this gives the only way (up to multiplicative constants) to scale the front in order to get non-trivial limits. We then construct scaling limits using the tightness arguments of Aizenman and Burchard. These scaling limits behave locally like off-critical interfaces.

2.6 Geometry of diffusion fronts

In this last chapter, we study some models of diffusion fronts: models of this type led Sapoval, Rosso and Gouyet [63] to introduce gradient percolation. Our main model is the following: on a lattice, we deposit some fixed number n of particles at the origin and let them perform independent simple random walks during time t. A concentration gradient appears (see Figure 6), and a Poissonian approximation turns out to be valid locally, allowing to describe the interface by using our study of gradient percolation.

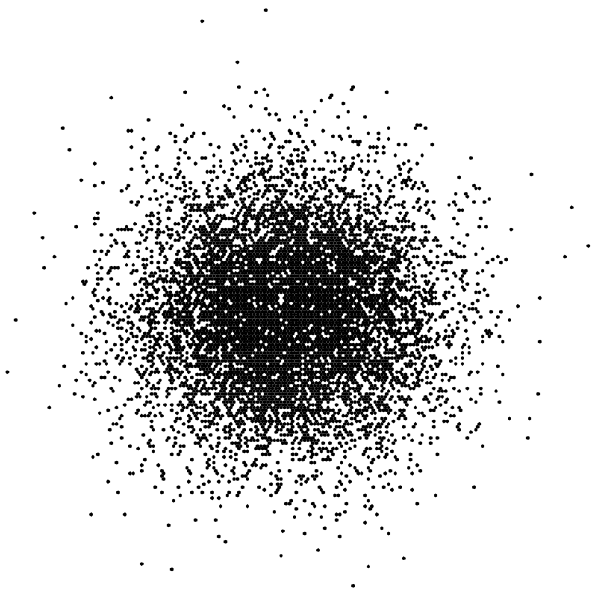


Figure 6: Picture obtained when one lets the n particles perform independent diffusions.

For n very large, two main regimes arise according to t: there exists some absolute constant λ_c such that at $t = \lambda n$,

- If $\lambda < \lambda_c$, there is a macroscopic cluster around the origin (*dense phase*). Its radius is of order $\sqrt{t}$, with fluctuations of order $(\sqrt{t})^{4/7} = t^{2/7}$. We also show that its radius increases up to some value $\lambda = \lambda_{\max} < \lambda_c$, and then decreases up to $\lambda = \lambda_c$ at which it vanishes.
- If $\lambda > \lambda_c$, the global picture is similar to sub-critical percolation: there are only very small clusters (*dilute phase*).

We also consider a model with a constant source of particles at the origin, which could be used for instance to model the growth of an ink stain: we start with no particles at all, and at each time t, new particles arrive at the origin with rate $\mu > 0$. We show that this model is similar to the previous one in dense phase (for instance the radius and the fluctuations remain comparable), for any $\mu > 0$.

Chapter 1

Near-critical percolation in two dimensions

Abstract: We give a self-contained and detailed presentation of Kesten's results that allow to relate critical and near-critical percolation on the triangular lattice. They constitute an important step in the derivation of the exponents describing the near-critical behavior of this model. For future use and reference, we also show how these results can be obtained in more general situations, and we state some new consequences.

1.1 Introduction

Since 2000, substantial progress has been made on the mathematical understanding of percolation on the triangular lattice. In fact, it is fair to say that it is now well-understood. Recall that performing a percolation with parameter p on a lattice means that each site is chosen independently to be black with probability p or white with probability $1-p$. We then look at the connectivity properties of the set of black sites (or the set of white ones). It is well-known that on the regular triangular lattice, when $p \leq 1/2$ there is almost surely no infinite black connected component, whereas when $p > 1/2$ there is almost surely one infinite black connected component. Its mean density can then be measured via the probability $\theta(p)$ that a given site belongs to this infinite black component.

Thanks to Smirnov's proof of conformal invariance of the percolation model at $p = 1/2$ [69], allowing to prove that critical percolation interfaces converge toward SLE_6, and to the derivation of the SLE_6 critical exponents [47, 48] by Lawler, Schramm and Werner, it is possible to prove results concerning the behavior of the model when p is exactly equal to $1/2$, that had been conjectured in the physics literature, such as the values of the arm exponents [71, 50]. See e.g. [78] for a survey and references.

More than ten years before the above-mentioned papers, Kesten had shown in his 1987 paper *Scaling relations for 2D-percolation* [43] that the behavior of percolation at criticality (*ie* when $p = 1/2$) and near criticality (*ie* when p is close to $1/2$) are very closely related. In particular, the exponents that describe the behavior of quantities such as $\theta(p)$ when $p \to 1/2^+$ and the arm exponents for percolation at $p = 1/2$ are related via relations known as scaling (or hyperscaling) relations. At that time, it was not proved that any of the exponents existed (not to mention their actual value) and Kesten's paper explains how the knowledge of the existence and the values of some arm exponents allows to deduce the existence and the value of the exponents that describe "near-critical" behavior. Therefore, by combining this with the derivation of the arm exponents, we can for instance conclude [71] that $\theta(p) = (p - 1/2)^{5/36+o(1)}$ as $p \to 1/2^+$.

Reading Kesten's paper in order to extract the statement that is needed to derive this result can turn out to be not so easy for a non-specialist, and the first goal of the present paper is to give a complete self-contained proof of Kesten's results that are used to describe near-critical percolation. We hope that this will be useful and help a wider community to have a clear and complete picture of this model.

It is also worth emphasizing that the proofs contain techniques (such as separation lemmas for arms) that are interesting in themselves and that can be applied to other situations. The second main purpose of the present paper is to state results in a more general setting than in [43], for possible future use. In particular, we will see that the "uniform estimates below the characteristic length" hold for an arbitrary number of arms and non-homogeneous percolation (see Theorem 11 on separation of arms, and Theorem 27 on arm events near criticality). Some technical difficulties arise due to these generalizations, but these new statements turn out to be useful. They are for instance instrumental in our study of gradient percolation in [54]. Other new statements in the present paper concern arms "with defects" or the fact that the finite-size scaling characteristic length $L_\epsilon(p)$ remains of the same order of magnitude when ϵ varies in $(0, 1/2)$ (Corollary 36) – and not only for ϵ small enough. This last fact is used in [56] to study the "off-critical" regime for percolation.

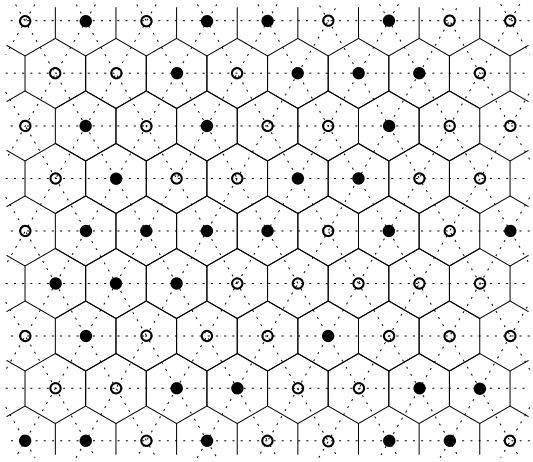


Figure 1.1: Percolation on the triangular lattice can be viewed as a random coloring of the dual hexagonal lattice.

1.2 Percolation background

Before turning to near-critical percolation in the next section, we review some general notations and properties concerning percolation. We also sketch the proof of some of them, for which small generalizations will be needed. We assume the reader is familiar with the standard fare associated with percolation, and we refer to the classic references [39, 35] for more details.

1.2.1 Notations

Setting

Unless otherwise stated, we will focus on site percolation in two dimensions, on the triangular lattice. This lattice will be denoted by $\mathbb{T} = (\mathbb{V}^T, \mathbb{E}^T)$, where $\mathbb{V}^T$ is the set of vertices (or "sites"), and $\mathbb{E}^T$ is the set of edges (or "bonds"), connecting adjacent sites. We restrict ourselves to this lattice because it is at present the only one for which the critical regime has been proved to be conformal invariant in the scaling limit.

The usual (homogeneous) site percolation process of parameter p can be defined by taking the different sites to be *black* (or occupied) with probability p, and *white* (vacant) with probability $1-p$, independently of each other. This gives rise to a product probability measure on the set of configurations, which is referred to as $\mathbb{P}_p$, the corresponding expectation being $\mathbb{E}_p$. We usually represent it as a random (black or white) coloring of the faces of the dual hexagonal lattice (see Figure 1.1).

More generally, we can associate to each family of parameters $\hat{p} = (\hat{p}_v)_v$ a product measure $\hat{\mathbb{P}}$ for which each site v is black with probability $\hat{p}_v$ and white with probability $1-\hat{p}_v$, independently of all other sites.

Coordinate system

We sometimes use complex numbers to position points in the plane, but we mostly use oblique coordinates, with the origin in 0 and the basis given by 1 and $e^{i\pi/3}$, *ie* we take the x–axis and its image under rotation of angle $\pi/3$ (see Figure 1.2). For $a_1 \leq a_2$ and $b_1 \leq b_2$, the parallelogram

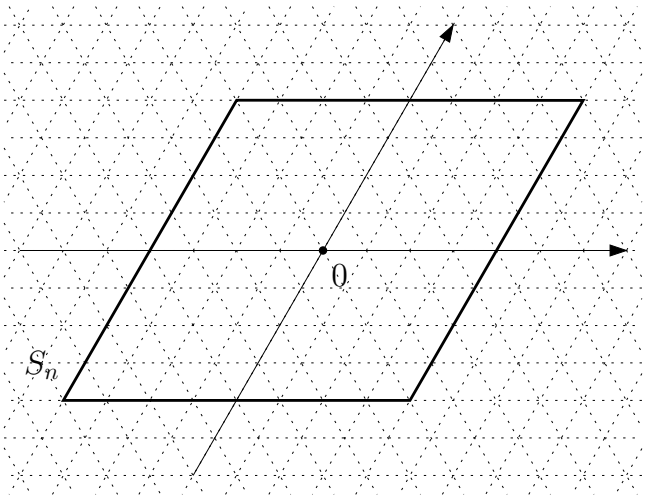


Figure 1.2: We refer to oblique coordinates, and we denote by S_n the "box of size n".

R of corners $a_j + b_k e^{i\pi/3}$ $(j,k = 1,2)$ is thus denoted by $[a_1, a_2] \times [b_1, b_2]$, its interior being $\mathring{R} :=]a_1, a_2[\times]b_1, b_2[= [a_1 + 1, a_2 - 1] \times [b_1 + 1, b_2 - 1]$ and its boundary $\partial R := R \setminus \mathring{R}$ the concatenation of the four boundary segments $\{a_i\} \times [b_1, b_2]$ and $[a_1, a_2] \times \{b_i\}$.

We denote by $\|z\|_\infty$ the infinity norm of a vertex z as measured with respect to these two axes, and by d the associated distance. For this norm, the set of points at distance at most N from a site z forms a rhombus $S_N(z)$ centered at this site and whose sides line up with the basis axes. Its boundary, the set of points at distance exactly N, is denoted by $\partial S_N(z)$, and its interior, the set of points at distance strictly less that N, by $\mathring{S}_N(z)$. To describe the percolation process, we often use $S_N := S_N(0)$ and call it the "box of size N". Note that it can also be written as $S_N = [-N, N] \times [-N, N]$. It will sometimes reveal useful to have noted that

$$|S_N(z)| \leq C_0 N^2 \tag{1.1}$$

for some universal constant C_0. For any two positive integers $n \leq N$, we also consider the annulus $S_{n,N}(z) := S_N(z) \setminus \mathring{S}_n(z)$, with the natural notation $S_{n,N} := S_{n,N}(0)$.

Connectivity properties

Two sites x and y are said to be connected if there exists a black path, *ie* a path consisting only of black sites, from x to y. We denote it by $x \rightsquigarrow y$. Similarly, if there exists a white path from x to y, these two sites are said to be $*$–connected, which we denote by $x \rightsquigarrow^* y$.

For notational convenience, we allow y to be "∞": we say that x is connected to infinity ($x \rightsquigarrow \infty$) if there exists an infinite, self-avoiding and black path starting from x. We denote by

$$\theta(p) := \mathbb{P}_p\big(0 \rightsquigarrow \infty\big) \tag{1.2}$$

the probability for 0 (or any other site by translation invariance) to be connected to ∞.

To study the connectivity properties of a percolation realization, we often are interested in the connected components of black or white sites: the set of black sites connected to a site x (empty if x is white) is called the cluster of x, denoted by $C(x)$. We can define similarly $C^*(x)$ the white cluster of x. Note that $x \rightsquigarrow \infty$ is equivalent to the fact that $|C(x)| = \infty$.

If A and B are two sets of vertices, the notation $A \rightsquigarrow B$ is used to denote the event that some site in A is connected to some site in B. If the connection is required to take place using exclusively the sites in some other set C, we write $A \overset{C}{\rightsquigarrow} B$.

Crossings

A left-right (or horizontal) crossing of the parallelogram $[a_1, a_2] \times [b_1, b_2]$ is simply a black path connecting its left side to its right side. However, this definition implies that the existence of a crossing in two boxes sharing just a side are not completely independent: it will actually be more convenient to relax (by convention) the condition on its extremities. In other words, we request such a crossing path to be composed only of sites in $]a_1, a_2[\times]b_1, b_2[$ which are black, with the exception of its two extremities on the sides of the parallelogram, which can be either black or white. The existence of such a horizontal crossing is denoted by $\mathcal{C}_H([a_1, a_2] \times [b_1, b_2])$. We define likewise top-bottom (or vertical) crossings and denote their existence by $\mathcal{C}_V([a_1, a_2] \times [b_1, b_2])$, and also white crossings, the existence of which we denote by $\mathcal{C}_H^*$ and $\mathcal{C}_V^*$.

More generally, the same definition applies for crossings of annuli $S_{n,N}(z)$, from the internal boundary $\partial S_n(z)$ to the external one $\partial S_N(z)$, or even in more general domains $\mathcal{D}$, from one part of the boundary to another part.

Asymptotic behavior

We use the standard notations to express that two quantities are asymptotically equivalent. For two positive functions f and g, the notation $f \asymp g$ means that f and g remain of the same order of magnitude, in other words that there exist two positive and finite constants C_1 and C_2 such that $C_1 g \leq f \leq C_2 g$ (so that the ratio between f and g is bounded away from 0 and $+\infty$), while $f \approx g$ means that $\log f / \log g \to 1$ ("logarithmic equivalence") – either when $p \to 1/2$ or when $n \to \infty$, which will be clear from the context. This weaker equivalence is generally the one obtained for quantities behaving like power laws.

1.2.2 General properties

On the triangular lattice, it is known since [38] that percolation features a phase transition at $p = 1/2$, called the *critical point*: this means that

- When $p < 1/2$, there is (a.s.) no infinite cluster (*sub-critical* regime), or equivalently $\theta(p) = 0$.
- When $p > 1/2$, there is (a.s.) an infinite cluster (*super-critical* regime), or equivalently $\theta(p) > 0$. Furthermore, the infinite cluster turns out to be *unique* in this case.

In sub- and super-critical regimes, "correlations" decay very fast, this is the so-called *exponential decay* property:

- For any $p < 1/2$,
$$\exists C_1, C_2(p) > 0, \quad \mathbb{P}_p(0 \rightsquigarrow \partial S_n) \leq C_1 e^{-C_2(p)n}.$$
- We can deduce from it that for any $p > 1/2$,
$$\begin{aligned}\mathbb{P}_p(0 \rightsquigarrow \partial S_n, |C(0)| < \infty) \\ &\leq \mathbb{P}_p(\exists \text{ white circuit surrounding 0 and a site on } \partial S_n) \\ &\leq C_1' e^{-C_2'(p)n}\end{aligned}$$

for some $C_1'(p), C_2'(p) > 0$.

We would like to stress the fact that the speed at which these correlations vanish is governed by a constant C_2 which depends on p – it becomes slower and slower as p approaches $1/2$. To study what happens near the critical point, we need to control this speed for different values of p: we will derive in Section 1.7.4 an exponential decay property that is uniform in p.

The intermediate regime at $p = 1/2$ is called the *critical* regime. It is known for the triangular lattice that there is no infinite cluster at criticality: $\theta(1/2) = 0$. Hence to summarize,

$$\theta(p) > 0 \quad \textit{iff} \quad p > 1/2.$$

Correlations no longer decay exponentially fast in this critical regime, but (as we will see) just like power laws. For instance, non trivial random scaling limits – with fractal structures – arise. This particular regime has some very strong symmetry property (conformal invariance) which allows to describe it very precisely.

1.2.3 Some technical tools

Monotone events

We use the standard terminology associated with events: an event A is increasing if "it still holds when we add black sites", and decreasing if it satisfies the same property, but when we add white sites.

Recall also the usual coupling of the percolation processes for different values of p: we associate to the different sites x i.i.d. random variables U_x uniform on $[0, 1]$, and for any p, we obtain the measure $\mathbb{P}_p$ by declaring each site x to be black if $U_x \leq p$, and white otherwise. This coupling shows for instance that

$$p \mapsto \mathbb{P}_p(A)$$

is a non-decreasing function of p when A is an increasing event. More generally, we can represent in this way any product measure $\hat{\mathbb{P}}$.

Correlation inequalities

The two most common inequalities for percolation concern monotone events: if A and B are increasing events, we have ([10, 35])

1. the Harris-FKG inequality:
$$\mathbb{P}(A \cap B) \geq \mathbb{P}(A)\mathbb{P}(B).$$

2. the BK inequality:
$$\mathbb{P}(A \circ B) \leq \mathbb{P}(A)\mathbb{P}(B)$$
if A and B depend only on sites in a finite set, $A \circ B$ meaning as usual that A and B occur "disjointly".

In the paper [10] where they proved the BK inequality, Van den Berg and Kesten also conjectured that this inequality holds in a more general fashion, for any pair of events A and B (depending on a finite number of sites): if we define $A \Box B$ the disjoint occurrence of A and B in this situation, we have

$$\mathbb{P}(A \Box B) \leq \mathbb{P}(A)\mathbb{P}(B). \tag{1.3}$$

This was later proved by Reimer [58], and it is now known as Reimer's inequality. We will also use the following inequality:

$$\mathbb{P}_{1/2}(A \circ B) \leq \mathbb{P}_{1/2}(A \cap \tilde{B}), \tag{1.4}$$

where $\tilde{B}$ is the event obtained by "flipping" the configurations in B. This inequality is an intermediate step in Reimer's proof. On this subject, the reader can consult the nice review [14].

Russo's formula

Russo's formula allows to study how probabilities of events vary when the percolation parameter p varies. Recall that for an increasing event A, the event "v is pivotal for A" is composed of the configurations ω such that if we make v black, A occurs, and if we make it white, A does not occur. Note that by definition, this event is independent of the particular state of v. An analog definition applies for decreasing events.

Theorem 1 (Russo's formula)**.** *Let A be an increasing event, depending only on the sites contained in some finite set S. Then*

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{v \in S} \mathbb{P}_p(v \textit{ is pivotal for } A). \tag{1.5}$$

We now quickly remind the reader how to prove this formula, since we will later (in Section 1.6) generalize it a little.

Proof. We allow the parameters $\hat{p}_v$ ($v \in S$) to vary independently, which amounts to consider the more general function $\mathcal{P} : \hat{p} = (\hat{p}_v)_{v \in S} \mapsto \hat{\mathbb{P}}(A)$. This is clearly a smooth function (it is polynomial), and $\mathbb{P}_p(A) = \mathcal{P}(p, \ldots, p)$. Now using the standard coupling, we see that for any site w, for a small variation $\epsilon > 0$ in w,

$$\hat{\mathbb{P}}^{+\epsilon}(A) - \hat{\mathbb{P}}(A) = \epsilon \times \hat{\mathbb{P}}(w \text{ is pivotal for } A), \tag{1.6}$$

so that

$$\frac{\partial}{\partial \hat{p}_w}\hat{\mathbb{P}}(A) = \hat{\mathbb{P}}(w \text{ is pivotal for } A).$$

Russo's formula now follows readily by expressing $\frac{d}{dp}\mathbb{P}_p(A)$ in terms of the previous partial derivatives:

$$\begin{aligned}\frac{d}{dp}\mathbb{P}_p(A) &= \sum_{v \in S}\left(\frac{\partial}{\partial \hat{p}_v}\hat{\mathbb{P}}(A)\right)_{\hat{p}=(p,\ldots,p)} \\ &= \sum_{v \in S}\mathbb{P}_p(v \text{ is pivotal for } A).\end{aligned}$$

□

Russo-Seymour-Welsh theory

For symmetry reasons, we have:

$$\forall n, \quad \mathbb{P}_{1/2}(\mathcal{C}_H([0,n] \times [0,n])) = 1/2. \tag{1.7}$$

In other words, the probability of crossing an $n \times n$ box is the same on every scale. In particular, this probability is bounded from below: this is the starting point of the so-called Russo-Seymour-Welsh theory (see [35, 39]), that provides lower bounds for crossings in parallelograms of fixed aspect ratio $\tau \times 1$ ($\tau \geq 1$) in the "hard direction".

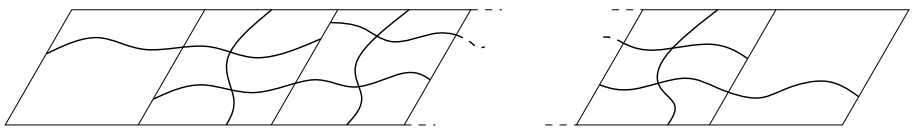


Figure 1.3: This construction shows that we can take $f_k(\delta) = \delta^{k-2}(f_2(\delta))^{k-1}$.

Theorem 2 (Russo-Seymour-Welsh)**.** *There exist universal non-decreasing functions $f_k(.)$ ($k \geq 2$), that stay positive on $(0,1)$ and verify: if for some parameter p the probability of crossing an $n \times n$ box is at least δ_1, then the probability of crossing a $kn \times n$ parallelogram is at least $f_k(\delta_1)$. Moreover, these functions can be chosen satisfying the additional property: $f_k(\delta) \to 1$ as $\delta \to 1$, with $f_k(1-\epsilon) = 1 - C_k\epsilon^{\alpha_k} + o(\epsilon^{\alpha_k})$ for some $C_k, \alpha_k > 0$.*

If for instance $p > 1/2$, we know that when n gets very large, the probability δ_1 of crossing an $n \times n$ rhombus becomes closer and closer to 1, and the additional property tells that the probability of crossing a $kn \times n$ parallelogram tends to 1 *as a function of* δ_1.

Combined with Eq.(1.7), the theorem entails:

Corollary 3. *For each $k \geq 1$, there exists some $\delta_k > 0$ such that*

$$\forall n, \quad \mathbb{P}_{1/2}(\mathcal{C}_H([0,kn] \times [0,n])) \geq \delta_k. \tag{1.8}$$

Note that only going from rhombi to parallelograms of aspect ratio slightly larger than 1 is difficult. For instance, once the result is known for $2n \times n$ parallelograms, the construction of Figure 1.3 shows that we can take

$$f_k(\delta) = \delta^{k-2}(f_2(\delta))^{k-1}. \tag{1.9}$$

Proof. The proof goes along the same lines as the one given by Grimmett [35] for the square lattice. We briefly review it to indicate the small adaptations to be made on the triangular lattice. We hope Figure 1.4 will make things clear. For an account on RSW theory in a general setting, the reader should consult Chapter 6 of [39].

We work with hexagons, since they exhibit more symmetries. Note that a crossing of an $N \times N$ rhombus induces a left-right crossing of a hexagon of side length $N/2$ (see Figure 1.4.1). We then apply the "square-root trick" – used recurrently during the proof – to the four events $\{i \rightsquigarrow j\}$: one of them occurs with probability at least $1-(1-\delta)^{1/4}$. This implies that

$$\mathbb{P}(l_1 \rightsquigarrow r_1) = \mathbb{P}(l_2 \rightsquigarrow r_2) \geq (1-(1-\delta)^{1/4})^2 =: \tau(\delta). \tag{1.10}$$

(if $\mathbb{P}(l_1 \rightsquigarrow r_1) = \mathbb{P}(l_2 \rightsquigarrow r_2) \geq 1-(1-\delta)^{1/4}$ we are OK, otherwise we just combine a crossing $l_1 \rightsquigarrow r_2$ and a crossing $l_2 \rightsquigarrow r_1$).

Now take two hexagons H, H' as on Figure 1.4.2 (with obvious notation for their sides). With probability at least $1-(1-\delta)^{1/2}$ there exists a left-right crossing in H whose last intersection with $l'_1 \cup l'_2$ is on l'_2. Assume this is the case, and condition on the lowest left-right crossing in H: with probability at least $1-(1-\tau(\delta))^{1/2}$ it is connected to t' in H'. We then use a crossing from l'_1 to $r'_1 \cup r'_2$, occurring with probability at least $1-(1-\delta)^{1/2}$, to obtain

$$\mathbb{P}(l_1 \cup l_2 \rightsquigarrow r'_1 \cup r'_2) \geq (1-(1-\delta)^{1/2}) \times (1-(1-\tau(\delta))^{1/2}) \times (1-(1-\delta)^{1/2}).$$

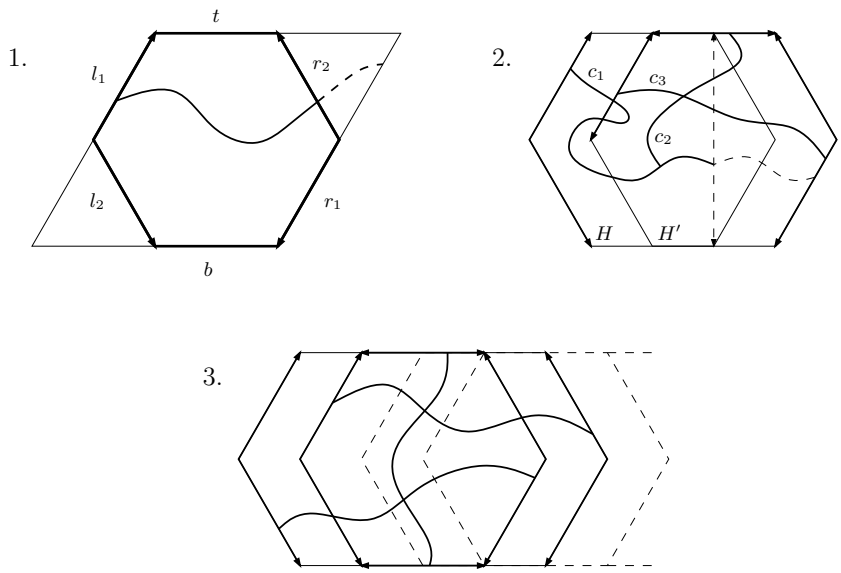


Figure 1.4: Proof of the RSW theorem on the triangular lattice.

The hard part is done: it now suffices to use t successive "longer hexagons", and $t-1$ top-bottom crossings of regular hexagons, for t large enough (see Figure 1.4.3). We construct in such a way a left right-crossing of a $2N \times N$ parallelogram, with probability at least

$$f_2(\delta) := (1-(1-\delta)^{1/2})^{2t}(1-(1-\tau(\delta))^{1/2})^{2t-1}. \tag{1.11}$$

When δ tends to 1, $\tau(\delta)$, and thus $f_2(\delta)$, also tend to 1. Moreover, it is not hard to convince oneself that f_2 admits near $\delta = 1$ an asymptotic development of the form

$$f_2(1-\epsilon) = 1 - C\epsilon^{1/8} + o(\epsilon^{1/8}). \tag{1.12}$$

Eq.(1.9) then provides the desired conclusion for any $k \geq 2$. □

1.3 Near-critical percolation overview

1.3.1 Characteristic length

We will use throughout the paper a certain "characteristic length" $L(p)$ defined in terms of crossing probabilities, or "sponge-crossing probabilities". This length is often convenient to work with, and it has been used in many papers concerning finite-size scaling, e.g. [23, 24, 12, 13].

Consider the rhombi $[0,n] \times [0,n]$. At $p = 1/2$, $\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) = 1/2$. When $p < 1/2$ (sub-critical regime), this probability tends to 0 when n goes to infinity, and it tends to 1 when $p > 1/2$ (super-critical regime). We introduce a quantity that measures the scale up to which these crossing probabilities remain bounded away from 0 and 1: for each $\epsilon_0 \in (0, 1/2)$, we define

$$L_{\epsilon_0}(p) = \begin{cases} \min\{n \text{ s.t. } \mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \leq \epsilon_0\} & \text{when } p < 1/2 \\ \min\{n \text{ s.t. } \mathbb{P}_p(\mathcal{C}^*_H([0,n] \times [0,n])) \leq \epsilon_0\} & \text{when } p > 1/2 \end{cases} \tag{1.1}$$

Hence by definition,

$$\mathbb{P}_p(\mathcal{C}_H([0, L_{\epsilon_0}(p)-1] \times [0, L_{\epsilon_0}(p)-1])) \geq \epsilon_0 \tag{1.2}$$

and

$$\mathbb{P}_p(\mathcal{C}_H([0, L_{\epsilon_0}(p)] \times [0, L_{\epsilon_0}(p)])) \leq \epsilon_0 \tag{1.3}$$

if $p < 1/2$, and the same with $*$'s if $p > 1/2$.

Note that by symmetry, we also have directly $L_{\epsilon_0}(p) = L_{\epsilon_0}(1-p)$. Since $\mathbb{P}_{1/2}(\mathcal{C}_H([0,n] \times [0,n]))$ is equal to $1/2$ on every scale, we will take the convention $L(1/2) = +\infty$, so that in the following, the expression "for any $n \leq L(p)$" must be interpreted as "for any n" when $p = 1/2$. This convention is also consistent with the following property.

Proposition 4. *For any fixed $\epsilon_0 \in (0, 1/2)$, $L_{\epsilon_0}(p) \to +\infty$ when $p \to 1/2$.*

Proof. Was it not the case, we could find an integer N and a sequence $p_k \to 1/2$, say $p_k < 1/2$, such that for each k, $L_{\epsilon_0}(p_k) = N$, which would imply

$$\mathbb{P}_{p_k}(\mathcal{C}_H([0, N] \times [0, N])) \leq \epsilon_0.$$

This contradicts the fact that

$$\mathbb{P}_{p_k}(\mathcal{C}_H([0, N] \times [0, N])) \to 1/2,$$

the function $p \mapsto \mathbb{P}_p(\mathcal{C}_H([0, N] \times [0, N]))$ being continuous (it is polynomial in p). □

1.3.2 Russo-Seymour-Welsh type estimates

When studying near-critical percolation, we will have to consider product measures $\hat{\mathbb{P}}$ more general than simply the measures $\mathbb{P}_p$ ($p \in [0, 1]$), with associated parameters $\hat{p}_v$ which are allowed to depend on the site v:

Definition 5. *A measure $\hat{\mathbb{P}}$ on configurations is said to be "between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$" if it is a product measure, and if its parameters $\hat{p}_v$ are all between p and $1-p$.*

The Russo-Seymour-Welsh theory implies that for each $k \geq 1$, there exists some $\delta_k = \delta_k(\epsilon_0) > 0$ (depending only on ϵ_0) such that for all p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$,

$$\forall n \leq L_{\epsilon_0}(p), \quad \hat{\mathbb{P}}(\mathcal{C}_H([0, kn] \times [0, n])) \geq \delta_k, \tag{1.4}$$

and for symmetry reasons this bound is also valid for horizontal white crossings.

These estimates for crossing probabilities will be the basic building blocks on which most further considerations are built. They imply that when n is not larger than $L(p)$, things can still be compared to critical percolation: roughly speaking, $L(p)$ is the scale up to which percolation can be considered as "almost critical".

In the other direction, we will see in Section 1.7.4 that $L(p)$ is also the scale at which percolation starts to look sub- or super-critical. Assume for instance that $p > 1/2$, we know that

$$\mathbb{P}_p(\mathcal{C}_H([0, L_{\epsilon_0}(p)] \times [0, L_{\epsilon_0}(p)])) \geq 1 - \epsilon_0.$$

Then using RSW (Theorem 2), we get that

$$\mathbb{P}_p(\mathcal{C}_H([0, 2L_{\epsilon_0}(p)] \times [0, L_{\epsilon_0}(p)])) \geq 1 - \epsilon_1,$$

where $1 - \epsilon_1 = f_2(1 - \epsilon_0)$ can be made arbitrarily close to 1 by taking ϵ_0 sufficiently small. This will be useful in the proof of Lemma 38 (but actually only for it).

1.3.3 Outline of the paper

In the following, we fix some value of ϵ_0 in $(0, 1/2)$. For notational convenience, we forget about the dependence on ϵ_0. We will see later (Corollary 36) that the particular choice of ϵ_0 is actually not relevant, in the sense that for any two ϵ_0, ϵ_0', the corresponding lengths are of the same order of magnitude.

In Section 1.4 we define the so-called arm events. On a scale $L(p)$, the RSW property, which we know remains true, allows to derive separation results for these arms. Section 1.5 is devoted to critical percolation, in particular how arm exponents – describing the asymptotic behavior of arm events – can be computed. In Section 1.6 we study how arm events are affected when we make vary the parameter p: if we stay on a scale $L(p)$, the picture does not change too much. It can be used to describe the characteristic functions, which we do in Section 1.7. Finally, Section 1.8 concludes the paper with some remarks and possible applications.

With the exception of this last section, the organization follows the implication between the different results: each section depends on the previous ones. A limited number of results can however be obtained directly, we will indicate it clearly when this is the case.

1.4 Arm separation

We will see that when studying critical and near-critical percolation, certain exceptional events play a central role: the arm events, referring to the existence of some number of crossings ("arms") of the annuli $S_{n,N}$ $(n < N)$, the color of each crossing (black or white) being prescribed. These events are useful because they can be combined together, and they will prove to be instrumental for studying more complex events. Their asymptotic behavior can be described precisely using SLE_6 (see next section), allowing to derive further estimates, especially on the characteristic functions.

1.4.1 Arm events

Let us consider an integer $j \geq 1$. A color sequence σ is a sequence $(\sigma_1, \ldots, \sigma_j)$ of "black" and "white" of length j. We use the letters "W" and "B" to encode the colors: the sequence (black, white, black) is thus denoted by "BWB". Only the cyclic order of the arms is relevant, and we identify two sequences if they are the same up to a cyclic permutation: for instance, the two sequences "$BWBW$" and "$WBWB$" are the same, but they are different from "$BBWW$". The resulting set is denoted by $\tilde{\mathfrak{S}}_j$. For any color sequence σ, we also introduce $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_j)$ the inverted sequence, where each color is replaced by its opposite.

For any two positive integers $n \leq N$, we define the event

$$A_{j,\sigma}(n, N) := \{\partial S_n \rightsquigarrow_{j,\sigma} \partial S_N\} \tag{1.1}$$

that there exist j disjoint monochromatic arms in the annulus $S_{n,N}$, whose colors are those prescribed by σ (when taken in counterclockwise order). We denote such an ordered set of crossings by $\mathcal{C} = \{c_i\}_{1 \leq i \leq j}$, and we say it to be "$\sigma$-colored". Recall that by convention, we have relaxed the color prescription for the extremities of the c_i's. Hence for $j = 1$ and $\sigma = B$, $A_{j,\sigma}(0, N)$ just denotes the existence of a black path $0 \rightsquigarrow \partial S_N$.

Note that a combinatorial objection due to discreteness can arise: if j is too large compared to n, the event $A_{j,\sigma}(n, N)$ can be void, just because the arms do not have enough space on ∂S_n to arrive all together. For instance $A_{j,\sigma}(0, N) = \varnothing$ if $j \geq 7$. In fact, we just have to check that n is large enough so that the number of sites touching the exterior of $|\partial S_n|$ (*ie* $|\partial S_{n+1}|$ with the acute

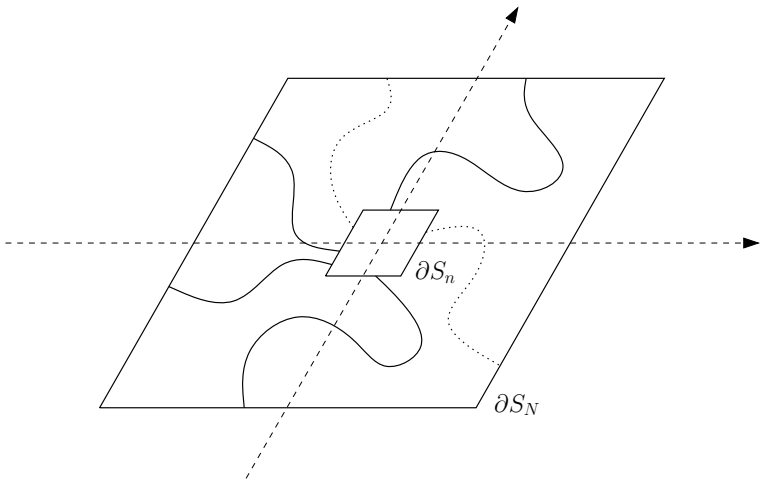


Figure 1.5: The event $A_{6,\sigma}(n, N)$, with $\sigma = BBBWBW$.

corners removed) is at least j: if this is true, we can then draw straight lines heading toward the exterior. For each positive integer j, we thus introduce $n_0(j)$ the least such nonnegative integer, and we have

$$\forall N \geq n_0(j), \quad A_{j,\sigma}(n_0(j), N) \neq \varnothing.$$

Note that $n_0(j) = 0$ for $j = 1, \ldots, 6$, and that $n_0(j) \leq j$. For asymptotics, the exact choice of n is not relevant since anyway, for any fixed $n_1, n_2 \geq n_0(j)$,

$$\hat{\mathbb{P}}(A_{j,\sigma}(n_1, N)) \asymp \hat{\mathbb{P}}(A_{j,\sigma}(n_2, N)).$$

Remark 6. *Note that Reimer's inequality implies that for any two integers j, j', and two color sequences σ, σ' of these lengths, we have:*

$$\hat{\mathbb{P}}(A_{j+j',\sigma\sigma'}(n, N)) \leq \hat{\mathbb{P}}(A_{j,\sigma}(n, N))\hat{\mathbb{P}}(A_{j',\sigma'}(n, N)) \tag{1.2}$$

for any $\hat{\mathbb{P}}$, $n \leq N$ (denoting by $\sigma\sigma'$ the concatenation of σ and σ').

1.4.2 Well-separateness

We now impose some restrictions on the events $A_{j,\sigma}(n, N)$. Our main goal is to prove that we can separate macroscopically (the extremities of) any sequence of arms: with this additional condition, the probability of $A_{j,\sigma}(n, N)$ does not decrease from more than a (universal) constant factor. This result is not really surprising, but we will need it recurrently for technical purposes.

Let us now give a precise meaning to the property of being "separated" for sets of crossings. In the following, we will actually consider crossings in different domain shapes. We first state the definition for a parallelogram of fixed $(1 \times \tau)$ aspect ratio, and explain how to adapt it in other cases.

We first require that the extremities of these crossings are distant from each other. We also need to add a condition ensuring that the crossings can easily be extended: we impose the existence of "free spaces" at their extremities, which will allow then to construct longer extensions. This leads to the following definition, similar to Kesten's "fences" [43].

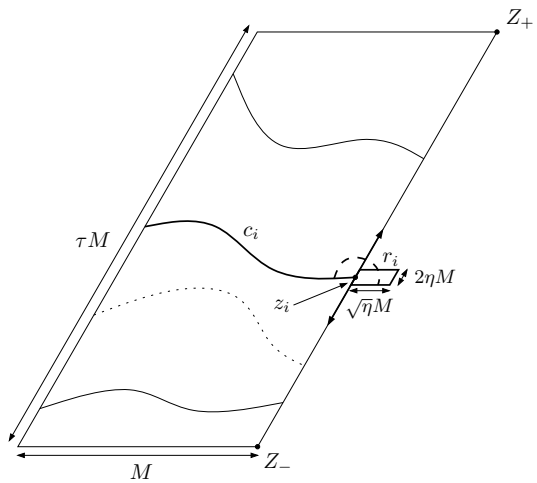


Figure 1.6: Well-separateness for a set of crossings $\mathcal{C} = \{c_i\}$.

Definition 7. *Consider some $M \times \tau M$ parallelogram $R = [a_1, a_1 + M] \times [b_1, b_1 + \tau M]$, and $\mathcal{C} = \{c_i\}_{1 \le i \le j}$ a (σ-colored) set of j disjoint left-right crossings. Introduce z_i the extremity of c_i on the right side of the parallelogram, and for some $\eta \in (0, 1]$, the parallelogram $r_i = z_i + [0, \sqrt{\eta}M] \times [-\eta M, \eta M]$, attached to R on its right side.*

We say that $\mathcal{C}$ is well-separated *at scale η (on the right side) if the two following conditions are satisfied:*

1. *The extremity z_i of each crossing is not too close from the other ones:*

$$\forall i \neq j, \quad dist(z_i, z_j) \ge 2\sqrt{\eta}M, \tag{1.3}$$

nor from the top and bottom right corners Z_+, Z_- of R:

$$\forall i, \quad dist(z_i, Z_\pm) \ge 2\sqrt{\eta}M. \tag{1.4}$$

2. *Each r_i is crossed vertically by some crossing $\tilde{c}_i$ of the same color as c_i, and*

$$c_i \rightsquigarrow \tilde{c}_i \quad in\ \mathring{S}_{\sqrt{\eta}M}(z_i). \tag{1.5}$$

For the second condition, we of course require the path connecting c_i and $\tilde{c}_i$ to be of the same color as these two crossings. The crossing $\tilde{c}_i$ is thus some small extension of c_i on the right side of R. The free spaces r_i will allow us to use locally an FKG-type inequality to further extend the c_i's on the right.

Definition 8. *We say that a set $\mathcal{C} = \{c_i\}_{1 \le i \le j}$ of j disjoint left-right crossings of R can be* made well-separated *on the right side if there exists another set $\mathcal{C}' = \{c'_i\}_{1 \le i \le j}$ of j disjoint crossings that is well-separated on the right side, such that c'_i has the same color as c_i, and the same extremity on the left side.*

The same definitions apply for well-separateness on the left side, and also for top-bottom crossings. Consider now a set of crossings of an annulus $S_{n,N}$. We can divide this set into four

subsets, according to the side of ∂S_N on which they arrive. Take for instance the set of crossings arriving on the right side: we say it to be *well-separated* if, as before, the extremities of these crossings on ∂S_N are distant from each other and from the top-right and bottom-right corners, and if there exist free spaces r_i that satisfy condition 2 of Definition 7. Then, we say that a set of crossings of $S_{n,N}$ is well-separated on the external boundary if each of the four previous sets is itself well-separated. Note that requiring the extremities to be not too close from the corners ensures that they are not too close from the extremities of the crossings arriving on the other sides either. We take the same definition for the internal boundary ∂S_n: in this case, taking the extremities away from the corners also ensures that the free spaces are included in S_n and do not intersect each other.

We are in position to define our first sub-event of $A_{j,\sigma}(n,N)$: for any $\eta, \eta' \in (0,1)$,

$$\tilde{A}^{\eta/\eta'}_{j,\sigma}(n,N) := \{\partial S_n \rightsquigarrow^{\eta/\eta'}_{j,\sigma} \partial S_N\} \tag{1.6}$$

denotes the event $A_{j,\sigma}(n,N)$ with the additional condition that the set of j arms is well-separated at scale η on ∂S_n, and at scale η' on ∂S_N.

We can even prescribe the "landing areas" of the different arms, *ie* the position of their extremities. We introduce for that some last definition:

Definition 9. *Consider ∂S_N for some integer N: a* landing sequence $\{I_i\}_{1\le i\le j}$ *on ∂S_N is a sequence of disjoint sub-intervals $I_1, \dots, I_j$ on ∂S_N in counterclockwise order. It is said to be* η-separated *if*[3]*,*

1. *dist$(I_i, I_{i+1}) \ge 2\sqrt{\eta}N$ for each i,*
2. *and dist$(I_i, Z) \ge 2\sqrt{\eta}N$ for each i and each corner Z of ∂S_N.*

It is called a landing sequence of size η *if the additional property*

3. *length$(I_i) \ge \eta N$ for each i*

is also satisfied.

We identify two landing sequences on ∂S_N and $\partial S_{N'}$ if they are identical up to a dilation. This leads to the following sub-event of $\tilde{A}^{\eta/\eta'}_{j,\sigma}(n,N)$: for two landing sequences $I = \{I_i\}_{1\le i\le j}$ and $I' = \{I'_i\}_{1\le i'\le j}$,

$$\tilde{\tilde{A}}^{\eta,I/\eta',I'}_{j,\sigma}(n,N) := \{\partial S_n \rightsquigarrow^{\eta,I/\eta',I'}_{j,\sigma} \partial S_N\} \tag{1.7}$$

denotes the event $\tilde{A}^{\eta/\eta'}_{j,\sigma}(n,N)$, with the additional requirement on the set of crossings $\{c_i\}_{1\le i\le j}$ that for each i, the extremities z_i and z'_i of c_i on (respectively) ∂S_n and ∂S_N satisfy $z_i \in I_i$ and $z'_i \in I'_i$.

We will also have use for another intermediate event between A and $\tilde{\tilde{A}}$: $\bar{A}^{I/I'}_{j,\sigma}(n,N)$, for which we only impose the landing areas I/I' of the j arms. We do not ask *a priori* the sub-intervals to be η-separated either, just to be disjoint. Note however that if they are η/η'-separated, then the extremities of the different crossings will be η/η'-separated too.

[3] As usual, we consider cyclic indices, so that here for instance $I_{j+1} = I_1$.

To summarize:

$$\begin{tikzcd}
 & A_{j,\sigma}(n,N) = \{j \text{ arms } \partial S_n \rightsquigarrow \partial S_N, \text{ color } \sigma\} \arrow[dl, "\text{separated at scale } \eta/\eta' + \text{small extensions}"'] \arrow[dr, "\text{landing areas } I/I'"] & \\
\tilde{A}^{\eta/\eta'}_{j,\sigma}(n,N) \arrow[dr, "\text{landing areas } I/I'"'] & & \bar{A}^{I/I'}_{j,\sigma}(n,N) \arrow[dl, "\text{small extensions (if } I/I' \text{ are } \eta/\eta'\text{-separated)}"] \\
 & \tilde{\tilde{A}}^{\eta,I/\eta',I'}_{j,\sigma}(n,N) &
\end{tikzcd}$$

Remark 10. *If we take for instance alternating colors ($\bar{\sigma} = BWBW$), and as landing areas $\bar{I}_1, \dots, \bar{I}_4$ the (resp.) right, top, left and bottom sides of ∂S_N, the 4-arm event $\bar{A}^{\cdot/\bar{I}}_{4,\bar{\sigma}}(0,N)$ (the "$\cdot$" meaning that we do not put any condition on the internal boundary) is then the event that 0 is pivotal for the existence of a left-right crossing of S_N.*

1.4.3 Statement of the results

Main result

Our main separation result is the following:

Theorem 11. *Fix an integer $j \geq 1$, some color sequence $\sigma \in \tilde{\mathfrak{S}}_j$ and $\eta_0, \eta_0' \in (0,1)$. Then we have*

$$\hat{\mathbb{P}}\big(\tilde{\tilde{A}}^{\eta,I/\eta',I'}_{j,\sigma}(n,N)\big) \asymp \hat{\mathbb{P}}\big(A_{j,\sigma}(n,N)\big) \tag{1.8}$$

uniformly in all landing sequences I/I' of size η/η', with $\eta \geq \eta_0$ and $\eta' \geq \eta_0'$, p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, $n \leq N \leq L(p)$.

First relations

Before turning to the proof of this theorem, we list some direct consequences of the RSW estimates that will be needed.

Proposition 12. *Fix $j \geq 1$, $\sigma \in \tilde{\mathfrak{S}}_j$ and $\eta_0, \eta_0' \in (0,1)$.*

1. "Extendability"*: We have*

$$\hat{\mathbb{P}}\big(\tilde{\tilde{A}}^{\eta,I/\tilde{\eta}',\tilde{I}'}_{j,\sigma}(n,2N)\big),\ \hat{\mathbb{P}}\big(\tilde{\tilde{A}}^{\tilde{\eta},\tilde{I}/\eta',I'}_{j,\sigma}(n/2,N)\big) \asymp \hat{\mathbb{P}}\big(\tilde{\tilde{A}}^{\eta,I/\eta',I'}_{j,\sigma}(n,N)\big)$$

uniformly in p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, $n \leq N \leq L(p)$, and all landing sequences I/I' (resp. $\tilde{I}/\tilde{I}'$) of size η/η' (resp. $\tilde{\eta}/\tilde{\eta}'$) larger than η_0/η_0'. In other words: "once well-separated, the arms can easily be extended".

2. "Quasi-multiplicativity"*: We have for some $C = C(\eta_0, \eta_0') > 0$*

$$\hat{\mathbb{P}}(A_{j,\sigma}(n_1,n_3)) \geq C\,\hat{\mathbb{P}}(\tilde{\tilde{A}}^{\cdot/\eta,I_\eta}_{j,\sigma}(n_1,n_2/4))\hat{\mathbb{P}}(\tilde{\tilde{A}}^{\eta',I_{\eta'}/\cdot}_{j,\sigma}(n_2,n_3))$$

uniformly in p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, $n_0(j) \leq n_1 < n_2 < n_3 \leq L(p)$ with $n_2 \geq 4n_1$, and all landing sequences I/I' of size η/η' larger than η_0/η_0'.

3. *For any $\eta, \eta' > 0$, there exists a constant $C = C(\eta, \eta') > 0$ with the following property: for any p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, $n \le N \le L(p)$, there exist two landing sequences I and I' of size η and η' (that may depend on all the parameters mentioned) such that*

$$\hat{\mathbb{P}}\big(\tilde{\tilde{A}}^{\eta,I/\eta',I'}_{j,\sigma}(n,N)\big) \ge C\,\hat{\mathbb{P}}\big(\tilde{A}^{\eta,\eta'}_{j,\sigma}(n,N)\big).$$

Proof. The proof relies on gluing arguments based on RSW constructions. However, the events considered are not monotone when σ is non-constant (there is at least one black arm and one white arm). We will thus need a slight generalization of the FKG inequality for events "locally monotone".

Lemma 13. *Consider A^+, $\tilde{A}^+$ two increasing events, and A^-, $\tilde{A}^-$ two decreasing events. Assume that there exist three disjoint finite sets of vertices $\mathcal{A}$, $\mathcal{A}^+$ and $\mathcal{A}^-$ such that A^+, A^-, $\tilde{A}^+$ and $\tilde{A}^-$ depend only on the sites in, respectively, $\mathcal{A} \cup \mathcal{A}^+$, $\mathcal{A} \cup \mathcal{A}^-$, $\mathcal{A}^+$ and $\mathcal{A}^-$. Then we have*

$$\hat{\mathbb{P}}(\tilde{A}^+ \cap \tilde{A}^- | A^+ \cap A^-) \ge \hat{\mathbb{P}}(\tilde{A}^+)\hat{\mathbb{P}}(\tilde{A}^-) \tag{1.9}$$

for any product measure $\hat{\mathbb{P}}$.

Proof. Conditionally on the configuration $\omega_{\mathcal{A}}$ in $\mathcal{A}$, the events $A^+ \cap \tilde{A}^+$ and $A^- \cap \tilde{A}^-$ are independent, so that

$$\hat{\mathbb{P}}(A^+ \cap \tilde{A}^+ \cap A^- \cap \tilde{A}^- | \omega_{\mathcal{A}}) = \hat{\mathbb{P}}(A^+ \cap \tilde{A}^+ | \omega_{\mathcal{A}})\hat{\mathbb{P}}(A^- \cap \tilde{A}^- | \omega_{\mathcal{A}}).$$

The FKG inequality implies that

$$\begin{aligned}\hat{\mathbb{P}}(A^+ \cap \tilde{A}^+ | \omega_{\mathcal{A}}) &\ge \hat{\mathbb{P}}(A^+ | \omega_{\mathcal{A}})\hat{\mathbb{P}}(\tilde{A}^+ | \omega_{\mathcal{A}}) \\ &= \hat{\mathbb{P}}(A^+ | \omega_{\mathcal{A}})\hat{\mathbb{P}}(\tilde{A}^+)\end{aligned}$$

and similarly with A^- and $\tilde{A}^-$. Hence,

$$\begin{aligned}\hat{\mathbb{P}}(A^+ \cap \tilde{A}^+ \cap A^- \cap \tilde{A}^- | \omega_{\mathcal{A}}) &\ge \hat{\mathbb{P}}(A^+ | \omega_{\mathcal{A}})\hat{\mathbb{P}}(\tilde{A}^+)\hat{\mathbb{P}}(A^- | \omega_{\mathcal{A}})\hat{\mathbb{P}}(\tilde{A}^-) \\ &= \hat{\mathbb{P}}(A^+ \cap A^- | \omega_{\mathcal{A}})\hat{\mathbb{P}}(\tilde{A}^+)\hat{\mathbb{P}}(\tilde{A}^-).\end{aligned}$$

The conclusion follows by summing over all configurations $\omega_{\mathcal{A}}$. □

Once this lemma at our disposal, items 1. and 2. are straightforward. For item 3., we consider a covering of ∂S_n (resp. ∂S_N) with at most $8\eta^{-1}$ (resp. $8\eta'^{-1}$) intervals (I) of length η (resp. (I') of length η'). Then for some I, I',

$$\hat{\mathbb{P}}\big(\tilde{\tilde{A}}^{\eta,I/\eta',I'}_{j,\sigma}(n,N)\big) \ge (8\eta^{-1})^{-1}(8\eta'^{-1})^{-1}\hat{\mathbb{P}}\big(\tilde{A}^{\eta,\eta'}_{j,\sigma}(n,N)\big).$$

□

We also have the following a-priori bounds for the arm events:

Proposition 14. *Fix some $j \ge 1$, $\sigma \in \tilde{\mathfrak{S}}_j$ and $\eta_0, \eta_0' \in (0,1)$. Then there exist some exponents $0 < \alpha_j, \alpha' < \infty$, as well as constants $0 < C_j, C' < \infty$, such that*

$$C_j\left(\frac{n}{N}\right)^{\alpha_j} \le \hat{\mathbb{P}}\big(\tilde{\tilde{A}}^{\eta,I/\eta',I'}_{j,\sigma}(n,N)\big) \le C'\left(\frac{n}{N}\right)^{\alpha'} \tag{1.10}$$

uniformly in p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, $n \le N \le L(p)$, and all landing sequences I/I' of size η/η' larger than η_0/η_0'.

The lower bound comes from iterating item 1. The upper bound can be obtained by using concentric annuli: in each of them, RSW implies that there is a probability bounded away from zero to observe a black circuit, preventing the existence of a white arm (consider a white circuit instead if $\sigma = BB\dots B$).

1.4.4 Proof of the main result

Assume that $A_{j,\sigma}(n,N)$ is satisfied: our goal to link this event to the event $\tilde{\tilde{A}}_{j,\sigma}^{\eta_0,I_{\eta_0}/\eta_0',I_{\eta_0'}}(n,N)$, for some fixed scales η_0, η_0'.

Proof. First note that it suffices to prove the result for n, N which are powers of two: then we would have, if k, K are such that $2^{k-1} < n \le 2^k$ and $2^K \le n < 2^{K+1}$,

$$\begin{aligned}\hat{\mathbb{P}}\big(A_{j,\sigma}(n,N)\big) &\le \hat{\mathbb{P}}\big(A_{j,\sigma}(2^k,2^K)\big)\\ &\le C_1\hat{\mathbb{P}}\big(\tilde{\tilde{A}}_{j,\sigma}^{\eta,I/\eta',I'}(2^k,2^K)\big)\\ &\le C_2\hat{\mathbb{P}}\big(\tilde{\tilde{A}}_{j,\sigma}^{\eta,I/\eta',I'}(n,N)\big).\end{aligned}$$

We have to deal with the extremities of the j arms on the internal boundary ∂S_n, and on the external boundary ∂S_N.

1. External extremities

Let us begin with the external boundary. In the course of proof, we will have use for the intermediate event $\tilde{A}_{j,\sigma}^{\cdot/\eta'}(n,N)$ that there exists a set of j arms that is well-separated on the external side ∂S_N only, and also the event $\tilde{\tilde{A}}_{j,\sigma}^{\cdot/\eta',I'}(n,N)$ associated to some landing sequence I' on ∂S_N. Each of the j arms induces in $S_{2^{K-1},2^K}$ a crossing of one of the four U-shaped regions $U_{2^{K-1}}^{1,\text{ext}},\dots,U_{2^{K-1}}^{4,\text{ext}}$ depicted in Figure 1.7. The "ext" indicates that a crossing of such a region connects the two marked parts of the boundary. For the internal extremities, we will use the same regions, but we distinguish different parts of the boundary. The key observation is the following:

In a U-shaped region, any set of disjoint crossings can be made well-separated with high probability.

More precisely, if we take such an $N \times 4N$ domain, the probability that any set of disjoint crossings can be made η-well-separated (on the external boundary) can be made arbitrarily close to 1 by choosing η sufficiently small, uniformly in N. We prove the following lemma, which implies that on *every* scale, with very high probability the j arms can be made well-separated.

Lemma 15. *For any $\delta > 0$, there exists a size $\eta(\delta) > 0$ such that for any p, any $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$ and any $N \le L(p)$: in the domain $U_N^{1,ext}$,*

$$\hat{\mathbb{P}}(\textit{Any set of disjoint crossings can be made } \eta\textit{-well-separated}) \ge 1-\delta. \tag{1.11}$$

Proof. First we note that there cannot be too many disjoint crossings in $U_N^{1,\text{ext}}$. Indeed, the probability of crossing this domain is less than some $1-\delta'$ (by RSW): combined with the BK inequality, this implies that the probability of observing at least h crossings is less than

$$(1-\delta')^h. \tag{1.12}$$

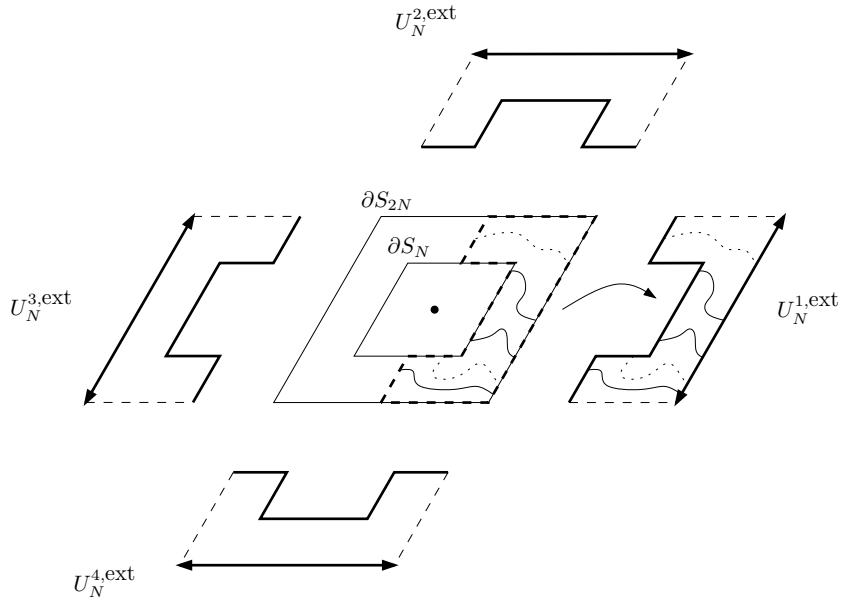


Figure 1.7: The four U-shaped regions that we use for the external extremities.

We thus take T such that this quantity is less than $\delta/4$.

Consider for the moment any $\eta \in (0,1)$ (we will see during the proof how to choose it). We note that we can put disjoint annuli around Z_- and Z_+ to prevent crossings from arriving there. Consider Z_- for instance, and look at the disjoint annuli centered on Z_- of the form $S_{2^{l-1},2^l}(Z_-)$, with $\sqrt{\eta}N \leq 2^{l-1} < 2^l \leq \eta^{3/8}N$ (see Figure 1.8). We can take at least $-C_4 \log \eta$ such disjoint annuli for some universal constant $C_4 > 0$, and with probability at least $1-(1-\delta'')^{-C_4 \log \eta}$ there exists a black circuit in one of the annuli. Consider then the annuli $S_{2^{l-1},2^l}(Z_-)$, with $\eta^{3/8}N \leq 2^{l-1} < 2^l \leq \eta^{1/4}N$: with probability at least $1-(1-\delta'')^{-C_4' \log \eta}$ we observe a white circuit in one of them. If two circuits as described exist, we say that Z_- is "protected". The same reasoning applies for Z_+.

Consider now the following construction: take c_1 the lowest (*ie* closest to the bottom side) monochromatic crossing (which can be either black or white), then c_2 the lowest monochromatic crossing disjoint from c_1, and so on. The process stops after t steps, and we denote by $\mathcal{C} = \{c_u\}_{1 \leq u \leq t}$ the set of crossings so-obtained. Of course, $\mathcal{C}$ can be void: we set $t = 0$ in this case. We have

$$\mathbb{P}(t \geq T) \leq (1-\delta')^T \leq \delta/4 \tag{1.13}$$

by definition of T. We denote by z_u the extremity of c_u on the right side, and by $\sigma_u \in \{B, W\}$ its color.

In order to get some independence and be able to apply the previous construction around the extremities of the crossings, we condition on the successive crossings. Consider some $u \in \{1, \dots, T\}$ and some ordered sequence of crossings $\tilde{c}_1, \tilde{c}_2, \dots, \tilde{c}_u$, together with colors $\tilde{\sigma}_1, \tilde{\sigma}_2, \dots, \tilde{\sigma}_u$. The event $E_u := \{t \geq u \text{ and } c_v = \tilde{c}_v,\ \sigma_v = \tilde{\sigma}_v \text{ for any } v \in \{1, \dots, u\}\}$ is independent from the status of the sites above $\tilde{c}_u$. Hence, if we condition on E_u, percolation there remains unbiased and we can use the RSW theorem.

We now do the same construction as before. Look at the disjoint annuli centered on z_u of the form $S_{2^{l-1},2^l}(z_u)$, with $\sqrt{\eta}N \leq 2^{l-1} < 2^l \leq \eta^{3/8}N$ on one hand, and with $\eta^{3/8}N \leq 2^{l-1} <$

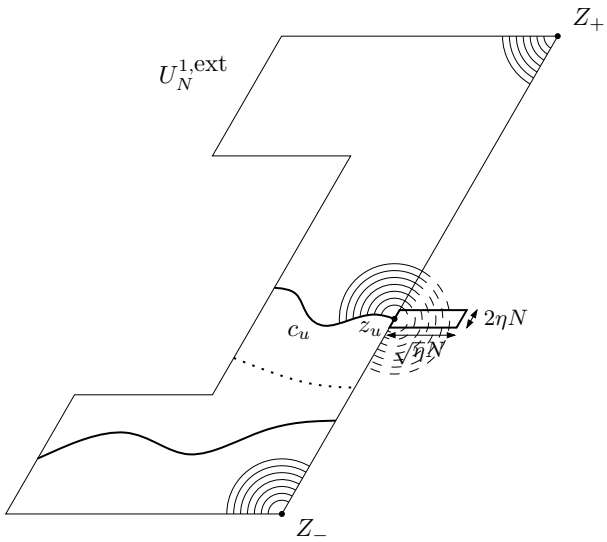


Figure 1.8: We apply RSW in concentric annuli around Z_- and Z_+, and then around the extremity z_u of each crossing c_u.

$2^l \leq \eta^{1/4}N$ on the other hand. Assume for instance that $\tilde{\sigma}_u = B$. With probability at least $1-(1-\delta'')^{-C_4''\log\eta}$ we observe a white circuit in one of the annuli in the first set, preventing other disjoint black crossings to arrive near z_u, and also a black one in the second set, preventing white crossings to arrive. Moreover, by considering a black circuit in the annuli $S_{2^{l-1},2^l}(z_u)$ with $\eta^{3/4}N \leq 2^{l-1} < 2^l \leq \sqrt{\eta}N$, we can construct a small extension of c_u. If the three circuits described exist, c_u is said to be "protected from above". Summing over all possibilities for $\tilde{c}_i$, $\tilde{\sigma}_i$ $(1 \leq i \leq u)$, we get that for some C_4''',

$$\mathbb{P}(t \geq u \text{ and } c_u \text{ is not protected from above}) \leq (1-\delta'')^{-C_4'''\log\eta}. \tag{1.14}$$

Now for our set of crossings $\mathcal{C}$,

$$\begin{aligned}\mathbb{P}(&\mathcal{C} \text{ is not } \eta\text{-well-separated})\\ &\leq \mathbb{P}(t \geq T) + \sum_{u=1}^{T-1} \mathbb{P}(t \geq u \text{ and } c_u \text{ is not protected from above})\\ &\quad + \mathbb{P}(Z_- \text{ is not protected}) + \mathbb{P}(Z_+ \text{ is not protected}).\end{aligned}$$

First, each term in the sum, as well as the last two terms, are less than $(1-\delta'')^{-C_4'''\log\eta}$. We also have $\mathbb{P}(t \geq T) \leq \delta/4$, so that the right-hand side is at most

$$(T+1)(1-\delta'')^{-C_4'''\log\eta} + \frac{\delta}{4}. \tag{1.15}$$

It is less than δ if we choose η sufficiently small (T is fixed).

We now assume that $\mathcal{C}$ is η-well-separated, and prove that any other set $\mathcal{C}' = \{c_u'\}_{1\leq u\leq t'}$ of t' $(\leq t)$ disjoint crossings (we take it ordered) can also be made η-well-separated. For that purpose, we replace recursively the tip of each c_v' by the tip of one of the c_u's. If we take c_1' for instance, it has to cross at least one of the c_v (by maximality of $\mathcal{C}$). Let us call c_{v_1} the lowest one: still

by maximality, c_1' cannot go below it. Take the piece of c_1' between its extremity z_1' and its last intersection a_1 with c_{v_1}, and replace it with the corresponding piece of c_{v_1}: this gives c_1''. This new crossing has the same extremity as c_{v_1} on the right-side, and it is not hard to check that it is connected to the small extension $\tilde{c}_{v_1}$ of c_{v_1} on the external side. Indeed, this extension is connected by a path that touches c_{v_1} in, say, b_1: either b_1 is between a_1 and z_1, in which case c_1'' is automatically connected to $\tilde{c}_{v_1}$, otherwise c_1' has to cross the connecting path before a_1 and c_1'' is also connected to $\tilde{c}_{v_1}$.

Consider then c_2, and c_{v_2} the lowest crossing it intersects: necessarily $v_2 > v_1$ (since c_1 stays above c_{v_1}), and the same reasoning applies. The claim follows by continuing this procedure until $c_{t'}'$.

□

The arms are well-separated with positive probability.

The idea is then to "go down" in successive concentric annuli, and to apply the lemma in each of them. We work with two different scales of separation:

- a fixed (macroscopic) scale η_0' that we will use to extend arms, associated to a *constant* extension cost.
- another scale η' which is *very* small ($\eta' \ll \eta_0'$), so that the j arms can be made well-separated at scale η' with very high probability.

The proof goes as follows. Take some $\delta > 0$ very small (we will see later how small), and some $\eta' > 0$ associated to it by the lemma. We start from the scale ∂S_{2^K} and look at the crossings induced by the j arms. The previous lemma implies that with very high probability, these j arms can be modified in $S_{2^{K-1},2^K}$ so that they are η'-well-separated. Otherwise, we go down to the next annulus: there still exist j arms, and what happens in $S_{2^{K-1},2^K}$ is independent of what happens in $S_{2^{K-1}}$. On each scale, we have a very low probability to fail, and once the arms are separated on scale η', we go backwards by using the scale η_0', for which the cost of extension is constant.

More precisely, after one step we get

$$A_{j,\sigma}(2^k, 2^K) \subseteq \tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k, 2^K) \cup \Big(\{\text{One of the four } U_{2^{K-1}}^{i,\text{ext}} \text{ fails}\} \cap A_{j,\sigma}(2^k, 2^{K-1})\Big).$$

Hence, by independence of the two latter events,

$$\hat{\mathbb{P}}(A_{j,\sigma}(2^k, 2^K)) \leq \hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k, 2^K)) + (4\delta)\hat{\mathbb{P}}(A_{j,\sigma}(2^k, 2^{K-1})).$$

We then iterate this argument: after $K - k$ steps,

$$\begin{aligned}
&\hat{\mathbb{P}}(A_{j,\sigma}(2^k, 2^K)) \\
&\quad \leq \hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k, 2^K)) + (4\delta)\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k, 2^{K-1})) + (4\delta)^2\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k, 2^{K-2})) + \dots \\
&\qquad + (4\delta)^{K-k-1}\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k, 2^{k+1})) + (4\delta)^{K-k}.
\end{aligned}$$

We then use the size η_0' to go backwards: if the crossings are η'-separated at some scale m, there exists some landing sequence $I_{\eta'}$ of size η' where the probability of landing is comparable to the probability of just being η'-well-separated, and then we can reach $I_{\eta_0'}$ of size η_0' on the next scale. More precisely, there exist universal constants $C_1(\eta')$, $C_2(\eta')$ depending only on η' such that for all $1 \leq i' \leq i$, we can choose some $I_{\eta'}$ (which can depend on i') such that

$$\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k, 2^{K-i'})) \leq C_1(\eta')\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\cdot/\eta',I_{\eta'}}(2^k, 2^{K-i'}))$$

and then go to $I_{\eta_0'}$ on the next scale with cost $C_2(\eta')$:

$$\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\cdot/\eta',I_{\eta'}}(2^k,2^{K-i'})) \leq C_2(\eta')\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\cdot/\eta_0',I_{\eta_0'}}(2^k,2^{K-i'+1})).$$

Now for the size η_0', going from ∂S_m to ∂S_{2m} has a cost C_0' depending *only* on η_0' on each scale m, we have thus

$$\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k,2^{K-i'})) \leq C_1(\eta')C_2(\eta')C_0^{i'-1}\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\cdot/\eta_0',I_{\eta_0'}}(2^k,2^K)).$$

There remains a problem with the first term $\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k,2^K))\ldots$ So assume that we have started from 2^{K-1} instead, so that the annulus $S_{2^{K-1},2^K}$ remains free:

$$\begin{aligned}
&\hat{\mathbb{P}}(A_{j,\sigma}(2^k,2^K)) \\
&\quad\leq \hat{\mathbb{P}}(A_{j,\sigma}(2^k,2^{K-1})) \\
&\quad\leq \hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k,2^{K-1})) + (4\delta)\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k,2^{K-2})) + (4\delta)^2\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k,2^{K-3})) + \ldots \\
&\qquad + (4\delta)^{K-k-2}\hat{\mathbb{P}}(\tilde{A}_{j,\sigma}^{\cdot/\eta'}(2^k,2^{k+1})) + (4\delta)^{K-k-1} \\
&\quad\leq C_1(\eta')C_2(\eta')\Big[1+(4\delta C_0)+\ldots+(4\delta C_0)^{K-k-1}\Big]\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\cdot/\eta_0',I_{\eta_0'}}(2^k,2^K)).
\end{aligned}$$

Now C_0 is fixed as was noticed before, so we may have taken δ such that $4\delta C_0 < 1/2$, so that

$$C_1(\eta')C_2(\eta')\Big[1+(4\delta C_0)+\ldots+(4\delta C_0)^{K-k-1}\Big] \leq C_3(\eta')$$

for some $C_3(\eta')$. We have thus reached the desired conclusion for external extremities:

$$\hat{\mathbb{P}}(A_{j,\sigma}(2^k,2^K)) \leq C_3(\eta')\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\cdot/\eta_0',I_{\eta_0'}}(2^k,2^K)).$$

2. Internal extremities

The reasoning is the same for internal extremities, except that we work in the other direction, from ∂S_{2^k} toward the interior. If we consider the domains $U_N^{i,\mathrm{int}}$ having the same shapes as the $U_N^{i,\mathrm{ext}}$ domains, but with different parts of the boundary distinguished (see Figure 1.9), then the lemma remains true. Hence,

$$\begin{aligned}
&\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\cdot/\eta_0',I_{\eta_0'}}(2^k,2^K)) \leq \hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\cdot/\eta_0',I_{\eta_0'}}(2^{k+1},2^K)) \\
&\quad\leq \hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\eta,\cdot/\eta_0',I_{\eta_0'}}(2^{k+1},2^K)) + (4\delta)\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\eta,\cdot/\eta_0',I_{\eta_0'}}(2^{k+2},2^K)) + \ldots \\
&\qquad + (4\delta)^{K-k-2}\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\eta,\cdot/\eta_0',I_{\eta_0'}}(2^{K-1},2^K)) + (4\delta)^{K-k-1} \\
&\quad\leq C_1(\eta)C_2(\eta)\Big[1+(4\delta C_0)+\ldots+(4\delta C_0)^{K-k-1}\Big]\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\eta_0,I_{\eta_0}/\eta_0',I_{\eta_0'}}(2^k,2^K))
\end{aligned}$$

and the conclusion follows. □

1.4.5 Some consequences

We now state some important consequences of the previous theorem.

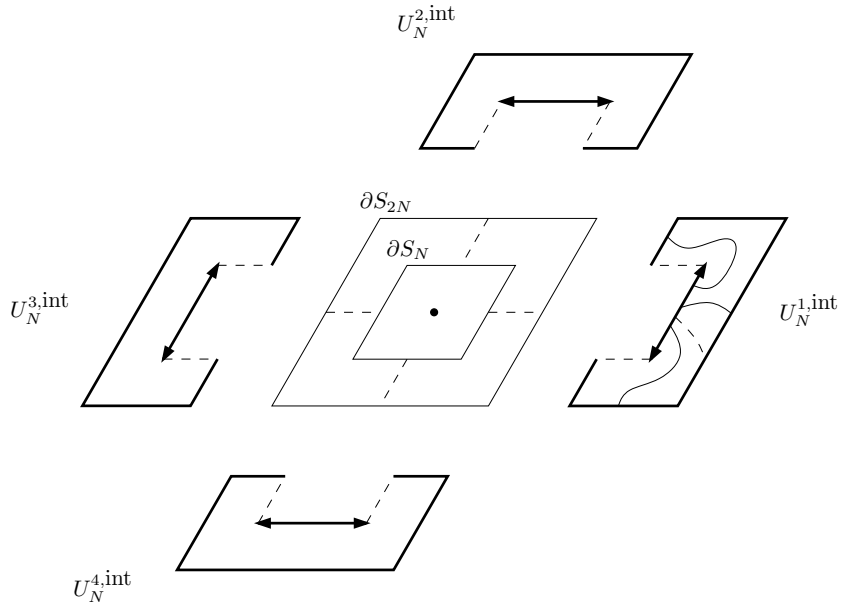


Figure 1.9: For the internal extremities, we consider the same domains but we mark different parts of the boundary.

Extendability

Proposition 16. *Take $j \geq 1$ and a color sequence $\sigma \in \tilde{\mathfrak{S}}_j$. Then*

$$\hat{\mathbb{P}}(A_{j,\sigma}(n, 2N)),\ \hat{\mathbb{P}}(A_{j,\sigma}(n/2, N)) \asymp \hat{\mathbb{P}}(A_{j,\sigma}(n, N)) \tag{1.16}$$

uniformly in p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$ and $n_0(j) \leq n \leq N \leq L(p)$.

Proof. This proposition comes directly from combining the arm separation theorem with the extendability property of the $\tilde{\tilde{A}}$ events (item 1. of Proposition 12). □

Quasi-multiplicativity

Proposition 17. *Take $j \geq 1$ and a color sequence $\sigma \in \tilde{\mathfrak{S}}_j$. Then*

$$\hat{\mathbb{P}}(A_{j,\sigma}(n_1, n_2))\hat{\mathbb{P}}(A_{j,\sigma}(n_2, n_3)) \asymp \hat{\mathbb{P}}(A_{j,\sigma}(n_1, n_3)) \tag{1.17}$$

uniformly in p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$ and $n_0(j) \leq n_1 < n_2 < n_3 \leq L(p)$.

Proof. On one hand, we have

$$\hat{\mathbb{P}}(A_{j,\sigma}(n_1, n_3)) \leq \hat{\mathbb{P}}(A_{j,\sigma}(n_1, n_2) \cap A_{j,\sigma}(n_2, n_3)) = \hat{\mathbb{P}}(A_{j,\sigma}(n_1, n_2))\hat{\mathbb{P}}(A_{j,\sigma}(n_2, n_3))$$

by independence of the events $A_{j,\sigma}(n_1, n_2)$ and $A_{j,\sigma}(n_2, n_3)$.

On the other hand, we may assume that $n_2 \geq 8n_1$. Then for some η_0, I_{η_0}, the previous results (separation and extendability) allow to use the quasi-multiplicativity for $\tilde{\tilde{A}}$ events (item 2. of Proposition 12):

$$\begin{aligned}
\hat{\mathbb{P}}(A_{j,\sigma}(n_1, n_2))\hat{\mathbb{P}}(A_{j,\sigma}(n_2, n_3)) &\asymp \hat{\mathbb{P}}(A_{j,\sigma}(n_1, n_2/4))\hat{\mathbb{P}}(A_{j,\sigma}(n_2, n_3)) \\
&\asymp \hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\cdot/\eta_0, I_{\eta_0}}(n_1, n_2/4))\hat{\mathbb{P}}(\tilde{\tilde{A}}_{j,\sigma}^{\eta_0, I_{\eta_0}/\cdot}(n_2, n_3)) \\
&\asymp \hat{\mathbb{P}}(A_{j,\sigma}(n_1, n_3)).
\end{aligned}$$

□

Arms with defects

In some situations, the notion of arms that are completely monochromatic is too restrictive, and the following question arises quite naturally: do the probabilities change if we allow the arms to present some (fixed) number of "defects", *ie* sites of the opposite color?

We define $A^{(d)}_{j,\sigma}(n,N)$ the event that there exist j disjoint arms $a_1,\dots,a_j$ from ∂S_n to ∂S_N with the property: for any $i\in\{1,\dots,j\}$, a_i contains at most d sites of color $\tilde\sigma_i$. The quasi-multiplicativity property entails the following result, which will be needed for the proof of Theorem 27:

Proposition 18. *Let $j\geq 1$ and $\sigma\in\tilde{\mathfrak{S}}_j$. Fix also some number d of defects. Then we have*

$$\hat{\mathbb{P}}\big(A^{(d)}_{j,\sigma}(n,N)\big)\asymp(1+\log(N/n))^d\hat{\mathbb{P}}\big(A_{j,\sigma}(n,N)\big) \tag{1.18}$$

uniformly in p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$ and $n_0(j)\leq n\leq N\leq L(p)$.

Actually, we will only need the upper bound on $\hat{\mathbb{P}}\big(A^{(d)}_{j,\sigma}(n,N)\big)$. For instance, we will see in the next section that the arm events decay like power laws at the critical point. This proposition thus implies, in particular, that the "arm with defects" events are described by the same exponents: allowing defects just adds a logarithmic correction.

Proof. We introduce a logarithmic division of the annulus $S_{n,N}$: we take k and K such that $2^{k-1}<n\leq 2^k$ and $2^K\leq N<2^{K+1}$. Roughly speaking, we "take away" the annuli where the defects take place, and "glue" the pieces of arms in the remaining annuli by using the quasi-multiplicativity property.

Let us begin with the upper bound: we proceed by induction on d. The property clearly holds for $d=0$. Take some $d\geq 1$: by considering the first annuli $S_{2^i,2^{i+1}}$ where a defect occurs, we get

$$\hat{\mathbb{P}}(A^{(d)}_{j,\sigma}(n,N))\leq\sum_{i=k}^{K-1}\hat{\mathbb{P}}(A_{j,\sigma}(2^k,2^i))\hat{\mathbb{P}}(A^{(d-1)}_{j,\sigma}(2^{i+1},2^K)). \tag{1.19}$$

We have $\hat{\mathbb{P}}(A^{(d-1)}_{j,\sigma}(2^{i+1},2^K))\leq C_{d-1}(1+\log(N/n))^{d-1}\hat{\mathbb{P}}(A_{j,\sigma}(2^{i+1},2^K))$ thanks to the induction hypothesis, and by quasi-multiplicativity,

$$\begin{aligned}\hat{\mathbb{P}}(A^{(d)}_{j,\sigma}(n,N))&\leq(1+\log(N/n))^{d-1}C_{d-1}\sum_{i=k}^{K-1}\hat{\mathbb{P}}(A_{j,\sigma}(2^k,2^i))\hat{\mathbb{P}}(A_{j,\sigma}(2^{i+1},2^K))\\&\leq C_{d-1}(1+\log(N/n))^{d-1}\sum_{i=k}^{K-1}C'\hat{\mathbb{P}}(A_{j,\sigma}(2^k,2^K))\\&\leq C_d(1+\log(N/n))^{d-1}(K-k)\hat{\mathbb{P}}(A_{j,\sigma}(2^k,2^K)),\end{aligned}$$

which gives the desired upper bound.

For the lower bound, note that for any $k\leq i_0<i_1<\ldots<i_d<i_{d+1}=K$, $A^{(d)}_{j,\sigma}(n,N)\supseteq A^{(d)}_{j,\sigma}(2^{k-1},2^{K+1})\supseteq A_{j,\sigma}(2^{k-1},2^{K+1})\cap\{$Each of the j arms has exactly one defect in each of the

annuli $S_{2^{i_r},2^{i_r+1}}\}$, so that for $K-k \geq d+1$,

$$\begin{aligned}\hat{\mathbb{P}}(A^{(d)}_{j,\sigma}(n,N)) &\geq \sum_{k=i_0<i_1<i_2<\ldots<i_d<i_{d+1}=K} C_d \prod_{r=0}^{d} \hat{\mathbb{P}}(A_{j,\sigma}(2^{i_r+1},2^{i_{r+1}})) \\ &\geq C'_d \binom{K-k-1}{d} \hat{\mathbb{P}}(A_{j,\sigma}(2^{k-1},2^{K+1})) \\ &\geq C''_d (K-k)^d \hat{\mathbb{P}}(A_{j,\sigma}(2^{k-1},2^{K+1})),\end{aligned}$$

and our lower bound follows. □

Remark: more general annuli

We will sometimes need to consider more general arm events, in annuli of the form $R \setminus r$, for non-necessarily concentric parallelograms $r \subseteq \mathring{R}$. Items 1. and 2. of Proposition 12 can easily be extended. Separateness and well-separateness can be defined in the same way for these arm events, and for any $\tau > 1$, we can get results uniform in the usual parameters and in parallelograms r, R such that $S_n \subseteq r \subseteq S_{\tau n}$ and $S_{N/\tau} \subseteq R \subseteq S_N$ for some $n, N \leq L(p)$:

$$\hat{\mathbb{P}}(\partial r \rightsquigarrow_{j,\sigma} \partial R) \asymp \hat{\mathbb{P}}(\partial S_n \rightsquigarrow_{j,\sigma} \partial S_N), \tag{1.20}$$

and similarly with separateness conditions on the external boundary or on the internal one.

1.4.6 Arms in the half-plane

So far, we have been interested in arm events in the whole plane: we can define in the same way the event $B_{j,\sigma}(n,N)$ that there exist j arms that *stay in the upper half-plane* $\mathbb{H}$, of colors prescribed by $\sigma \in \tilde{\mathfrak{S}}_j$ and connecting $\partial S'_n$ to $\partial S'_N$, with the notation $\partial S'_n = (\partial S_n) \cap \mathbb{H}$. These events appear naturally when we look at arms near a boundary.

For the sake of completeness, let us just mention that all the results stated here remain true for arms in the half-plane. In fact, there is a natural way to order the different arms, which makes this case easier. We will not use these events in the following, and we leave the details to the reader.

1.5 Description of critical percolation

When studying the phase transition of percolation, the critical regime plays a very special role. It possesses a strong property of *conformal invariance* in the scaling limit. This particularity, first observed by physicists ([57, 7, 8]), has been proved by Smirnov in [69], and later extended by Camia and Newman in [16]. It allows to link the critical regime to the SLE processes (with parameter 6 here) introduced by Schramm in [67], and thus to use computations made for these processes ([47, 48]).

In the next sections, we will see why our description of critical percolation yields in turn a good description of near-critical percolation (which does not feature a priori any sort of conformal invariance), in particular how the characteristic functions behave through the phase transition.

1.5.1 Arm exponents for critical percolation

Color switching

We focus here on the probabilities of arm events at the critical point. For arms in the half-plane, a nice combinatorial argument (noticed in [3, 71]) shows that once fixed the number j of arms, prescribing the color sequence σ does not change the probability. This is the so-called "color exchange trick":

Proposition 19. *Let $j \geq 1$ be any fixed integer. If σ, σ' are two color sequences, then for any $n_0'(j) \leq n \leq N$,*

$$\mathbb{P}_{1/2}(B_{j,\sigma}(n,N)) = \mathbb{P}_{1/2}(B_{j,\sigma'}(n,N)). \tag{1.1}$$

Proof. The proof relies on the fact that there is a canonical way to order the arms. If we condition on the i left-most arms, percolation in the remaining domain is unbiased, so that we can "flip" the sites there: for any color sequence σ, if we denote by

$$\tilde{\sigma}^{(i)} = (\sigma_1, \ldots, \sigma_i, \tilde{\sigma}_{i+1}, \ldots, \tilde{\sigma}_j)$$

the sequence with the same i first colors, and the remaining ones flipped, then

$$\mathbb{P}_{1/2}(B_{j,\sigma}(n,N)) = \mathbb{P}_{1/2}(B_{j,\tilde{\sigma}^{(i)}}(n,N)).$$

It is not hard to convince oneself that for any two sequences σ, σ', we can go from σ to σ' in a finite number of such operations. □

This result is not as direct in the whole plane case, since there is no canonical ordering any more. However, the argument can be adapted to prove that the probabilities change only by a constant factor, as long as there is an interface, *ie* as long as σ contains at least one white arm and one black arm.

Proposition 20. *Let $j \geq 1$ be any fixed integer. If $\sigma, \sigma' \in \tilde{\mathfrak{S}}_j$ are two* non-constant *color sequences (*ie *both colors are present), then*

$$\mathbb{P}_{1/2}(A_{j,\sigma}(n,N)) \asymp \mathbb{P}_{1/2}(A_{j,\sigma'}(n,N)) \tag{1.2}$$

uniformly in $n_0(j) \leq n \leq N$.

Proof. Assume that $\sigma_1 = B$ and $\sigma_2 = W$, and fix some landing sequence I. If we replace the event $A_{j,\sigma}(n,N)$ by the strengthened event $\bar{A}^{I/\cdot}_{j,\sigma}(n,N)$, we are allowed to condition on the black arm arriving on I_1 and on the white arm arriving on I_2 that are closest to each other: if we choose for instance I such that the point $(N,0)$ is between I_1 and I_2, these two arms can be determined via an exploration process starting at $(N,0)$. We can then "flip" the remaining region. More generally, we can condition on any set of consecutive arms including these two arms, and the result follows for the same reasons as in the half-plane case. □

We would like to stress the fact that for the reasoning, we crucially need two arms of opposite colors. In fact, the preceding result is expected to be false if σ is constant and σ' non-constant (the two probabilities *not* being of the same order of magnitude), which is quite surprising at first sight.

Derivation of the exponents

The link with SLE_6 makes it possible to prove the existence of the (multichromatic) "arm exponents", and derive their values ([50, 71]), that had been predicted in the physics literature (see e.g. [3] and the references therein).

Theorem 21. *Fix some $j \geq 1$. Then for any non-constant color sequence $\sigma \in \tilde{\mathfrak{S}}_j$,*

$$\mathbb{P}_{1/2}\big(A_{j,\sigma}(n_0(j), N)\big) \approx N^{-\alpha_j} \tag{1.3}$$

when $N \to \infty$, with

- $\alpha_1 = 5/48$,
- *and for $j \geq 2$, $\alpha_j = (j^2 - 1)/12$.*

Let us sketch very briefly how it is proved. Consider the discrete (radial) exploration process in a unit disc: using the property of conformal invariance in the scaling limit, we can prove that this process converges toward a radial SLE_6, for which we can compute disconnection probabilities. It implies that

$$\mathbb{P}_{1/2}(A_{j,\sigma}(\eta n, n)) \to g_j(\eta),$$

for some function $g_j(\eta) \sim \eta^{\alpha_j}$ as $\eta \to 0$. Then, the quasi-multiplicativity property in concentric annuli of fixed modulus provides the desired result.

As mentioned, this theorem is believed to be *false* for constant σ, *ie* when the arms are all of the same color. In this case, the probability should be smaller, or equivalently the exponent (assuming its existence) larger. Hence for each $j = 2, 3, \ldots$, there are two different arm exponents, the multichromatic j-arm exponent α_j given by the previous formula (most often simply called the j-arm exponent) and the monochromatic j-arm exponent α'_j, for which no closed formula is currently known, nor even predicted. The only result proved so far concerns the case $j = 2$: as shown in [50], the monochromatic 2-arm exponent can be expressed as the leading eigenvalue of some (complicated) differential operator. Numerically, it has been found (see [3]) to be approximately $\alpha'_2 \simeq 0.35\ldots$

Note also that the derivation using SLE_6 only provides a logarithmic equivalence. However, there are reasons to believe that a stronger equivalence holds, a "$\asymp$": for instance we know that this is the case for the "universal exponents" computed in the next sub-section.

We will often relate events to combinations of arm events, that in turn can be linked (see next section) to arm events at the critical point $p = 1/2$. It will thus be convenient to introduce the following notation, with $\sigma_j = BWBW\ldots$: for any $n_0(j) \leq n < N$,

$$\pi_j(n, N) := \mathbb{P}_{1/2}(A_{j,\sigma_j}(n, N)) \tag{1.4}$$

($\asymp \mathbb{P}_{1/2}(A_{j,\sigma}(n, N))$ for any non-constant σ), and in particular

$$\pi_j(N) := \mathbb{P}_{1/2}(A_{j,\sigma_j}(n_0(j), N)) \quad (\approx N^{-\alpha_j}). \tag{1.5}$$

Note that with this notation, the a-priori bound and the quasi-multiplicativity property take the aesthetic forms

$$C(n/N)^{\alpha_j} \leq \pi_j(n, N) \leq C'(n/N)^{\alpha'}, \tag{1.6}$$

$$\text{and } \pi_j(n_1, n_2)\pi_j(n_2, n_3) \asymp \pi_j(n_1, n_3). \tag{1.7}$$

Let us mention that we can derive in the same way exponents for arms in the upper half-plane, the "half-plane exponents":

Theorem 22. *Fix some $j \geq 1$. Then for any sequence of colors σ,*

$$\mathbb{P}_{1/2}\big(B_{j,\sigma}(n_0'(j), N)\big) \approx N^{-\beta_j} \tag{1.8}$$

when $N \to \infty$, with

$$\beta_j = j(j+1)/6.$$

Remark 23. *As mentioned earlier, the triangular lattice is at present the only lattice for which conformal invariance in the scaling limit has been proved, and as a consequence the only lattice for which the existence and the values of the arm exponents have been established – with the noteworthy exception of the three "universal" exponents that we are going to derive.*

Note: fractality of various sets

These arm exponents can be used to measure the size (Hausdorff dimension) of various sets describing percolation clusters. In physics literature for instance (see e.g. [3]), a set S is said to be fractal of dimension D_S if the density of points in S within a box of size n decays as n^{-x_S}, with $x_S = 2 - D_S$ (in 2D). The co-dimension x_S is related to arm exponents in many cases:

- The 1-arm exponent is related to the existence of long connections, from the center of a box to its boundary. It will thus measure the size of "big" clusters, like the incipient infinite cluster (IIC) as defined by Kesten ([40]), which scales as $n^{(2-5/48)} = n^{91/48}$.
- The monochromatic 2-arm exponent describes the size of the "backbone" of a cluster. The fact that this backbone is much thinner than the cluster itself was used by Kesten [42] to prove that the random walk on the IIC is sub-diffusive (while it has been proved to converge toward a Brownian Motion on a super-critical infinite cluster).
- The multichromatic 2-arm exponent is related to the boundaries (hulls) of big clusters, which are thus of fractal dimension $2 - \alpha_2 = 7/4$.
- The 3-arm exponent concerns the external (accessible) perimeter of a cluster, which is the accessible part of the boundary: one excludes "fjords" which are connected to the exterior only by 1-site wide passages. The dimension of this frontier is $2 - \alpha_3 = 4/3$. These two latter exponents can be observed on random interfaces, numerically and in "real-life" experiments as well (see [63, 29] for instance).
- As mentioned earlier, the 4-arm exponent with alternating colors counts the pivotal (singly-connecting) sites (often called "red" sites in physics literature). This set can be viewed as the contact points between two distinct (large) clusters, its dimension is $2 - \alpha_4 = 3/4$. We will relate this exponent to the correlation length exponent ν in Section 1.7.

1.5.2 Universal exponents

We will now examine as a complement some particular exponents, for which heuristic predictions and elementary derivations exist, namely $\beta_2 = 1$, $\beta_3 = 2$ and $\alpha_5 = 2$. They are all integers, and they were established before the complete derivation using the SLE_6 (and actually they provide crucial a-priori estimates to prove the convergence toward SLE_6). Moreover, the equivalence that we get is stronger: we can replace the "$\approx$" by a "$\asymp$".

Theorem 24. *When $N \to \infty$,*

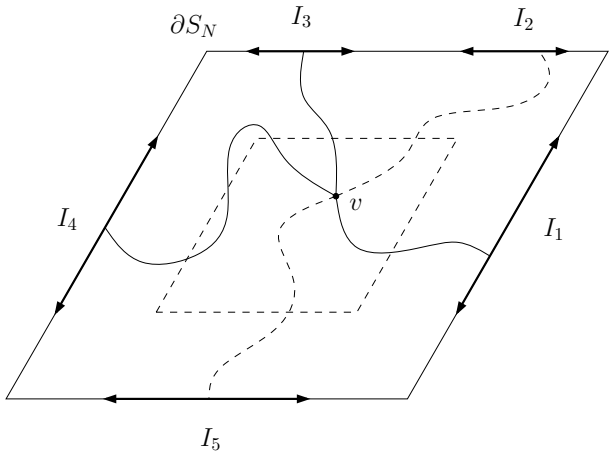


Figure 1.10: The landing sequence $I_1, \dots, I_5$.

1. *For any* $\sigma \in \mathfrak{S}_2$,
$$\mathbb{P}_{1/2}\big(B_{2,\sigma}(0,N)\big) \asymp N^{-1}.$$

2. *For any* $\sigma \in \mathfrak{S}_3$,
$$\mathbb{P}_{1/2}\big(B_{3,\sigma}(0,N)\big) \asymp N^{-2}.$$

3. *For any non-constant* $\sigma \in \tilde{\mathfrak{S}}_5$,
$$\mathbb{P}_{1/2}\big(A_{5,\sigma}(0,N)\big) \asymp N^{-2}.$$

Proof. We give a complete proof only for item 3., since we will not need the two first ones – we will however sketch at the end how to derive them.

Heuristically, we can prove that the 5-arm sites can be seen as particular points on the boundary of two big black clusters, and that consequently their number is of order 1 in $S_{N/2}$. Then it suffices to use that the different sites in $S_{N/2}$ produce contributions of the same order. This argument can be made rigorous by proving that the number of "macroscopic" clusters has an exponential tail: we refer to the first exercise sheet in [78] for more details. We propose here a more direct – but less elementary – proof using the separation lemmas.

By color switching, it is sufficient to prove the claim for $\sigma = BWBBW$. In light of our previous results, it is clear that
$$\mathbb{P}_{1/2}\big(v \rightsquigarrow_{5,\sigma} \partial S_N\big) \asymp \mathbb{P}_{1/2}\big(0 \rightsquigarrow_{5,\sigma} \partial S_N\big)$$
uniformly in N, $v \in S_{N/2}$. It is thus enough to prove that the number of such 5-arm sites in $S_{N/2}$ is of order 1.

Let us consider the upper bound first. Take the particular landing sequence $I_1, \dots, I_5$ depicted on the figure, and consider the event
$$A_v := \{v \rightsquigarrow^{I}_{5,\sigma} \partial S_N\} \cap \{v \text{ is black}\}.$$

Note that $\mathbb{P}_{1/2}(A_v) = \frac{1}{2}\mathbb{P}(v \rightsquigarrow^I_{5,\sigma} \partial S_N)$ since the existence of the arms is independent of the status of v, so that $\mathbb{P}_{1/2}(A_v) \asymp \mathbb{P}_{1/2}\big(0 \rightsquigarrow_{5,\sigma} \partial S_N\big)$. We claim that A_v can occur for at most one site v. Indeed, assume that A_v and A_w occur, and denote by $r_1, \dots, r_5$ and $r'_1, \dots, r'_5$ the corresponding arms. Since $r_1 \cup r_4 \cup \{v\}$ separates I_3 from I_5, necessarily $w \in r_1 \cup r_4 \cup \{v\}$. Similarly, $w \in r_2 \cup r_4 \cup \{v\}$: since $r_1 \cap r_2 = \varnothing$, we get that $w \in r_4 \cup \{v\}$. But only one arm can "go through" $r_3 \cup r_5$: the arm $r'_1 \cup \{w\}$ from w to I_1 has to contain v, and so does $r'_2 \cup \{w\}$. Since $r'_1 \cap r'_2 = \varnothing$, we get finally $v = w$.

Consequently,

$$1 \geq \mathbb{P}_{1/2}\big(\cup_{v \in S_{N/2}} A_v\big) = \sum_{v \in S_{N/2}} \mathbb{P}_{1/2}(A_v) \asymp N^2 \mathbb{P}_{1/2}\big(0 \rightsquigarrow_{5,\sigma} \partial S_N\big), \tag{1.9}$$

which provides the upper bound.

Let us turn to the lower bound. We perform a construction showing that a 5-arm site appears with positive probability, by using multiple applications of RSW. With probability at least $\delta_{16}^2 > 0$, there is a black horizontal crossing in the strip $[-N, N] \times [0, N/8]$, together with a white one in $[-N, N] \times [-N/8, 0]$. Assume this is the case, and condition on the lowest black left-right crossing c. We note that any site on this crossing has already 3 arms, 2 black arms and a white one. On the other hand, the percolation in the region above it remains unbiased.

Now, still using RSW, with positive probability c is connected to the top side by a black path included in $[-N/8, 0] \times [-N, N]$, and another white path included in $[0, N/8] \times [-N, N]$. Assume these paths exist, and denote by v_1 and v_2 the respective sites on c where they arrive. Follow c from left to right, and consider the last vertex v before v_2 that is connected to the top side: it is not hard to see that there is a white arm from v to the top side, and that $v \in S_{N/2}$, since v is between v_1 and v_2. Hence,

$$\mathbb{P}_{1/2}\big(\cup_{v \in S_{N/2}} \{v \rightsquigarrow_{5,\sigma} \partial S_N\}\big) \geq C \tag{1.10}$$

for some universal constant $C > 0$. Since we also have

$$\begin{aligned}\mathbb{P}_{1/2}\big(\cup_{v \in S_{N/2}} \{v \rightsquigarrow_{5,\sigma} \partial S_N\}\big) &\leq \sum_{v \in S_{N/2}} \mathbb{P}_{1/2}\big(v \rightsquigarrow_{5,\sigma} \partial S_N\big) \\ &\leq C' N^2 \mathbb{P}_{1/2}\big(0 \rightsquigarrow_{5,\sigma} \partial S_N\big),\end{aligned}$$

the desired lower bound follows.

We now explain briefly how to obtain the two half-plane exponents (items 1. and 2.). We again use the arm separation theorem, but note that [78] contains elementary proofs for them too. For the 2-arm exponent in the half-plane, we take $\sigma = BW$ and remark that if we fix two landing areas I_1 and I_2 on $\partial S'_N$, at most one site on the segment $[-N/2, N/2] \times \{0\}$ is connected by two arms to I_1 and I_2. On the other hand, a 2-arm site can be constructed by considering a black path from $[-N/2, 0] \times \{0\}$ to I_1 and a white path from $[0, N/2] \times \{0\}$ to I_2. Then the right-most site on $[-N/2, N/2] \times \{0\}$ connected by a black arm to I_1 is a 2-arm site. Several applications of RSW allow to conclude.

For the 3-arm exponent, we take three landing areas I_1, I_2 and I_3, and $\sigma = BWB$. It is not hard to construct a 3-arm site by taking a black crossing from I_1 to I_3 and considering the closest to I_2. We can then force it to be in $S_{N/2} \cap \mathbb{H}$ by a RSW construction. For the upper bound, we first notice that if we require the arms to stay strictly positive, the probability remains of the same order of magnitude. We then use that at most one site in $S_{N/2} \cap \mathbb{H}$ is connected to the landing areas by three positive arms. $\square$

The proofs given here only require RSW-type considerations (including separation of arms). As a consequence, they also apply to near-critical percolation. It is clear for $\mathbb{P}_p$, on scales $N \leq L(p)$, but a priori only for the color sequences we have used in the proofs (resp. $\sigma = BW$, BWB and $BWBBW$ – and of course those we can deduce from them by the symmetry $p \leftrightarrow 1-p$): it is indeed not obvious that $\mathbb{P}_p\big(0 \rightsquigarrow_{5,\sigma} \partial S_N\big) \asymp \mathbb{P}_p\big(0 \rightsquigarrow_{5,\sigma'} \partial S_N\big)$ for two distinct non-constant σ and σ'. This is essentially Theorem 27, its proof occupies a large part of the next section.

For a general measure $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, we similarly have to be careful: we do not know whether $\hat{\mathbb{P}}(v \rightsquigarrow_{5,\sigma} \partial S_N)$ remains of the same order of magnitude when v varies. This also comes from Theorem 27, but in the course of its proof we will need an a-priori estimate on the probability of 5 arms, so temporarily we will be content with a weaker statement that does not use its conclusion:

Lemma 25. *For $\sigma = BWBBW$ $(= \sigma_5)$, we have uniformly in p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$ and $N \leq L(p)$:*

$$\sum_{v \in S_{N/2}} \hat{\mathbb{P}}\big(v \rightsquigarrow_{5,\sigma} \partial S_N\big) \asymp 1. \tag{1.11}$$

Remark 26. *We would like to mention that these estimates for critical and near-critical percolation remain valid on other lattices too, like the square lattice (see the discussion in the last section) – at least for the color sequences that we have used in the proofs, no analog of the color exchange trick being available (to our knowledge).*

1.6 Arm events near criticality

1.6.1 Statement of the theorem

We would like now to study how the events $A_{j,\sigma}(n,N)$ are affected by a variation of the parameter p. We have defined $L(p)$ in terms of crossing events to be the scale on which percolation can be considered as (approximately) critical, we would thus expect the probabilities of these events not to vary too much if n, N remain below $L(p)$. This is what happens:

Theorem 27. *Let $j \geq 1$, $\sigma \in \tilde{\mathfrak{S}}_j$ be as usual. Then we have*

$$\hat{\mathbb{P}}\big(A_{j,\sigma}(n,N)\big) \asymp \hat{\mathbb{P}}'\big(A_{j,\sigma}(n,N)\big) \tag{1.1}$$

uniformly in p, $\hat{\mathbb{P}}$ and $\hat{\mathbb{P}}'$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, and $n_0(j) \leq n \leq N \leq L(p)$.

Note that if we take in particular $\hat{\mathbb{P}}' = \mathbb{P}_{1/2}$, we get that below the scale $L(p)$, the arm events remain roughly the same as at criticality:

$$\hat{\mathbb{P}}\big(A_{j,\sigma}(n,N)\big) \asymp \mathbb{P}_{1/2}\big(A_{j,\sigma}(n,N)\big).$$

This will be important to derive the critical exponents for the characteristic functions from the arm exponents at criticality.

Remark 28. *Note that the property of exponential decay with respect to $L(p)$ (Lemma 38), proved in Section 1.7.4, shows that we cannot hope for a similar result on a much larger range, so that $L(p)$ is the appropriate scale here: consider for instance $\mathbb{P}_p$ with $p > 1/2$, the probability to observe a white arm tends to 0 exponentially fast (and thus much faster than at the critical point), while the probability to observe a certain number of disjoint black arms tends to a positive constant.*

1.6.2 Proof of the theorem

We want to compare the value of $\hat{\mathbb{P}}(A_{j,\sigma}(n,N))$ for different measures $\hat{\mathbb{P}}$. A natural way of doing this is to go from one to the other by using Russo's formula (Theorem 1). But since for $j \geq 2$ and non-constant σ, the event $A_{j,\sigma}(n,N)$ is not monotone, we need a slight generalization of this formula, for events that can be expressed as the intersection of two monotone events, one increasing and one decreasing. We also allow the parameters p_v to be differentiable functions of $t \in [0,1]$.

Lemma 29. *Let A^+ and A^- be two monotone events, respectively increasing and decreasing, depending only on the sites contained in some finite set of vertices S. Let $(\hat{p}_v)_{v\in S}$ be a family of differentiable functions $\hat{p}_v : t \in [0,1] \mapsto \hat{p}_v(t) \in [0,1]$, and denote by $(\hat{\mathbb{P}}_t)_{0\leq t\leq 1}$ the associated product measures. Then*

$$\begin{aligned}\frac{d}{dt}&\hat{\mathbb{P}}_t(A^+ \cap A^-)\\ &= \sum_{v\in S} \frac{d}{dt}\hat{p}_v(t)\Big[\hat{\mathbb{P}}_t(v \textit{ is pivotal for } A^+ \textit{ but not for } A^-, \textit{ and } A^- \textit{ occurs})\\ &\qquad - \hat{\mathbb{P}}_t(v \textit{ is pivotal for } A^- \textit{ but not for } A^+, \textit{ and } A^+ \textit{ occurs})\Big].\end{aligned}$$

Proof. We adapt the proof of standard Russo's formula. We use the same function $\mathcal{P}$ of the parameters $(\hat{p}_v)_{v\in S}$, and we note that for a small variation $\epsilon > 0$ in w,

$$\begin{aligned}\hat{\mathbb{P}}^{+\epsilon}&(A^+ \cap A^-) - \hat{\mathbb{P}}(A^+ \cap A^-)\\ &= \epsilon \times \hat{\mathbb{P}}(w \text{ is pivotal for } A^+ \text{ but not for } A^-, \text{ and } A^- \text{ occurs})\\ &\quad - \epsilon \times \hat{\mathbb{P}}(w \text{ is pivotal for } A^- \text{ but not for } A^+, \text{ and } A^+ \text{ occurs}).\end{aligned}$$

Now, it suffices to compute the derivative of the function $t \mapsto \hat{\mathbb{P}}_t(A^+ \cap A^-)$ by writing it as the composition of $t \mapsto (\hat{p}_v(t))$ and $(\hat{p}_v)_{v\in S} \mapsto \hat{\mathbb{P}}(A)$. □

Proof of the theorem. We now turn to the proof itself. It is divided into three main steps.

1. First simplifications

Note first that by quasi-multiplicativity, we can restrict ourselves to $n = n_0(j)$. It also suffices to prove the result for some fixed $\hat{\mathbb{P}}'$, with $\hat{\mathbb{P}}$ varying: we thus assume that $p < 1/2$, and take $\hat{\mathbb{P}}' = \mathbb{P}_p$. Denoting by $\hat{p}_v$ the parameters of $\hat{\mathbb{P}}$, we have by hypothesis $\hat{p}_v \geq p$ for each site v. For technical reasons, we suppose that the sizes of annuli are powers of two: take k_0, K such that $2^{k_0-1} < n_0 \leq 2^{k_0}$ and $2^K \leq N < 2^{K+1}$, then

$$\mathbb{P}_p(A_{j,\sigma}(n_0,N)) \asymp \mathbb{P}_p(A_{j,\sigma}(2^{k_0},2^K))$$

and the same is true for $\hat{\mathbb{P}}$.

To estimate the change in probability when p is replaced by $\hat{p}_v$, we will use the observation that the pivotal sites give rise to 4 alternating arms locally (see Figure 1.11). However, this does not work so nicely for the sites v which are close to $\partial S_{2^{k_0}}$ or ∂S_{2^K}, so for the sake of simplicity we treat apart these sites. We perform the change $p \rightsquigarrow \hat{p}_v$ in $S_{2^{k_0},2^K} \setminus S_{2^{k_0+3},2^{K-3}}$. Note that the intermediate measure $\tilde{\mathbb{P}}$ so obtained is between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, and that $\tilde{\mathbb{P}}(A_{j,\sigma}(2^{k_0+3},2^{K-3})) = \mathbb{P}_p(A_{j,\sigma}(2^{k_0+3},2^{K-3}))$. We have

$$\tilde{\mathbb{P}}(A_{j,\sigma}(2^{k_0},2^K)) \asymp \tilde{\mathbb{P}}(A_{j,\sigma}(2^{k_0+3},2^{K-3})) \tag{1.2}$$

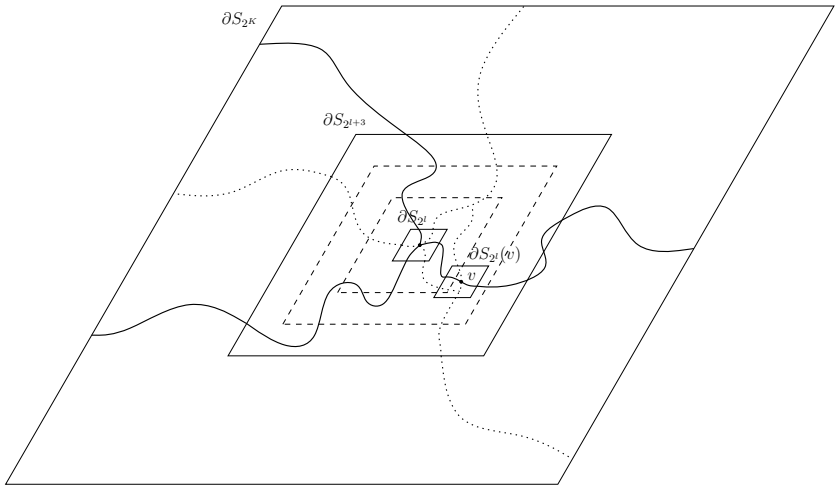


Figure 1.11: If v is pivotal, 4 alternating arms arise locally.

and also

$$\mathbb{P}_p(A_{j,\sigma}(2^{k_0}, 2^K)) \asymp \mathbb{P}_p(A_{j,\sigma}(2^{k_0+3}, 2^{K-3})), \tag{1.3}$$

which shows that it would be enough to prove the result with $\tilde{\mathbb{P}}$ instead of $\mathbb{P}_p$.

2. Make appear the logarithmic derivative of the probability by applying Russo's formula

The event $A_{j,\sigma}(2^{k_0}, 2^K)$ cannot be directly written as an intersection like in Russo's formula, since the order of the different arms is prescribed. To fix this difficulty, we impose the landing areas of the different arms on ∂S_{2^K}, *ie* we fix some landing sequence $I' = I'_1, \dots, I'_j$ and we consider the event $\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0}, 2^K)$. Since we know that

$$\tilde{\mathbb{P}}\big(A_{j,\sigma}(2^{k_0}, 2^K)\big) \asymp \tilde{\mathbb{P}}\big(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0}, 2^K)\big), \tag{1.4}$$

and also with $\hat{\mathbb{P}}$ instead of $\tilde{\mathbb{P}}$, it is enough to prove the result for this particular landing sequence.

We study successively three cases. We begin with the case of one arm, which is slightly more direct than the two next ones – however, only small adaptations are needed. We then consider the special case where j is even and σ alternating: due to the fact that any arm is surrounded by two arms of opposite color, the local four arms are always of the right length. We finally prove the result for any j and any σ: a technical complication arises in this case, for which the notion of "arms with defects" is needed.

Case 1: $j = 1$

We consider first the case of one arm, and assume for instance $\sigma = B$. We introduce the family of measures $(\tilde{\mathbb{P}}_t)_{t\in[0,1]}$ with parameters

$$\tilde{p}_v(t) = t\hat{p}_v + (1-t)p \tag{1.5}$$

in $S_{2^{k_0+3},2^{K-3}}$, corresponding to a linear interpolation between p and $\hat{p}_v$. For future use, note that $\tilde{\mathbb{P}}_t$ is between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$ for any $t \in [0,1]$. We have $\frac{d}{dt}\tilde{p}_v(t) = \hat{p}_v - p$ if $v \in S_{2^{k_0+3},2^{K-3}}$ (and

0 otherwise...), generalized Russo's formula (with just an increasing event – take for instance $A^- = \Omega$) thus gives:

$$\frac{d}{dt}\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)\big) = \sum_{v\in S_{2^{k_0}+3,2^K-3}} (\hat{p}_v - p)\tilde{\mathbb{P}}_t(v \text{ is pivotal for } \bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)).$$

The key remark is that the summand can be expressed in terms of arm events: for probabilities, being pivotal is approximately the same as having a black arm, and four arms locally around v. Indeed, $\big[v$ is pivotal for $\bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)\ \big]$ *iff*

(1) There exists an arm r_1 from $\partial S_{2^{k_0}}$ to I'_1, with $v \in r_1$; r_1 is black, with a possible exception in v ($\bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)$ *occurs when v is black*).

(2) There exists a path c_1 passing through v and separating $\partial S_{2^{k_0}}$ from I'_1 (c_1 may be either a circuit around $\partial S_{2^{k_0}}$ or a path with extremities on ∂S_{2^K}); c_1 is white, except possibly in v (*there is no black arm from $\partial S_{2^{k_0}}$ to I'_1 when v is white*).

Put now a rhombus $R(v)$ around v: if it does not contain 0, then v is connected to $\partial R(v)$ by 4 arms of alternating colors. Indeed, r_1 provides two black arms, and c_1 two white arms.

Look at the pieces of the black arm outside of $R(v)$: if $R(v)$ is not too large, we can expect them to be sufficiently large to enable us to reconstitute the whole arm. We would like that the two white arms are a good approximation of the whole circuit too. We thus take $R(v)$ of size comparable to the distance $d(0,v)$: if $2^{l+1} < \|v\|_\infty \le 2^{l+2}$, we take $R(v) = S_{2^l}(v)$. It is not hard to check that $R(v) \subseteq S_{2^l,2^{l+3}}$ for this particular choice of $R(v)$ (see Figure 1.11), so that for any $t \in [0,1]$,

$$\begin{aligned}
&\tilde{\mathbb{P}}_t(v \text{ is pivotal for } \bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)) \\
&\qquad \le \tilde{\mathbb{P}}_t\big(\{\partial S_{2^{k_0}} \rightsquigarrow \partial S_{2^l}\} \cap \{\partial S_{2^{l+3}} \rightsquigarrow \partial S_{2^K}\} \cap \{v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\}\big) \\
&\qquad = \tilde{\mathbb{P}}_t\big(\partial S_{2^{k_0}} \rightsquigarrow \partial S_{2^l}\big)\tilde{\mathbb{P}}_t\big(\partial S_{2^{l+3}} \rightsquigarrow \partial S_{2^K}\big)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big)
\end{aligned}$$

by independence of the three events, since they are defined in terms of sites in disjoint sets (recall that $\sigma_4 = BWBW$). We can then make appear the original event by joining together the two first terms, using quasi-multiplicativity and extendability[4]:

$$\tilde{\mathbb{P}}_t\big(\partial S_{2^{k_0}} \rightsquigarrow \partial S_{2^l}\big)\tilde{\mathbb{P}}_t\big(\partial S_{2^{l+3}} \rightsquigarrow \partial S_{2^K}\big) \le C_2\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)\big) \tag{1.6}$$

for some C_2 universal. Hence[5],

$$\tilde{\mathbb{P}}_t(v \text{ is pivotal for } \bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)) \le C_2\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)\big)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big). \tag{1.7}$$

We thus get

$$\begin{aligned}
&\frac{d}{dt}\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)\big) \\
&\qquad \le C_2 \sum_{v\in S_{2^{k_0}+3,2^K-3}} (\hat{p}_v - p)\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)\big)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big).
\end{aligned}$$

[4]Note that in the case of one arm, the extendability property, as well as the quasi-multiplicativity, are direct consequences of RSW and do not require the separation lemmas.

[5]As we will see in the next sub-section (Proposition 31), the converse bound also holds: the estimate obtained gives the exact order of magnitude for the summand.

Now dividing by $\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)\big)$, we make appear its logarithmic derivative in the left-hand side,

$$\frac{d}{dt}\log\big[\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{1,\sigma}(2^{k_0},2^K)\big)\big] \le C_2 \sum_{v\in S_{2^{k_0+3},2^{K-3}}} (\hat{p}_v - p)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big), \tag{1.8}$$

it thus suffices to show that for some C_3 universal,

$$\int_0^1 \sum_{v\in S_{2^{k_0+3},2^{K-3}}} (\hat{p}_v - p)\,\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big)dt \le C_3. \tag{1.9}$$

We will prove it in the next step, but before that, we turn to the two other cases: even if the computations need to be modified, it is still possible to reduce the proof to this inequality.

Case 2: j even and σ alternating

In this case,

$$\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0},2^K) = A^+ \cap A^- \tag{1.10}$$

with $A^+ = A^+(2^{k_0},2^K) = \{$There exist $j/2$ disjoint black arms $r_1 : \partial S_{2^{k_0}} \rightsquigarrow I'_1$, $r_3 : \partial S_{2^{k_0}} \rightsquigarrow I'_3 \ldots\}$ and $A^- = A^-(2^{k_0},2^K) = \{$There exist $j/2$ disjoint white arms $r_2 : \partial S_{2^{k_0}} \rightsquigarrow^* I'_2$, $r_4 : \partial S_{2^{k_0}} \rightsquigarrow^* I'_4 \ldots\}$.

We then perform the change $p \rightsquigarrow \hat{p}_v$ in $S_{2^{k_0+3},2^{K-3}}$ linearly as before (Eq.(1.5)), which gives rise to the family of measures $(\tilde{\mathbb{P}}_t)_{t\in[0,1]}$, and generalized Russo's formula reads

$$\begin{aligned}
&\frac{d}{dt}\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0},2^K)\big)\\
&\quad= \sum_{v\in S_{2^{k_0+3},2^{K-3}}} (\hat{p}_v - p)\Big[\tilde{\mathbb{P}}_t(v \text{ is pivotal for } A^+ \text{ but not for } A^-, \text{ and } A^- \text{ occurs})\\
&\qquad\qquad - \tilde{\mathbb{P}}_t(v \text{ is pivotal for } A^- \text{ but not for } A^+, \text{ and } A^+ \text{ occurs})\Big].
\end{aligned}$$

We note that $\big[v$ is pivotal for $A^+(2^{k_0},2^K)$ but not $A^-(2^{k_0},2^K)$, and $A^-(2^{k_0},2^K)$ occurs $\big]$ *iff* for some $i' \in \{1,3\ldots,j-1\}$,

(1) There exist j disjoint monochromatic arms $r_1,\ldots,r_j$ from $\partial S_{2^{k_0}}$ to $I'_1,\ldots,I'_j$, with $v \in r_{i'}$; $r_2,r_4,\ldots$ are white, and $r_1,r_3,\ldots$ are black, with a possible exception for $r_{i'}$ in v (*the event* $\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0},2^K)$ *is satisfied when* v *is black*).

(2) There exists a path $c_{i'}$ separating $\partial S_{2^{k_0}}$ from $I'_{i'}$; this path is white, except possibly in v ($\partial S_{2^{k_0}}$ *and* $I'_{i'}$ *are separated when* v *is white*).

If we take the same rhombus $R(v) \subseteq S_{2^l,2^{l+3}}$ around v, then v is still connected to $\partial R(v)$ by 4 arms of alternating colors. Indeed, $r_{i'}$ provides two black arms, and $c_{i'}$ (which can contain parts of $r_{i'-1}$ or $r_{i'+1}$ – see Figure 1.11) provides the two white arms.

Hence for any $t \in [0,1]$,

$$\begin{aligned}
&\tilde{\mathbb{P}}_t(v \text{ is pivotal for } A^+ \text{ but not } A^-, \text{ and } A^- \text{ occurs})\\
&\qquad\le \tilde{\mathbb{P}}_t\big(A_{j,\sigma}(2^{k_0},2^l) \cap \bar{A}^{\cdot/I'}_{j,\sigma}(2^{l+3},2^K) \cap \{v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\}\big)\\
&\qquad= \tilde{\mathbb{P}}_t\big(A_{j,\sigma}(2^{k_0},2^l)\big)\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{l+3},2^K)\big)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big)
\end{aligned}$$

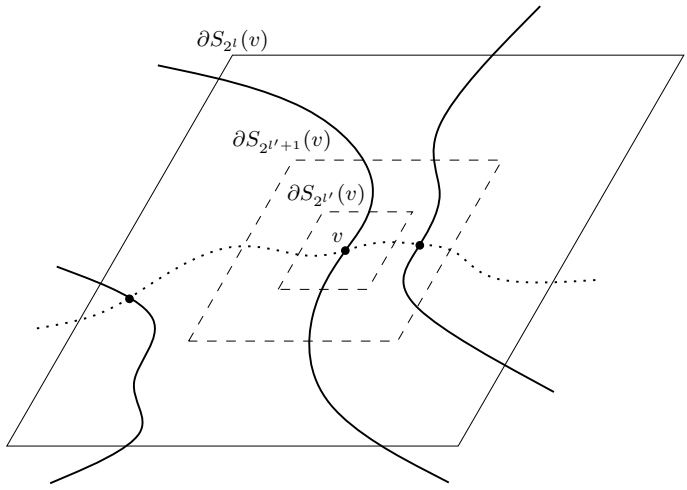


Figure 1.12: More complex events may arise when σ is not alternating.

by independence of the three events. We join together the two first terms using extendability and quasi-multiplicativity:

$$\tilde{\mathbb{P}}_t\big(A_{j,\sigma}(2^{k_0},2^l)\big)\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{l+3},2^K)\big) \le C_1\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0},2^K)\big) \tag{1.11}$$

for some C_2 universal. We thus obtain

$$\begin{aligned}\tilde{\mathbb{P}}_t(&v \text{ is pivotal for } A^+ \text{ but not } A^-, \text{ and } A^- \text{ occurs})\\ &\le C_1\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0},2^K)\big)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big).\end{aligned}$$

If we then do the same manipulation on the second term of the sum, we get

$$\begin{aligned}&\left|\frac{d}{dt}\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0},2^K)\big)\right|\\ &\qquad\le 2C_1\sum_{v\in S_{2^{k_0+3},2^{K-3}}}(\hat{p}_v-p)\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0},2^K)\big)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big),\end{aligned}$$

and if we divide by $\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0},2^K)\big)$,

$$\left|\frac{d}{dt}\log\big[\tilde{\mathbb{P}}_t\big(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0},2^K)\big)\big]\right| \le 2C_1\sum_{v\in S_{2^{k_0+3},2^{K-3}}}(\hat{p}_v-p)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big). \tag{1.12}$$

As promised, we have thus reduced this case to Eq.(1.9).

Case 3: Any j, σ

In the general case, a minor complication may arise, coming from consecutive arms of the same color: indeed, the property of being pivotal for a site v does not always give rise to four arms in $R(v)$, but to some more complex event $E(v)$ (see Figure 1.12). If v is on r_i, and this arm is black for instance, there are still two black arms coming from r_i, but the two white arms do not necessarily reach $\partial R(v)$, since they can encounter neighboring black arms.

We first introduce an event for which the property of being pivotal is easier to formulate. We group consecutive arms of the same color in "packs": if $(r_{i_q}, r_{i_q+1}, \dots, r_{i_q+l_q-1})$ is such a sequence of arms, say black, we take an interval $\tilde{I}_q$ covering all the I_i for $i_q \le i \le i_q + l_q - 1$ and replace the condition "$r_i \rightsquigarrow I_i$ for all $i_q \le i \le i_q + l_q - 1$" by "$r_i \rightsquigarrow \tilde{I}_q$ for all $i_q \le i \le i_q + l_q - 1$". We construct in this way an event $\tilde{A} = \tilde{A}^+ \cap \tilde{A}^-$: since it is intermediate between $\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0}, 2^K)$ and $A_{j,\sigma}(2^{k_0}, 2^K)$, we have

$$\tilde{\mathbb{P}}_t(\tilde{A}) \asymp \tilde{\mathbb{P}}_t(\bar{A}^{\cdot/I'}_{j,\sigma}(2^{k_0}, 2^K)).$$

This new definition allows to use Menger's theorem (see [30], Theorem 3.3.1): $\big[v$ is pivotal for $\tilde{A}^+$ but not $\tilde{A}^-$, and $\tilde{A}^-$ occurs $\big]$ *iff* for some arm $r_{i'}$ in a black pack $(r_{i_q}, r_{i_q+1}, \dots, r_{i_q+l_q-1})$,

(1) There exist j disjoint monochromatic arms $r_1, \dots, r_j$ from $\partial S_{2^{k_0}}$ to the $\tilde{I}_q$ (an appropriate number of arms for each of these intervals), with $v \in r_{i'}$; all of these arms are of the prescribed color, with a possible exception for $r_{i'}$ in v ($\tilde{A}$ *occurs when* v *is black*).

(2) There exists a path $c_{i'}$ separating $\partial S_{2^{k_0}}$ from $\tilde{I}_q$; this path is white, except in (at most) $l_q - 1$ sites, and also possibly in v ($\partial S_{2^{k_0}}$ *and* $\tilde{I}_q$ *are separated when* v *is white*).

Now we take the same rhombus $R(v) \subseteq S_{2^l, 2^{l+3}}$ around v: if there are four arms $v \rightsquigarrow_{4,\sigma_4} \partial S_{2^{l-1}}(v)$, we are OK. Otherwise, if l', $1 \le l' \le l-2$, is such that the defect on $c_{i'}$ closest to v is in $S_{2^{l'+1}}(v) \setminus S_{2^{l'}}(v)$, then there are 4 alternating arms $v \rightsquigarrow_{4,\sigma_4} \partial S_{2^{l'}}(v)$, and also 6 arms $\partial S_{2^{l'+1}}(v) \rightsquigarrow^{(j)}_{6,\sigma'_6} \partial S_{2^l}(v)$ having at most j defects, with the notation $\sigma'_6 = BBWBBW$. We denote by $E(v)$ the corresponding event: $E(v) := \{$There exists $l' \in \{1, \dots, l-2\}$ such that $v \rightsquigarrow_{4,\sigma_4} \partial S_{2^{l'}}(v)$ and $\partial S_{2^{l'+1}}(v) \rightsquigarrow^{(j)}_{6,\sigma'_6} \partial S_{2^l}(v)\} \cup \{v \rightsquigarrow_{4,\sigma_4} \partial S_{2^{l-1}}(v)\}$.

For the 6 arms with defects, Proposition 18 applies and the probability remains roughly the same, with just an extra logarithmic correction:

$$\begin{aligned}
&\tilde{\mathbb{P}}_t\big(\partial S_{2^{l'+1}}(v) \rightsquigarrow^{(j)}_{6,\sigma'_6} \partial S_{2^l}(v)\big) \\
&\quad \le C_1 (l-l')^j \tilde{\mathbb{P}}_t\big(\partial S_{2^{l'+1}}(v) \rightsquigarrow_{6,\sigma'_6} \partial S_{2^l}(v)\big) \\
&\quad \le C_1 (l-l')^j \tilde{\mathbb{P}}_t\big(\partial S_{2^{l'+1}}(v) \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big) \tilde{\mathbb{P}}_t\big(\partial S_{2^{l'+1}}(v) \rightsquigarrow_{2,BB} \partial S_{2^l}(v)\big) \\
&\quad \le C_2 (l-l')^j \tilde{\mathbb{P}}_t\big(\partial S_{2^{l'+1}}(v) \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big) 2^{-\alpha'(l-l')}
\end{aligned}$$

using Reimer's inequality (its consequence Eq.(1.2)) and the a-priori bound for one arm (Eq.(1.10)).

It implies that

$$\tilde{\mathbb{P}}_t\big(E(v)\big) \le C_5 \tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big) \tag{1.13}$$

for some universal constant C_5: indeed, by quasi-multiplicativity,

$$\begin{aligned}
&\sum_{l'=1}^{l-2} \tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^{l'}}(v)\big) \tilde{\mathbb{P}}_t\big(\partial S_{2^{l'+1}}(v) \rightsquigarrow^{(j)}_{6,\sigma'_6} \partial S_{2^l}(v)\big) \\
&\quad \le C_2 \sum_{l'=1}^{l-2} \tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^{l'}}(v)\big) \tilde{\mathbb{P}}_t\big(\partial S_{2^{l'+1}}(v) \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big) (l-l')^j 2^{-\alpha'(l-l')} \\
&\quad \le C_3 \tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big) \sum_{l'=1}^{l-2} (l-l')^j 2^{-\alpha'(l-l')} \\
&\quad \le C_4 \tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big),
\end{aligned}$$

since $\sum_{l'=1}^{l-2}(l-l')^j 2^{-\alpha'(l-l')} \le \sum_{r=1}^{\infty} r^j 2^{-\alpha' r} < \infty$.

The reasoning is then identical:

$$\begin{aligned}\tilde{\mathbb{P}}_t(v \text{ is pivotal for } \tilde{A}^+ \text{ but not } \tilde{A}^-, \text{ and } \tilde{A}^- \text{ occurs})\\ &\le \tilde{\mathbb{P}}_t\big(A_{j,\sigma}(2^{k_0},2^l)\big)\tilde{\mathbb{P}}_t\big(A_{j,\sigma}(2^{l+3},2^K)\big)\tilde{\mathbb{P}}_t\big(E(v)\big)\\ &\le C_6\tilde{\mathbb{P}}_t\big(A_{j,\sigma}(2^{k_0},2^K)\big)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big),\end{aligned}$$

and using $\tilde{\mathbb{P}}_t\big(A_{j,\sigma}(2^{k_0},2^K)\big) \le C_7\tilde{\mathbb{P}}_t(\tilde{A})$, we get

$$\left|\frac{d}{dt}\log\left[\tilde{\mathbb{P}}_t(\tilde{A})\right]\right| \le C_8 \sum_{v\in S_{2^{k_0+3},2^{K-3}}} (\hat{p}_v - p)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big). \tag{1.14}$$

Once again, Eq.(1.9) would be sufficient.

3. Final summation

We now prove Eq.(1.9), *ie* that for some universal constant C_1,

$$\int_0^1 \sum_{v\in S_{2^{k_0+3},2^{K-3}}} (\hat{p}_v - p)\,\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big)dt \le C_1. \tag{1.15}$$

Recall that Russo's formula allows to count 4-arm sites: for any N and any measure $\bar{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$,

$$\int_0^1 \sum_{v\in S_N} (\bar{p}_v - p)\,\bar{\mathbb{P}}_t\big(v \rightsquigarrow^{/\bar{I}}_{4,\sigma_4} \partial S_N\big)dt = \bar{\mathbb{P}}(\mathcal{C}_H(S_N)) - \mathbb{P}_p(\mathcal{C}_H(S_N)) \le 1 \tag{1.16}$$

(we remind that $\bar{I}$ consists of the different sides of ∂S_N). This is essentially the only relation we have at our disposal, the end of the proof consists in using it in a clever way.

Roughly speaking, when applied to $N = L(p)$, this relation gives that $(p-1/2)N^2\pi_4(N) \le 1$, since all the sites give a contribution of order

$$\bar{\mathbb{P}}_t\big(0 \rightsquigarrow_{4,\sigma_4} \partial S_{N/2}\big) \asymp \pi_4(N). \tag{1.17}$$

This corresponds more or less to the sites in the "external annulus" in Eq.(1.15). Now each time we get from an annulus to the next inside it, the probability to have 4 arms is multiplied by $2^{\alpha_4} \approx 2^{5/4}$, while the number of sites is divided by 4, so that things decay exponentially fast, and the sum of Eq.(1.15) is bounded by something like

$$\sum_{j=3}^{K-k_0-4} (2^{5/4-2})^{K-j} \le \sum_{q=0}^{\infty}(2^{-3/4})^q < \infty.$$

We have to be more cautious, in particular Eq.(1.17) does not trivially hold, since we do not know at this point that the probability of having 4 arms remains of the same order on a scale $L(p)$, and the estimate for 4 arms only gives a logarithmic equivalence. The a-priori estimate coming from the 5-arm exponent will allow us to circumvent these difficulties. We also need to take care of the boundary effects.

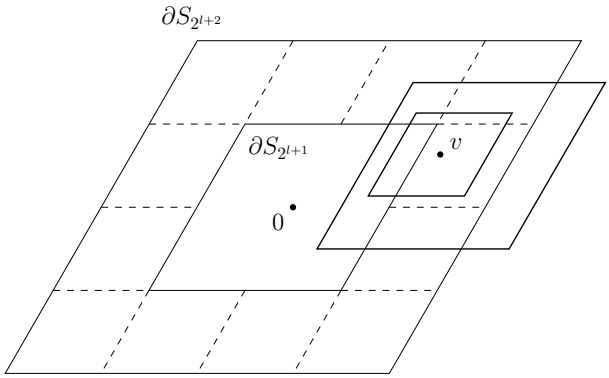


Figure 1.13: We replace $R(v) = S_{2^l}(v)$ by one of the $R'^i_{2^{l+1}}$ $(i = 1, \dots, 12)$.

Assume that $v \in S_{2^{l+1},2^{l+2}}$ as before. We subdivide this annulus into 12 sub-boxes of size 2^{l+1} (see figure) $R^i_{2^{l+1}}$ $(i = 1, \dots, 12)$. We then associate to each of these boxes a slightly enlarged box $R'^i_{2^{l+1}}$, of size 2^{l+2}. At least one of these boxes contains v: we denote it by $R'(v)$. Since

$$\{v \rightsquigarrow_{4,\sigma_4} \partial S_{2^{l+1}}(v)\} \subseteq \{v \rightsquigarrow_{4,\sigma_4} \partial R'(v)\} \subseteq \{v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\},$$

we have

$$\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^l}(v)\big) \asymp \tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial R'(v)\big).$$

We thus have to find an upper bound for

$$\sum_{j=k_0+3}^{K-4} \sum_{i=1}^{12} \int_0^1 \sum_{v \in R'^i_{2^j}} (\hat{p}_v - p)\, \tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial R'^i_{2^j}\big) dt. \tag{1.18}$$

For that purpose, we will prove that for $i = 1, \dots, 12$,

$$S^{i,(4)}_j := \sum_{v \in R'^i_{2^j}} (\hat{p}_v - p)\, \tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial R'^i_{2^j}\big)$$

indeed decays fast when, starting from $j = K - 4$, we make j decrease. For that, we duplicate the parameters in the box $R'^i_{2^j}$ periodically inside $S_{2^{K-3}}$: this gives rise to a new measure $\bar{\mathbb{P}}$ inside S_{2^K} (to completely define it, simply take $\bar{p}_v = p$ outside of $S_{2^{K-3}}$). This measure contains $2^{2(K-j-3)}$ copies (R'') of the original box, and we know that

$$\int_0^1 \sum_{v \in S_{2^{K-3}}} (\bar{p}_v - p)\, \bar{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^K}\big) dt \le 1. \tag{1.19}$$

We have

$$\begin{aligned}
&\sum_{v \in S_{2^{K-3}}} (\bar{p}_v - p) \bar{\mathbb{P}}_t(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^K}) \\
&\qquad \asymp \sum_{v \in S_{2^{K-3}}} (\bar{p}_v - p) \bar{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial R'(v)\big) \bar{\mathbb{P}}_t\big(\partial R'(v) \rightsquigarrow_{4,\sigma_4} \partial S_{2^K}\big) \\
&\qquad \asymp \Big(\sum_{R''} \bar{\mathbb{P}}_t\big(\partial R'' \rightsquigarrow_{4,\sigma_4} \partial S_{2^K}\big) \Big) S^{i,(4)}_j.
\end{aligned}$$

Hence, using Reimer's inequality and the a-priori bound for one arm,

$$\begin{aligned}\sum_{v\in S_{2^{K-3}}} &(\bar{p}_v - p)\bar{\mathbb{P}}_t(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^K}) \\ &\geq C_1\Big(\sum_{R''} \bar{\mathbb{P}}_t\big(\partial R'' \rightsquigarrow_{5,\sigma_5} \partial S_{2^K}\big)\bar{\mathbb{P}}_t\big(\partial R'' \rightsquigarrow \partial S_{2^K}\big)^{-1}\Big) S_j^{i,(4)} \\ &\geq C_2 2^{\alpha'(K-j)} S_j^{i,(4)}\Big(\sum_{R''} \bar{\mathbb{P}}_t\big(\partial R'' \rightsquigarrow_{5,\sigma_5} \partial S_{2^K}\big)\Big).\end{aligned}$$

The same manipulation for 5 arms gives, with $\tilde{S}_j^{i,(5)} = \sum_{v\in R'^i_{2^j}} \tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{5,\sigma_5} \partial R'^i_{2^j}\big)$,

$$\sum_{v\in S_{2^{K-3}}} \bar{\mathbb{P}}_t(v \rightsquigarrow_{5,\sigma_5} \partial S_{2^K}) \asymp \Big(\sum_{R''} \bar{\mathbb{P}}_t\big(\partial R'' \rightsquigarrow_{5,\sigma_5} \partial S_{2^K}\big)\Big)\tilde{S}_j^{i,(5)}. \tag{1.20}$$

We know from Lemma 25 that $\sum_{v\in S_{2^{K-3}}} \bar{\mathbb{P}}_t(v \rightsquigarrow_{5,\sigma_5} \partial S_{2^K})$ and $\tilde{S}_j^{i,(5)} \asymp 1$, and thus

$$\sum_{R''} \bar{\mathbb{P}}_t\big(\partial R'' \rightsquigarrow_{5,\sigma_5} \partial S_{2^K}\big) \asymp 1. \tag{1.21}$$

This entails that

$$S_j^{i,(4)} \leq C_3 2^{-\alpha'(K-j)} \sum_{v\in S_{2^{K-3}}} (\bar{p}_v - p)\bar{\mathbb{P}}_t(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^K}),$$

and finally, by integrating and using Eq.(1.19),

$$\int_0^1 \sum_{v\in R'^i_{2^j}} (\hat{p}_v - p)\, \tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial R'^i_{2^j}\big)dt \leq C_3 2^{-\alpha'(K-j)}.$$

The sum of Eq.(1.18) is thus less than

$$\sum_{j=k_0+3}^{K-4} 12 C_3 2^{-\alpha'(K-j)} \leq C_4 \sum_{r=0}^{\infty} 2^{-\alpha' r} < \infty,$$

which completes the proof. □

Remark 30. *We will use this theorem in the next section to relate the so-called "characteristic functions" to the arm exponents at criticality. We will have use in fact only for the two cases $j = 1$ and $j = 4$, $\sigma = \sigma_4$: the general case (3rd case in the previous proof) will thus not be needed there. It is however of interest for other applications, for instance to say that for an interface in near-critical percolation, the dimension of the accessible perimeter is the same as at criticality: this requires the case $j = 3$, $\sigma = \sigma_3$.*

1.6.3 Some complements

Theorem for more general annuli

We will sometimes need a version of Theorem 27 with non concentric rhombi:

$$\hat{\mathbb{P}}(\partial r \rightsquigarrow \partial R) \asymp \hat{\mathbb{P}}(\partial S_n \rightsquigarrow \partial S_N). \tag{1.22}$$

for $S_{N/\tau} \subseteq R \subseteq S_N$ and $r \subseteq S_{N/2\tau}$. It results from the remark on more general annuli (Eq.(1.20)) combined with Theorem 27 applied to $\hat{\mathbb{P}}$ and translations of it. In particular, for any fixed $\eta > 0$,

$$\hat{\mathbb{P}}(\partial S_n(v) \rightsquigarrow \partial S_N) \asymp \hat{\mathbb{P}}(\partial S_n \rightsquigarrow \partial S_N) \tag{1.23}$$

uniformly in $v \in S_{(1-\eta)N}$.

A complementary bound

Following the same lines as in the previous proof, we can get a bound in the other direction:

Proposition 31. *There exists some universal constant $\tilde{C} > 1$ such that for all $p > 1/2$,*

$$\mathbb{P}_p\big(0 \rightsquigarrow \partial S_{L(p)}\big) \geq \tilde{C}\, \mathbb{P}_{1/2}\big(0 \rightsquigarrow \partial S_{L(p)}\big). \tag{1.24}$$

In other words, the one-arm probability varies of a non-negligible amount, like the crossing probability: there is a macroscopic difference with the critical regime.

Proof. Take K such that $2^K \leq L(p) < 2^{K+1}$ and $(\hat{\mathbb{P}}_t)$ the linear interpolation between $\mathbb{P}_{1/2}$ and $\mathbb{P}_p$. By gluing arguments, for $A = \{0 \rightsquigarrow \partial S_{L(p)}\}$, for any $v \in S_{2^{K-4},2^{K-3}}$,

$$\begin{aligned}
&\tilde{\mathbb{P}}_t(v \text{ is pivotal for } A) \\
&\qquad \geq C_1 \tilde{\mathbb{P}}_t\big(0 \rightsquigarrow \partial S_{2^{K-5}}\big)\tilde{\mathbb{P}}_t\big(\partial S_{2^{K-2}} \rightsquigarrow \partial S_{L(p)}\big)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^{K-5}}(v)\big) \\
&\qquad \geq C_2 \tilde{\mathbb{P}}_t\big(0 \rightsquigarrow \partial S_{2^K}\big)\tilde{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^{K-5}}(v)\big),
\end{aligned}$$

so that

$$\begin{aligned}
\frac{d}{dt}\log\big[\tilde{\mathbb{P}}_t(A)\big] &\geq \sum_{v \in S_{2^{K-4},2^{K-3}}} (p-1/2)\hat{\mathbb{P}}_t\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{2^{K-5}}(v)\big) \\
&\geq C_3(p-1/2)L(p)^2 \hat{\mathbb{P}}_t\big(0 \rightsquigarrow_{4,\sigma_4} \partial S_{L(p)}\big),
\end{aligned}$$

since each of the sites $v \in S_{2^{K-4},2^{K-3}}$ produces a contribution of order $\hat{\mathbb{P}}_t\big(0 \rightsquigarrow_{4,\sigma_4} \partial S_{L(p)}\big)$. Proposition 33, proved later[6], allows to conclude. □

1.7 Consequences for the characteristic functions

1.7.1 Different characteristic lengths

Roughly speaking, a *characteristic length* is a quantity intended to measure a "typical" scale of the system. There may be several natural definitions of such a length, but we usually expect the different possible definitions to produce lengths that are of the same order of magnitude. For two-dimensional percolation, the three most common definitions are the following:

Finite-size scaling

The lengths L_ϵ that we have used throughout the paper, introduced in [23], are known as "finite-size scaling characteristic lengths":

$$L_\epsilon(p) = \begin{cases} \min\{n \text{ s.t. } \mathbb{P}_p(\mathcal{C}_H([0,n]\times[0,n])) \leq \epsilon\} & \text{when } p < 1/2, \\ \min\{n \text{ s.t. } \mathbb{P}_p(\mathcal{C}^*_H([0,n]\times[0,n])) \leq \epsilon\} & \text{when } p > 1/2. \end{cases} \tag{1.1}$$

[6]This does not raise any problem since we have included this complementary bound only for the sake of completeness, and we will not use it later.

Mean radius of a finite cluster

The (quadratic) mean radius measures the "typical" size of a finite cluster. It can be defined by the formula

$$\xi(p) = \left[\frac{1}{\mathbb{E}_p\big[|C(0)|; |C(0)| < \infty\big]} \sum_x \|x\|_\infty^2 \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big) \right]^{1/2}. \tag{1.2}$$

Connection probabilities

A third possible definition would be via the rate of decay of correlations. Take first $p < 1/2$ for example. For two sites x and y, we consider the connection probability between them

$$\tau_{x,y} := \mathbb{P}_p\big(x \rightsquigarrow y\big), \tag{1.3}$$

and then

$$\tau_n := \sup_{x \in \partial S_n} \tau_{0,x}, \tag{1.4}$$

the maximum connection probability between sites at distance n (using translation invariance). For any $n, m \geq 0$, we have

$$\tau_{n+m} \geq \tau_n \tau_m,$$

in other words $(-\log \tau_n)_{n \geq 0}$ is sub-additive, which implies the existence of a constant $\tilde{\xi}(p)$ such that

$$-\frac{\log \tau_n}{n} \longrightarrow \frac{1}{\tilde{\xi}(p)} = \inf_m \left(-\frac{\log \tau_m}{m} \right) \tag{1.5}$$

when $n \to \infty$. Note the following a-priori bound:

$$\mathbb{P}_p\big(0 \rightsquigarrow x\big) \leq e^{-\|x\|_\infty / \tilde{\xi}(p)}. \tag{1.6}$$

For $p > 1/2$, we simply use the symmetry $p \leftrightarrow 1-p$: we consider

$$\tau_n^* := \sup_{x \in \partial S_n} \mathbb{P}_p\big(0 \rightsquigarrow^* x\big) \tag{1.7}$$

and then $\tilde{\xi}(p)$ in the same way. We have in this case

$$\mathbb{P}_p\big(0 \rightsquigarrow^* x\big) \leq e^{-\|x\|_\infty / \tilde{\xi}(p)}. \tag{1.8}$$

Note that the symmetry $p \leftrightarrow 1-p$ gives immediately

$$\tilde{\xi}(p) = \tilde{\xi}(1-p).$$

Relation between the different lengths

As expected, these characteristic lengths turn out to be all of the same order of magnitude: we will prove in Section 1.7.3 that $L_\epsilon \asymp L_{\epsilon'}$ for any two $\epsilon, \epsilon' \in (0, 1/2)$, in Section 1.7.4 that $L \asymp \tilde{\xi}$, and in Section 1.7.5 that $L \asymp \xi$.

1.7.2 Main critical exponents

We focus here on three functions commonly used to describe the macroscopic behavior of percolation. We have already encountered some of them:

(i) $\xi(p) = \left[\frac{1}{\mathbb{E}_p\left[|C(0)|;|C(0)|<\infty\right]} \sum_x \|x\|_\infty^2 \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big)\right]^{1/2}$ the mean radius of a finite cluster.

(ii) $\theta(p) := \mathbb{P}_p(0 \rightsquigarrow \infty)$. This function can be viewed as the density of the infinite cluster C_∞, in the following sense:

$$\frac{1}{|S_N|}\big|S_N \cap C_\infty\big| \xrightarrow{\text{a.s.}} \theta(p) \tag{1.9}$$

when $N \to \infty$.

(iii) $\chi(p) = \mathbb{E}_p\big[|C(0)|; |C(0)| < \infty\big]$ the average size of a finite cluster.

Theorem 32 (Critical exponents)**.** *The following power-law estimates hold:*

(i) When $p \to 1/2$,

$$\xi(p) \asymp L(p) \approx |p - 1/2|^{-4/3}. \tag{1.10}$$

(ii) When $p \to 1/2^+$,

$$\theta(p) \approx (p - 1/2)^{5/36}. \tag{1.11}$$

(iii) When $p \to 1/2$,

$$\chi(p) \approx |p - 1/2|^{-43/18}. \tag{1.12}$$

The corresponding exponents are usually denoted by (respectively) ν, β and γ. This theorem is proved in the next sub-sections by combining the arm exponents for critical percolation with the estimates established for near-critical percolation.

1.7.3 Critical exponent for L

We derive here the exponent for $L_\epsilon(p)$ by counting the sites which are pivotal for the existence of a crossing in a box of size $L_\epsilon(p)$. These pivotal sites are exactly those for which the 4-arm event $\bar{A}_{4,\sigma_4}^{\cdot/\bar{I}}$ with alternating colors ($\sigma_4 = BWBW$) and sides ($\bar{I}$ = right, top, left and bottom sides):

Proposition 33 ([43, 71])**.** *For any fixed $\epsilon \in (0, 1/2)$, the following equivalence holds:*

$$|p - 1/2|\big(L_\epsilon(p)\big)^2 \pi_4(L_\epsilon(p)) \asymp 1. \tag{1.13}$$

Recall now the value $\alpha_4 = 5/4$ of the 4-arm exponent, stated in Theorem 21. If we plug it into Eq.(1.13), we get the value of the characteristic length exponent: when $p \to 1/2$,

$$1 \approx |p - 1/2|\big(L_\epsilon(p)\big)^2\big(L_\epsilon(p)\big)^{-5/4} = |p - 1/2|\big(L_\epsilon(p)\big)^{3/4},$$

so that indeed

$$L_\epsilon(p) \approx |p - 1/2|^{-4/3}.$$

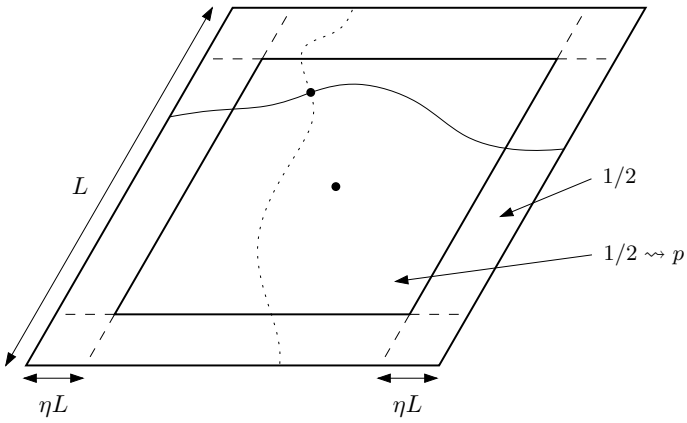


Figure 1.14: We restrict to the sites at distance at least ηL from the boundary of $[0, L]^2$: these sites produce contributions of the same order, since the 4 arms stay comparable in size.

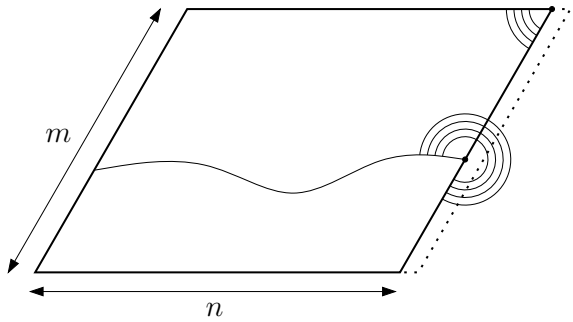


Figure 1.15: We extend a crossing of $[0, n] \times [0, m]$ into a crossing of $[0, (1 + \eta)n] \times [0, m]$ by applying RSW in concentric annuli.

Proof. For symmetry reasons, we can assume that $p > 1/2$. The proof goes as follows. We first apply Russo's formula to estimate the variation in probability of the event $\mathcal{C}_H([0, L(p)] \times [0, L(p)])$ between $1/2$ and p, which makes appear the events $\bar{A}^{\cdot/\bar{I}}_{4,\sigma_4}$. By construction of $L(p)$, the variation of the crossing event is of order 1, and the sites that are "not too close to the boundary" (such that none of the 4 arms can become too small – see Figure 1.14) each produce a contribution of the same order by Theorem 27: proving that they all together produce a non-negligible variation in the crossing probabilities will thus imply the result. For that, we need the following lemma:

Lemma 34. *For any $\delta > 0$, there exists $\eta_0 > 0$ such that for all p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, we have: for any parallelogram $[0, n] \times [0, m]$ with sides $n, m \leq L(p)$ and aspect ratio less than 2 (*ie *such that $1/2 \leq n/m \leq 2$), for any $\eta \leq \eta_0$,*

$$\left|\hat{\mathbb{P}}(\mathcal{C}_H([0, n] \times [0, m])) - \hat{\mathbb{P}}(\mathcal{C}_H([0, (1 + \eta)n] \times [0, m]))\right| \leq \delta. \tag{1.14}$$

Proof of lemma. First, we clearly have

$$\hat{\mathbb{P}}(\mathcal{C}_H([0, n] \times [0, m])) \geq \hat{\mathbb{P}}(\mathcal{C}_H([0, (1 + \eta)n] \times [0, m])).$$

For the converse bound, we use the same idea as for Lemma 15, we apply RSW in concentric annuli (see Figure 1.15). By considering (parts of) annuli centered on the top right corner of

$[0,n]\times[0,m]$, with radii between $\eta^{3/4}n$ and $\sqrt{\eta}n$, we see that the probability for a crossing to arrive at a distance less than $\eta^{3/4}n$ from this corner is at most $\delta/100$ for η_0 small enough. Assume this is not the case, condition on the lowest crossing and apply RSW in annuli between scales ηn and $\eta^{3/4}n$: if η_0 is sufficiently small, with probability at least $1-\delta/100$, this crossing can be extended into a crossing of $[0,(1+\eta)n]\times[0,m]$. □

Let us return to the proof of the proposition. Take η_0 associated to $\delta=\epsilon/100$ by the lemma, and assume that instead of performing the change $1/2\rightsquigarrow p$ in the whole box $[0,L(p)]^2$, we make it only for the sites in the sub-box $[\eta L(p),(1-\eta)L(p)]^2$, for $\eta=\eta_0/4$. It amounts to consider the measure $\hat{\mathbb{P}}^{(\eta)}$ with parameters

$$\hat{p}_v^{(\eta)}=\left|\begin{array}{ll} p & \text{if } v\in[\eta L(p),(1-\eta)L(p)]^2,\\ 1/2 & \text{otherwise.}\end{array}\right. \tag{1.15}$$

We are going to prove that $\hat{\mathbb{P}}^{(\eta)}(\mathcal{C}_H([0,L(p)]^2))$ and $\mathbb{P}_p(\mathcal{C}_H([0,L(p)]^2))$ are very close by showing that they are both very close to $\mathbb{P}_p(\mathcal{C}_H([\eta L(p),(1-\eta)L(p)]^2))=\hat{\mathbb{P}}^{(\eta)}(\mathcal{C}_H([\eta L(p),(1-\eta)L(p)]^2))$. Indeed, for any $\tilde{\mathbb{P}}\in\{\hat{\mathbb{P}}^{(\eta)},\mathbb{P}_p\}$, we have by the lemma

$$\begin{aligned}\tilde{\mathbb{P}}(\mathcal{C}_H([0,L(p)]^2)) &\le \tilde{\mathbb{P}}(\mathcal{C}_H([\eta L(p),(1-\eta)L(p)]\times[0,L(p)]))\\ &= 1-\tilde{\mathbb{P}}(\mathcal{C}_V^*([\eta L(p),(1-\eta)L(p)]\times[0,L(p)]))\\ &\le 1-\big(\tilde{\mathbb{P}}(\mathcal{C}_V^*([\eta L(p),(1-\eta)L(p)]^2))-2\delta\big)\\ &= \tilde{\mathbb{P}}(\mathcal{C}_H([\eta L(p),(1-\eta)L(p)]^2))+2\delta\end{aligned}$$

and in the other way,

$$\begin{aligned}\tilde{\mathbb{P}}(\mathcal{C}_H([0,L(p)]^2)) &\ge \tilde{\mathbb{P}}(\mathcal{C}_H([\eta L(p),(1-\eta)L(p)]\times[0,L(p)]))-2\delta\\ &= 1-\tilde{\mathbb{P}}(\mathcal{C}_V^*([\eta L(p),(1-\eta)L(p)]\times[0,L(p)]))-2\delta\\ &\ge 1-\tilde{\mathbb{P}}(\mathcal{C}_V^*([\eta L(p),(1-\eta)L(p)]^2))-2\delta\\ &= \tilde{\mathbb{P}}(\mathcal{C}_H([\eta L(p),(1-\eta)L(p)]^2))-2\delta.\end{aligned}$$

The claim follows readily, in particular

$$\hat{\mathbb{P}}^{(\eta)}(\mathcal{C}_H([0,L(p)]^2))\ge\mathbb{P}_p(\mathcal{C}_H([0,L(p)]^2))-4\delta, \tag{1.16}$$

which is at least $(1/2+\epsilon)-4\delta\ge 1/2+\epsilon/2$ by the very definition of $L(p)$. It shows as desired that the sites in $[\eta L(p),(1-\eta)L(p)]^2$ produce all together a non-negligible contribution.

Now, Russo's formula applied to the interpolating measures $(\hat{\mathbb{P}}_t^{(\eta)})_{t\in[0,1]}$ (with parameters $\hat{p}_v^{(\eta)}(t)=t\times\hat{p}_v^{(\eta)}+(1-t)\times 1/2$) and the event $\mathcal{C}_H([0,L(p)]^2)$ gives

$$\begin{aligned}\int_0^1 \sum_{v\in[\eta L(p),(1-\eta)L(p)]^2} &\big(p-1/2\big)\,\hat{\mathbb{P}}_t^{(\eta)}\big(v\rightsquigarrow^{\bar{I}}_{4,\sigma_4}\partial[0,L(p)]^2\big)dt\\ &= \hat{\mathbb{P}}^{(\eta)}(\mathcal{C}_H([0,L(p)]^2))-\mathbb{P}_{1/2}(\mathcal{C}_H([0,L(p)]^2)),\end{aligned}$$

and this quantity is at least $\epsilon/2$, and thus of order 1.

Finally, it is not hard to see that once η fixed, we have (uniformly in p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, and $v \in [\eta L(p), (1-\eta)L(p)]^2$)

$$\begin{aligned}\hat{\mathbb{P}}\big(v \rightsquigarrow^{\bar{I}}_{4,\sigma_4} \partial[0, L(p)]^2\big) &\asymp \hat{\mathbb{P}}\big(v \rightsquigarrow_{4,\sigma_4} \partial S_{\eta L(p)}(v)\big)\\ &\asymp \mathbb{P}_{1/2}(0 \rightsquigarrow_{4,\sigma_4} \partial S_{\eta L(p)})\\ &\asymp \mathbb{P}_{1/2}(0 \rightsquigarrow_{4,\sigma_4} \partial S_{L(p)}),\end{aligned}$$

which yields the desired conclusion. □

Remark 35. *Note that the intermediate lemma was required for the lower bound only, the upper bound can be obtained directly from Russo's formula. To get the lower bound, we could also have proved that for $n \leq L(p)$,*

$$\sum_{x \in S_n} \hat{\mathbb{P}}\big(x \rightsquigarrow_{4,\sigma_4} \partial S_n\big) \asymp n^2 \pi_4(n). \tag{1.17}$$

Basically, it comes from the fact that when we get closer to ∂S_N, one of the arms may be shorter, but the remaining arms also have less space - and the 3-arm exponent in the half-plane appears.

All the results we have seen so far hold for any fixed value of ϵ in $(0, 1/2)$, in particular the last Proposition. Combining it with the estimate for 4 arms, we get an important corollary, that the behavior of L_ϵ does not depend on the value of ϵ.

Corollary 36. *For any $\epsilon, \epsilon' \in (0, 1/2)$,*

$$L_\epsilon(p) \asymp L_{\epsilon'}(p). \tag{1.18}$$

Proof. To fix ideas, assume that $\epsilon \leq \epsilon'$, so that $L_\epsilon(p) \geq L_{\epsilon'}(p)$, and we need to prove that $L_\epsilon(p) \leq C L_{\epsilon'}(p)$ for some constant C. We know that

$$|p - 1/2|\big(L_\epsilon(p)\big)^2 \pi_4(L_\epsilon(p)) \asymp 1 \asymp |p - 1/2|\big(L_{\epsilon'}(p)\big)^2 \pi_4(L_{\epsilon'}(p)),$$

hence for some constant C_1,

$$\frac{\big(L_\epsilon(p)\big)^2 \pi_4(L_\epsilon(p))}{\big(L_{\epsilon'}(p)\big)^2 \pi_4(L_{\epsilon'}(p))} \leq C_1.$$

This yields

$$\left(\frac{L_\epsilon(p)}{L_{\epsilon'}(p)}\right)^2 \leq C_1 \frac{\pi_4(L_{\epsilon'}(p))}{\pi_4(L_\epsilon(p))} \leq C_2\big(\pi_4(L_{\epsilon'}(p), L_\epsilon(p))\big)^{-1}$$

by quasi-multiplicativity. Now we use the a-priori bound for 4 arms given by the 5-arm exponent:

$$\pi_4(L_{\epsilon'}(p), L_\epsilon(p)) \geq \left(\frac{L_{\epsilon'}(p)}{L_\epsilon(p)}\right)^{-\alpha'} \pi_5(L_{\epsilon'}(p), L_\epsilon(p)) \geq C_3 \left(\frac{L_{\epsilon'}(p)}{L_\epsilon(p)}\right)^{2-\alpha'}.$$

Together with the previous equation, it implies the result:

$$L_\epsilon(p) \leq (C_4)^{1/\alpha'} L_{\epsilon'}(p).$$

□

Remark 37. *In the other direction, a RSW construction shows that we can increase L_ϵ by any constant factor by choosing ϵ small enough.*

1.7.4 Uniform exponential decay, critical exponent for θ

Up to now, our reasonings (separation of arms, arm events in near-critical percolation, critical exponent for L) were based on RSW considerations on scales $n \leq L(p)$, so that critical and near-critical percolation could be handled simultaneously. In the other direction, the definition of $L(p)$ also implies that when $n > L(p)$, the picture starts to look like super/sub-critical percolation, supporting the choice of $L_(p)$ as the characteristic scale of the model.

More precisely, we prove a property of exponential decay uniform in p. This property will then be used to link L with the other characteristic functions, and we will derive the following expressions of θ, χ and ξ as functions of L:

(i) $\theta(p) \asymp \pi_1(L(p))$,

(ii) $\chi(p) \asymp L(p)^2 \pi_1^2(L(p))$,

(iii) $\xi(p) \asymp L(p)$.

The critical exponents for these three functions will follow readily, since we already know the exponent for L.

Uniform exponential decay

The following lemma shows that correlations decay exponentially fast with respect to $L(p)$. It allows to control the speed for p varying:

Lemma 38. *For any $\epsilon \in (0, 1/2)$, there exist constants $C_i = C_i(\epsilon) > 0$ such that for all $p < 1/2$, all n,*

$$\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \leq C_1 e^{-C_2 n / L_\epsilon(p)}. \tag{1.19}$$

Proof. We use a block argument: for each integer n,

$$\mathbb{P}_p(\mathcal{C}_H([0,2n] \times [0,4n])) \leq C' [\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,2n]))]^2, \tag{1.20}$$

with $C' = 10^2$ some universal constant.

It suffices for that (see Figure 1.16) to divide the parallelogram $[0,2n] \times [0,4n]$ into 4 horizontal sub-parallelograms $[0,2n] \times [in,(i+1)n]$ $(i = 0, \ldots, 3)$ and 6 vertical ones $[in,(i+1)n] \times [jn,(j+2)n]$ $(i = 0,1, j = 0,1,2)$. Indeed, consider a horizontal crossing of the big parallelogram: by considering its pieces in the two regions $0 < x < n$ and $n < x < 2n$, we can extract from it two sub-paths crossing one of the sub-parallelograms "in the easy way". They are disjoint by construction, so the claim follows by using the BK inequality.

We then obtain by iterating:

$$C' \mathbb{P}_p(\mathcal{C}_H([0, 2^k L(p)] \times [0, 2^{k+1} L(p)])) \leq (C' \epsilon_1)^{2^k} \tag{1.21}$$

as soon as $\epsilon_1 \geq \mathbb{P}_p(\mathcal{C}_H([0, L(p)] \times [0, 2L(p)]))$.

Recall that by definition, $\mathbb{P}_p(\mathcal{C}_H([0, L(p)] \times [0, L(p)])) \leq \epsilon_0$ if $\epsilon \leq \epsilon_0$. The RSW theory thus entails (Theorem 2) that for all fixed $\epsilon_1 > 0$, we can take ϵ_0 sufficiently small to get automatically (and independently of p) that

$$\mathbb{P}_p(\mathcal{C}_H([0, L(p)] \times [0, 2L(p)])) \leq \epsilon_1. \tag{1.22}$$

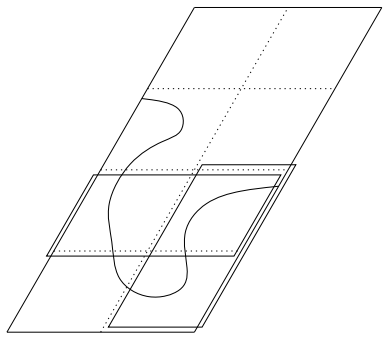


Figure 1.16: Two of the small sub-parallelograms are crossed in the "easy" way.

We now choose $\epsilon_1 = 1/(e^2 C')$. For each integer $n \geq L(p)$, we can define $k = k(n)$ such that $2^k \leq n/L(p) < 2^{k+1}$, and then,

$$\begin{aligned}\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) &\leq \mathbb{P}_p(\mathcal{C}_H([0, 2^k L(p)] \times [0, 2^{k+1} L(p)])) \\ &\leq e^{-2^{k+1}} \\ &\leq e \times e^{-n/L(p)},\end{aligned}$$

which is also valid for $n < L(p)$, thanks to the extra factor e.

Hence, we have proved the property for any ϵ below some fixed value ϵ_0 (given by RSW). The result for any $\epsilon \in (0, 1/2)$ follows readily by using the equivalence of lengths for different values of ϵ (Corollary 36). □

We would like to stress the fact that we have not used any of the previous results until the last step, this exponential decay property could thus have been derived much earlier – but only for values of ϵ small enough. It would for instance provide a more direct way to prove that

$$L_\epsilon(p) \asymp L_{\epsilon'}(p),$$

but still only for ϵ, ϵ' less than some fixed value.

Remark 39. *It will sometimes reveal more useful to know this property for crossings of longer parallelograms "in the easy way": we also have for any $k \geq 1$,*

$$\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,kn])) \leq C_1^{(k)} e^{-C_2^{(k)} n/L(p)} \tag{1.23}$$

for some constants $C_i^{(k)}$ (depending on k and ϵ). This can be proved by combining the previous lemma with the fact that in Theorem 2, we can take f_k satisfying $f_k(1-\epsilon) = 1 - C_k \epsilon^{\alpha_k} + o(\epsilon^{\alpha_k})$ for some $C_k, \alpha_k > 0$.

Consequence for θ

When $p > 1/2$, we now show that at distance $L(p)$ from the origin, we are already "not too far from infinity": once we have reached this distance, there is a positive probability (bounded away from 0 uniformly in p) to reach infinity.

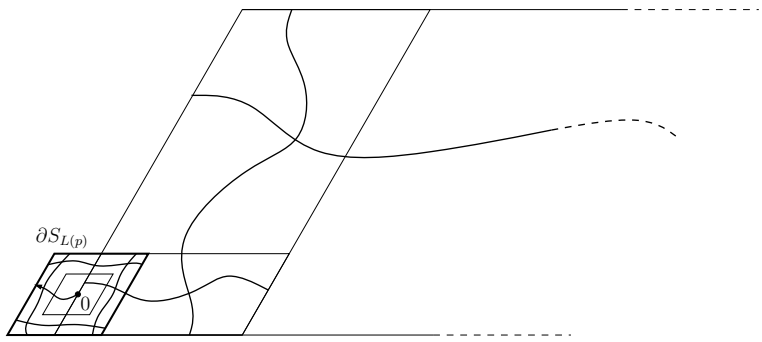


Figure 1.17: We consider overlapping parallelograms, with size doubling at each step.

Corollary 40. *We have*

$$\theta(p) = \mathbb{P}_p\big(0 \rightsquigarrow \infty\big) \asymp \mathbb{P}_p\big(0 \rightsquigarrow \partial S_{L(p)}\big) \tag{1.24}$$

uniformly in $p > 1/2$.

Proof. It suffices to consider overlapping parallelograms like in Figure 1.17, each parallelogram twice larger than the previous one, so that the kth of them has a probability at least $1 - C_1 e^{-C_2 2^k}$ to present a crossing in the "hard" direction (thanks to the previous remark). Since $\prod_k (1 - C_1 e^{-C_2 2^k}) > 0$, we are done. □

Now, combining Eq.(1.24) with Theorem 27 gives (for $p > 1/2$)

$$\theta(p) \asymp \mathbb{P}_p\big(0 \rightsquigarrow \partial S_{L(p)}\big) \asymp \mathbb{P}_{1/2}\big(0 \rightsquigarrow \partial S_{L(p)}\big) = \pi_1(L(p)). \tag{1.25}$$

Using the 1-arm exponent $\alpha_1 = 5/48$ stated in Theorem 21, we get

$$\theta(p) \approx \big(L(p)\big)^{-5/48} \tag{1.26}$$

as $p \to 1/2^+$. Together with the critical exponent for L derived previously, this provides the critical exponent for θ:

$$\theta(p) \approx \big((p - 1/2)^{-4/3}\big)^{-5/48} \approx (p - 1/2)^{5/36}. \tag{1.27}$$

Equivalence of L and $\tilde{\xi}$

To fix ideas, we assume in this sub-section that $p < 1/2$. Performing a RSW-type construction like in Figure 1.3 yields

$$\mathbb{P}_p(\mathcal{C}_H([0, kL(p)] \times [0, L(p)])) \geq \delta_2^{k-1}\delta_1^{k-2} = C_1 e^{-C_2 kL(p)/L(p)}, \tag{1.28}$$

so that $L(p)$ measures *exactly* the speed of decaying. Once knowing this, it is easy to compare L and $\tilde{\xi}$.

Corollary 41. *For any fixed* $\epsilon \in (0, 1/2)$,

$$\tilde{\xi}(p) \asymp L_\epsilon(p). \tag{1.29}$$

Proof. We exploit the previous remark: on one hand L measures the speed of decaying for crossings of rhombi, and on the other hand $\tilde{\xi}$ was defined to give the optimal bound for point-to-point connections.

More precisely, for any $x \in \partial S_n$ we have

$$\begin{aligned}\tau_{0,x} = \mathbb{P}_p(0 \rightsquigarrow x) &\leq \mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,2n])) \\ &\leq C_1^{(2)} e^{-C_2^{(2)} n/L(p)}\end{aligned}$$

so that $\tau_{kL(p)} \leq C_1^{(2)} e^{-C_2^{(2)} k}$ and

$$-\frac{\log \tau_{kL(p)}}{kL(p)} \geq -\frac{1}{kL(p)}\big(\log C_1^{(2)} - C_2^{(2)} k\big) \xrightarrow[k\to\infty]{} \frac{C_2^{(2)}}{L(p)}.$$

Hence,

$$\frac{1}{\tilde{\xi}(p)} \geq \frac{C_2^{(2)}}{L(p)}$$

and finally $\tilde{\xi}(p) \leq CL(p)$.

Conversely, we know that $\mathbb{P}_p(\mathcal{C}_H([0,kL(p)]\times[0,kL(p)])) \geq \tilde{C}_1 e^{-\tilde{C}_2 k}$ for some $\tilde{C}_i > 0$ (Eq.(1.28)). Consequently,

$$\tau_{kL(p)} \geq \frac{1}{(kL(p))^2}\mathbb{P}_p(\mathcal{C}_H([0,kL(p)] \times [0,kL(p)])) \geq \frac{1}{(kL(p))^2}\tilde{C}_1 e^{-\tilde{C}_2 k},$$

which implies

$$-\frac{\log \tau_{kL(p)}}{kL(p)} \leq -\frac{1}{kL(p)}\big(\log \tilde{C}_1 - 2\log(kL(p)) - \tilde{C}_2 k\big) \xrightarrow[k\to\infty]{} \frac{\tilde{C}_2}{L(p)},$$

whence the conclusion: $\tilde{\xi}(p) \geq C'L(p)$. □

1.7.5 Further estimates, critical exponents for χ and ξ

Estimates from critical percolation

We start by stating some estimates that we will need. These estimates were originally derived for critical percolation (see e.g. [40, 41]), but for exactly the same reasons they also hold for near-critical percolation on scales $n \leq L(p)$:

Lemma 42. *Uniformly in p, $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$ and $n \leq L(p)$, we have*

1. $\hat{\mathbb{E}}\big[|x \in S_n : x \rightsquigarrow \partial S_n|\big] \asymp n^2 \pi_1(n)$.
2. *For any $t \geq 0$,*

$$\sum_{x\in S_n} \|x\|_\infty^t \hat{\mathbb{P}}\big(0 \rightsquigarrow x\big) \asymp \sum_{x\in S_n} \|x\|_\infty^t \hat{\mathbb{P}}\big(0 \overset{S_n}{\rightsquigarrow} x\big) \asymp n^{t+2}\pi_1^2(n).$$

Note that item 2. implies in particular for $t = 0$ that

$$\hat{\mathbb{E}}\big[|x \in S_n : x \rightsquigarrow 0|\big] \asymp \hat{\mathbb{E}}\big[|x \in S_n : x \overset{S_n}{\rightsquigarrow} 0|\big] \asymp n^2\pi_1^2(n).$$

Proof. We will have use for the fact that we can take $\alpha = 1/2$ for $j = 1$ in Eq.(1.6) (actually any $\alpha < 1$ would be enough for our purpose): for any integers $n < N$,

$$\pi_1(n, N) \geq C(n/N)^{1/2}. \tag{1.30}$$

This can be proved like (3.15) of [10]: just use blocks of size n instead of individual sites to obtain that $\frac{N}{n}\pi_1^2(n, N)$ is bounded below by a constant.

Proof of item 1. We will use that

$$\hat{\mathbb{E}}\big[|x \in S_n : x \rightsquigarrow \partial S_n|\big] = \sum_{x \in S_n} \hat{\mathbb{P}}(x \rightsquigarrow \partial S_n). \tag{1.31}$$

For the lower bound, it suffices to note that for any $x \in S_n$,

$$\hat{\mathbb{P}}(x \rightsquigarrow \partial S_n) \geq \hat{\mathbb{P}}(x \rightsquigarrow \partial S_{2n}(x)) = \hat{\mathbb{P}}^x(0 \rightsquigarrow \partial S_{2n}) \tag{1.32}$$

(where $\hat{\mathbb{P}}^x$ is the measure $\hat{\mathbb{P}}$ translated by x), and that

$$\hat{\mathbb{P}}^x(0 \rightsquigarrow \partial S_{2n}) \geq C_1 \hat{\mathbb{P}}^x(0 \rightsquigarrow \partial S_n) \geq C_2 \pi_1(n) \tag{1.33}$$

by extendability and Theorem 27 for one arm.

For the upper bound, we sum over concentric rhombi around 0:

$$\begin{aligned}\sum_{x \in S_n} \hat{\mathbb{P}}(x \rightsquigarrow \partial S_n) &\leq \sum_{x \in S_n} \hat{\mathbb{P}}(x \rightsquigarrow \partial S_{d(x, \partial S_n)}(x)) \\ &\leq \sum_{j=1}^{n} C_1 n \hat{\mathbb{P}}(0 \rightsquigarrow \partial S_j)\end{aligned}$$

using that there are at most $C_1 n$ sites at distance j from ∂S_n. By Theorem 27, this last sum is at most

$$\begin{aligned}C_2 n \sum_{j=1}^{n} \pi_1(j) &\leq C_2 n \pi_1(n) \sum_{j=1}^{n} \frac{\pi_1(j)}{\pi_1(n)} \\ &\leq C_3 n \pi_1(n) \sum_{j=1}^{n} \pi_1(j, n)^{-1}\end{aligned}$$

by quasi-multiplicativity. Now Eq.(1.30) says that $\pi_1(j, n) \geq (j/n)^{1/2}$, so that

$$\sum_{j=1}^{n} \pi_1(j, n)^{-1} \leq \sum_{j=1}^{n} (j/n)^{-1/2} = n^{1/2} \sum_{j=1}^{n} j^{-1/2} \leq C_4 n,$$

which gives the desired upper bound.

Proof of item 2.

Since

$$\sum_{x \in S_n} \|x\|_\infty^t \hat{\mathbb{P}}\big(0 \overset{S_n}{\rightsquigarrow} x\big) \leq \sum_{x \in S_n} \|x\|_\infty^t \hat{\mathbb{P}}\big(0 \rightsquigarrow x\big),$$

it suffices to prove the desired lower bound for the left-hand side, and the upper bound for the right-hand side.

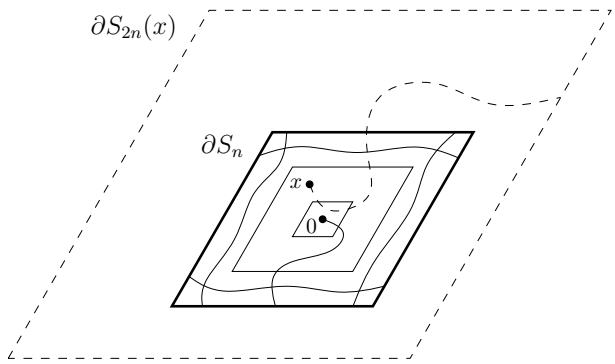


Figure 1.18: With this construction, any site x in $S_{n/3,2n/3}$ connected to a site at distance $2n$ is also connected to 0 in S_n.

Consider the lower bound first. We note (see Figure 1.18) that if 0 is connected to ∂S_n and if there exists a black circuit in $S_{2n/3,n}$ (which occurs with probability at least δ_6^4 by RSW), then any $x \in S_{n/3,2n/3}$ connected to $\partial S_{2n}(x)$ will be connected to 0 in S_n. Using the FKG inequality, we thus get for such an x:

$$\hat{\mathbb{P}}(0 \overset{S_n}{\rightsquigarrow} x) \geq \delta_6^4 \hat{\mathbb{P}}(0 \rightsquigarrow \partial S_n)\hat{\mathbb{P}}(x \rightsquigarrow \partial S_{2n}(x))$$

which is at least (still using extendability and Theorem 27) $C_1\pi_1^2(n)$. Consequently,

$$\begin{aligned}\sum_{x\in S_n} \|x\|_\infty^t \hat{\mathbb{P}}\big(0 \overset{S_n}{\rightsquigarrow} x\big) &\geq \sum_{x\in S_{n/3,2n/3}} \|x\|_\infty^t C_1\pi_1^2(n) \\ &\geq C_2 n^2 (n/3)^t \pi_1^2(n).\end{aligned}$$

Let us turn to the upper bound. We take a logarithmic division of S_n: define $k = k(n)$ so that $2^k < n \leq 2^{k+1}$, we have

$$\sum_{x\in S_n} \|x\|_\infty^t \hat{\mathbb{P}}\big(0 \rightsquigarrow x\big) \leq C_1 + \sum_{j=3}^{k+1} \sum_{x\in S_{2^{j-1},2^j}} \|x\|_\infty^t \hat{\mathbb{P}}\big(0 \rightsquigarrow x\big). \tag{1.34}$$

Now for $x \in S_{2^{j-1},2^j}$, take the two boxes $S_{2^{j-2}}(0)$ and $S_{2^{j-2}}(x)$: since they are disjoint,

$$\hat{\mathbb{P}}\big(0 \rightsquigarrow x\big) \leq \hat{\mathbb{P}}\big(0 \rightsquigarrow S_{2^{j-2}}(0)\big)\hat{\mathbb{P}}\big(x \rightsquigarrow S_{2^{j-2}}(x)\big), \tag{1.35}$$

which is at most $C_2\pi_1^2(2^{j-1})$ using the same arguments as before. Our sum is thus less than (since $|S_{2^{j-1},2^j}| \leq C_3 2^{2j}$)

$$\sum_{j=3}^{k+1} C_3 2^{2j} \times (2^j)^t \times (C_2\pi_1^2(2^{j+1})) \leq C_4 2^{(2+t)k}\pi_1^2(2^k) \times \left[\sum_{j=3}^{k+1} 2^{(2+t)(j-k)} \frac{\pi_1^2(2^{j-1})}{\pi_1^2(2^k)}\right]. \tag{1.36}$$

Now $2^{(2+t)k}\pi_1^2(2^k) \leq C_5 n^{2+t}\pi_1^2(n)$, and this yields the desired result, using as previously $\frac{\pi_1(2^j)}{\pi_1(2^k)} \leq$

$C_6\pi_1(2^j,2^k)^{-1} \le C_6 2^{-(j-k)/2}$:

$$\sum_{j=3}^{k+1} 2^{(2+t)(j-k)} \frac{\pi_1^2(2^{j-1})}{\pi_1^2(2^k)} \le C_6 \sum_{j=3}^{k+1} 2^{(2+t)(j-k)} 2^{-(j-k)}$$
$$\le C_7 \sum_{l=0}^{k-3} 2^{-(1+t)l},$$

and this sum is bounded by $\sum_{l=0}^{\infty} 2^{-(1+t)l} < \infty$. □

Main estimate

The following lemma will allow us to link directly χ and ξ with L. Roughly speaking, it relies on the fact that the sites at distance much larger than $L(p)$ from the origin have a negligible contribution, due to the exponential decay property, so that the sites in $S_{L(p)}$ produce a positive fraction of the total sum:

Lemma 43. *For any $t \ge 0$, we have*

$$\sum_x \|x\|_\infty^t \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big) \asymp L(p)^{t+2}\pi_1^2(L(p)) \tag{1.37}$$

uniformly in p.

Proof. **Lower bound.** The lower bound is a direct consequence of item 2. above: indeed,

$$\sum_x \|x\|_\infty^t \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big)$$
$$\ge \mathbb{P}_p(\exists \text{ white circuit in } S_{L,2L}) \sum_{x\in S_L} \|x\|_\infty^t \mathbb{P}_p\big(0 \overset{S_L}{\rightsquigarrow} x\big)$$
$$\ge \delta_4^4 \sum_{x\in S_L} \|x\|_\infty^t \mathbb{P}_p\big(0 \overset{S_L}{\rightsquigarrow} x\big)$$

by RSW, and item 2. gives

$$\sum_{x\in S_L} \|x\|_\infty^t \mathbb{P}_p\big(0 \overset{S_L}{\rightsquigarrow} x\big) \ge CL^{t+2}\pi_1^2(L).$$

Upper bound. To get the upper bound, we cover the plane by translating S_L: we consider the family of rhombi $S_L(2n_1L, 2n_2L)$, for any two integers n_1 and n_2. By isolating the contribution of S_L, we get:

$$\sum_x \|x\|_\infty^t \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big)$$
$$\le \sum_{x\in S_L} \|x\|_\infty^t \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big)$$
$$+ \sum_{(n_1,n_2)\neq(0,0)} \sum_{x\in S_L(2n_1L,2n_2L)} \|x\|_\infty^t \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big).$$

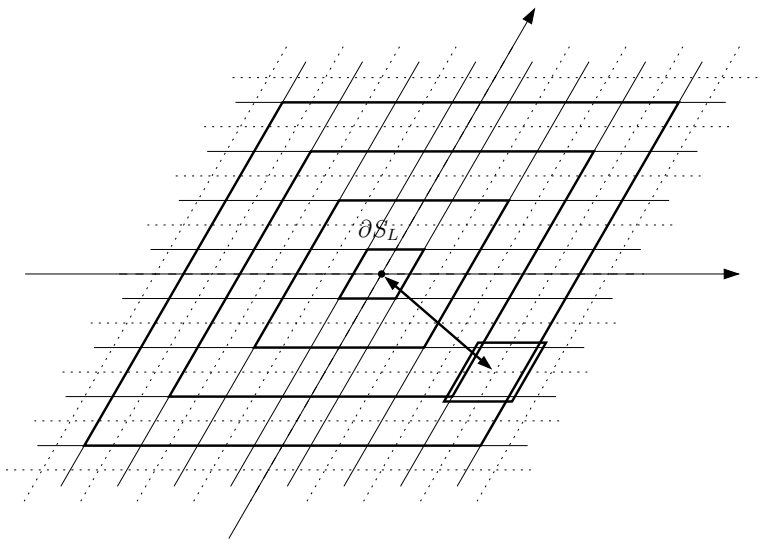


Figure 1.19: For the upper bound, we cover the plane with rhombi of size $2L$ and sum their different contributions.

Using item 2. above, we see that the rhombus S_L gives a contribution

$$\sum_{x\in S_L} \|x\|_\infty^t \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big) \le CL^{t+2}\pi_1^2(L),$$

which is of the right order of magnitude.

We now prove that each small rhombus outside of S_L at distance kL gives a contribution of order $\pi_1(L) \times L^t \times \mathbb{E}_p\big[|x \in S_L : x \rightsquigarrow \partial S_L|\big] \asymp L^{t+2}\pi_1^2(L)$ (using item 1.), multiplied by some quantity which decays exponentially fast in k and will thus produce a series of finite sum. More precisely, if we regroup the rhombi into concentric annuli around S_L, we get that the previous summation is at most

$$\begin{aligned}
&\sum_{k=1}^{\infty} \sum_{\substack{(n_1,n_2)\\ \|(n_1,n_2)\|_\infty = k}} \sum_{x\in S_L(2n_1L,2n_2L)} \|x\|_\infty^t \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big) \\
&\qquad \le \sum_{k=1}^{\infty} \sum_{\substack{(n_1,n_2)\\ \|(n_1,n_2)\|_\infty = k}} \sum_{x\in S_L(2n_1L,2n_2L)} [(2k+1)L]^t \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big) \\
&\qquad \le \sum_{k=1}^{\infty} \sum_{\substack{(n_1,n_2)\\ \|(n_1,n_2)\|_\infty = k}} C' k^t L^t\, \mathbb{E}_p\big[|C(0) \cap S_L(2n_1L, 2n_2L)|; |C(0)| < \infty\big].
\end{aligned}$$

Now we have to distinguish between the sub-critical and the super-critical cases: we are going to prove that in both cases,

$$\mathbb{E}_p\big[|C(0) \cap S_L(2n_1L, 2n_2L)|; |C(0)| < \infty\big] \le C_1 L^2 \pi_1^2(L) e^{-C_2 k}$$

for some constants $C_1, C_2 > 0$. When $p < 1/2$, we will use that

$$\mathbb{P}_p(\partial S_L \rightsquigarrow \partial S_{kL}) \le C_3 e^{-C_4 k}, \tag{1.38}$$

which is a direct consequence of the exponential decay property Eq.(1.23) for "longer" parallelograms. When $p > 1/2$, we have an analog result, which can be deduced from the sub-critical case just like in the discrete case (just replace sites by translations of S_L):

$$\begin{aligned}
&\mathbb{P}_p(\partial S_L \rightsquigarrow \partial S_{kL}, |C(0)| < \infty) \\
&\qquad \le \mathbb{P}_p(\exists \text{ white circuit surrounding a site on } \partial S_L \text{ and a site on } \partial S_{kL}) \\
&\qquad \le C_5 e^{-C_6 k}.
\end{aligned}$$

Assume first that $p < 1/2$. By independence, we have ($\|(n_1, n_2)\|_\infty = k$)

$$\begin{aligned}
&\mathbb{E}_p\big[|C(0) \cap S_L(2n_1L, 2n_2L)|; |C(0)| < \infty\big] \\
&\qquad \le \mathbb{P}_p(0 \rightsquigarrow \partial S_L)\mathbb{E}_p\big[|x \in S_L(2n_1L, 2n_2L) : x \rightsquigarrow \partial S_L(2n_1L, 2n_2L)|\big] \\
&\qquad\qquad \times \mathbb{P}_p(\partial S_L \rightsquigarrow \partial S_{(2k-1)L}) \\
&\qquad \le \pi_1(L) \times (CL^2\pi_1(L)) \times C_3' e^{-C_4' k}.
\end{aligned}$$

If $p > 1/2$, we write similarly (here we use FKG to separate the existence of a white circuit (decreasing) from the other terms (increasing), and then independence of the remaining terms)

$$\begin{aligned}
&\mathbb{E}_p\big[|C(0) \cap S_L(2n_1L, 2n_2L)|; |C(0)| < \infty\big] \\
&\qquad \le \mathbb{P}_p(0 \rightsquigarrow \partial S_L)\mathbb{E}_p\big[|x \in S_L(2n_1L, 2n_2L) : x \rightsquigarrow \partial S_L(2n_1L, 2n_2L)|\big] \\
&\qquad\quad \times \mathbb{P}_p(\exists \text{ white circuit surrounding a site on } \partial S_L \text{ and a site on } \partial S_{(2k-1)L}) \\
&\qquad \le \pi_1(L) \times (CL^2\pi_1(L)) \times C_5' e^{-C_6' k}.
\end{aligned}$$

Since there are at most $C''k$ rhombi at distance k for some constant C'', the previous summation is in both cases less than

$$\sum_{k=1}^{\infty} C''k \times C'k^t L^t \times C_1 L^2 \pi_1^2(L) e^{-C_2 k} \le C'''\Big(\sum_{k=1}^{\infty} k^{t+1} e^{-C_2 k}\Big) L^{t+2}\pi_1^2(L),$$

which yields the desired upper bound, as $\sum_{k=1}^{\infty} k^{t+1} e^{-C_2 k} < \infty$.

□

Critical exponents for χ and ξ

The previous lemma reads for $t = 0$:

Proposition 44. *We have*

$$\chi(p) = \mathbb{E}_p\big[|C(0)|; |C(0)| < \infty\big] \asymp L(p)^2 \pi_1^2(L(p)). \tag{1.39}$$

In other words, "$\chi(p) \asymp \chi^{\text{near}}(p)$". It provides the critical exponent for χ:

$$\chi(p) \approx L(p)^2 \big[L(p)^{-5/48}\big]^2 \approx \big[|p - 1/2|^{-4/3}\big]^{86/48} \approx |p - 1/2|^{-43/18}. \tag{1.40}$$

Recall that ξ was defined via the formula

$$\xi(p) = \left[\frac{1}{\mathbb{E}_p\big[|C(0)|; |C(0)| < \infty\big]} \sum_x \|x\|_\infty^2 \mathbb{P}_p\big(0 \rightsquigarrow x, |C(0)| < \infty\big)\right]^{1/2}.$$

Using the last proposition and the lemma for $t = 2$, we get

$$\xi(p) \asymp \left[\frac{L(p)^4 \pi_1^2(L(p))}{L(p)^2 \pi_1^2(L(p))}\right]^{1/2} = L(p). \tag{1.41}$$

We thus obtain the following proposition, announced in Section 1.7.1:

Proposition 45. *We have*

$$\xi(p) \asymp L(p). \tag{1.42}$$

This implies in particular that

$$\xi(p) \approx |p - 1/2|^{-4/3}. \tag{1.43}$$

1.8 Concluding remarks

1.8.1 Other lattices

Most of the results presented here (the separation of arms, the theorem concerning arm events on a scale $L(p)$, the "universal" arm exponents, the relations between the different characteristic functions, etc.) come from RSW considerations or the exponential decay property, and remain true on other regular lattices like the square lattice. The triangular lattice has a property of self-duality which makes life easier, in general we have to consider the original lattice together with the matching lattice (obtained by "filling" each face with a complete graph): instead of black or white connections, we thus talk about primal and dual connections. We can also handle bond percolation in this way. We refer the reader to the original paper of Kesten [43] for more details, where results are proved in this more general setting. The only obstruction to get the critical exponents is actually the derivation of the arm exponents at the critical point $p = p_c$ (besides only two exponents are needed, for 1 arm and 4 alternating arms).

Consider site percolation on $\mathbb{Z}^2$ for instance. We know that $0 < \alpha_j, \alpha'_j, \beta_j < \infty$ for any $j \geq 1$. Hence the a-priori estimate

$$\mathbb{P}_{p_c}(0 \rightsquigarrow_{4,\sigma_4} \partial S_N) \geq N^{-2+\alpha}$$

for some $\alpha > 0$, coming from the 5-arm exponent, remains true: $\alpha_4 < 2$ (and in the same way $\alpha_6 > 2$). Combined with Proposition 33, this leads to the weaker but nonetheless interesting statement

$$L(p) \leq |p - p_c|^{-A} \tag{1.1}$$

for some $A > 0$. Hence $\nu < \infty$, and then $\gamma < \infty$ (if these exponents exist). Using $\alpha_1 < \infty$, we also get $\beta < \infty$.

If we use a RSW construction in a box, we can make appear 3-arm sites on the lowest crossing and deduce that $\alpha_1 \leq 1/3$. Here are rigorous bounds for the critical exponents in two dimensions:

triangular lattice	general rigorous bounds
$\beta = 5/36$	$0 < \beta < 1$
$\gamma = 43/18$	$8/5 \leq \gamma < \infty$
$\nu = 4/3$	$1 < \nu < \infty$

For more details, the reader can consult [43] and the references therein.

1.8.2 Some related issues

For the sake of completeness, let us just mention finally that the way the correlation length L was defined also allows to use directly the compactness results of [2]. Indeed, the a-priori estimates on arm events coming from RSW considerations are exactly the hypothesis (H1) of this paper. This hypothesis entails that the curve cannot cross too many times any annulus, and thus cannot be too "intricate": this is Theorem 1, asserting the existence of Hölder parametrizations with high probability.

This regularity property then implies tightness, using (a version of) Arzela-Ascoli's theorem for continuous functions on a compact subset of the plane. We can thus show in this way the existence of scaling limits for near-critical percolation interfaces.

As a conclusion, let us also mention that the techniques presented here are important to study various models related to the critical regime, for instance Incipient Infinite Clusters [25, 36, 37], Dynamical Percolation [68], Gradient Percolation [54]...

Chapter 2

Large scale properties of the IIIC for 2D percolation

Abstract: We reinvestigate the 2D problem of the *inhomogeneous incipient infinite cluster* where, in an independent percolation model, the density decays to p_c with an inverse power, λ, of the distance to the origin. Assuming the existence of critical exponents (as is known in the case of the triangular site lattice) if the power is less than $1/\nu$, with ν the correlation length exponent, we demonstrate an infinite cluster with scale dimension given by $D_H = 2 - \beta\lambda$. Further, we investigate the critical case $\lambda_c = 1/\nu$ and show that *iterated* logarithmic corrections will tip the balance between the possibility and impossibility of an infinite cluster.

This chapter is joint work with Lincoln Chayes.

2.1 Introduction

A while ago, one of us – in collaboration with others – introduced a notion of *inhomogeneous percolation* [25] that was demonstrated to have some interesting properties. The model is defined by allowing the density parameter to vary, e.g. with the distance to the origin, in such a way that the system will just barely house an infinite cluster. Explicitly, one looks at

$$p(r) \cong p_c + \frac{1}{r^\lambda} \tag{2.1}$$

where r denotes distance to the origin (and it should be assumed r is large enough so that the right–hand side makes sense). For $d = 2$, under the assumption of the existence of critical exponents, it was found that if $\lambda < \lambda_c = 1/\nu$ the origin belongs to an infinite cluster with positive probability, while this probability vanishes if $\lambda > \lambda_c$. In the preceding, ν is the correlation length exponent – precise definitions later – and, in fact, an equivalent but more awkward statement can be made without reference to exponents. For $\lambda < \lambda_c$, we will refer to the infinite object as the *inhomogeneous incipient infinite cluster* (IIIC).

In the ensuing time, there have been two landmarks in 2D percolation, namely the works of Kesten in the late 1980's ([40], [41], [43]) wherein critical scaling relations were established modulo the existence of certain critical exponents, and the more recent works by (various combinations of) Lawler, Schramm, Smirnov and Werner ([69], [50], [71]) where the existence of these exponents – and their values – was established for the case of the triangular site lattice using the connection, in the scaling limit, to the Schramm-Loewner Evolution (SLE) with parameter 6. Thus, most of the original results can be sharpened at least in certain cases. However, such matters are largely automatic.

The main result of this note concerns the large scale structure of the percolating cluster. In particular, it turns out that these objects have a well–defined Hausdorff dimension (more precisely scale–dimension) that is given by

$$D_H = 2 - \beta\lambda \tag{2.2}$$

for $0 < \lambda < \lambda_c$, where β is the percolation density, or order parameter exponent. It is noted that as $\lambda \downarrow \lambda_c$ this dimension matches that of the standard IIC as discussed in e.g. [73], [74] and proved, modulo the existence of exponents, in [40]. Further, we discuss the borderline case, informally $p(r) - p_c = r^{-1/\nu}K(r)$ where $K(r)$ is a "correction". It turns out that at the border, the balance is *very* delicate and

$$K(r) \sim [\log\log r]^{1/\nu} \tag{2.3}$$

will determine the presence or absence of infinite structures. All results save the latter can be stated without apology for the triangular site model; a statement along the lines of Eq.(2.3) requires *strong* existence of power laws which, at this time, has not been established, and we will be content with a statement that circumvents this necessity.

2.2 Setup and statement of theorems

2.2.1 Setting

We consider any of the standard 2D percolation models – explicitly any model for which the results of [40] – [43] can be established. In particular, what is needed is reflection symmetry about one of the coordinate axes and overall rotational invariance by any angle in $(0, \pi)$. However, it is

sufficient that the reader keeps in mind only the bond or site problems on the square or triangular lattice (unfortunately, the latter requires the use of parallelograms rather than rectangles and, since the triangular site model is where the strongest results are known, we are forced to carry this terminology).

For the purposes of this note, it is assumed that the reader is familiar with the standard fare associated with these sorts of percolation problems; additional background material can be found in the reference [35].

Let us now fix some working notation/definitions: we take the vertical axis to be the axis of reflection symmetry and $r(z) = \|z\|_\infty$ (abv. $\|z\|$ since, in any case, all norms are equivalent) will denote the infinity norm of a site z as measured with respect to the x–axis and the axis related to this by the angle of rotation symmetry. The set of points at distance at most N from a site z is a rhombus centered at this site and whose sides line up with the above mentioned axes. It will be denoted by $S_N(z)$, its boundary being the set $\partial S_N(z)$ of points at distance exactly N from z. We will refer to $S_N(0)$ simply as S_N. We will often use the fact that

$$|S_N(z)| \leq C_0 N^2 \tag{2.1}$$

for some constant C_0 that may depend on the lattice.

Bonds or sites (as appropriate) will be *occupied* with probability p and *vacant* with probability $1-p$, independently of each other. We denote by $P_\infty(p)$ the probability that the site at the origin is connected to infinity, and by $p_c \in (0,1)$ the percolation threshold: $P_\infty(p) > 0$ *iff* $p > p_c$. If A and B are sets (which, for convenience, will include the case "infinity"), then we use the notation $A \rightsquigarrow B$ to denote the event that some site in A is connected to some site in B. If the connection is required to take place using exclusively the sites of some other set C, we write $A \overset{C}{\rightsquigarrow} B$. Finally, all quantities adorned by an $*$ will pertain to the *dual model*.

We will make use of the following one–arm probability

$$\pi(N) := \mathbb{P}_{p_c}(\{0 \rightsquigarrow \partial S_N\}) \tag{2.2}$$

and, in addition,

$$\pi(n|N) := \mathbb{P}_{p_c}(\{\partial S_n \rightsquigarrow \partial S_N\}). \tag{2.3}$$

The so-called Russo-Seymour-Welsh theory (see e.g. [35]) implies that

$$\pi(n|2N), \pi(\lfloor n/2 \rfloor|N) \geq D_1 \pi(n|N) \tag{2.4}$$

and

$$D_2 \left[\frac{n}{N}\right]^{\mu} \leq \pi(n|N) \leq D_3 \left[\frac{n}{N}\right]^{\mu'} \tag{2.5}$$

for some constants $0 < D_1, D_2, D_3, \mu, \mu' < \infty$. We will later have use for $\mu < 2$ so we may as well take $\mu = \frac{1}{2}$ (this may be derived by a variant of the "example" (3.15) in [10] where one now uses blocks of size n instead of individual sites to obtain that $\frac{N}{n}\pi^2(n|N)$ is bounded below by a constant). Finally, we also have

$$D_4 \pi(n_0|n_2) \leq \pi(n_0|n_1) \times \pi(n_1|n_2) \leq D_5 \pi(n_0|n_2) \tag{2.6}$$

whenever $n_0 < n_1 < n_2$, for some $0 < D_4, D_5 < \infty$.

2.2.2 Correlation lengths

We will assume throughout that $p > p_c$, as this is the only case we are interested in. The primary correlation length used in this note, describing connection probabilities, will be defined via the dual model: let z^* denote a site on the dual lattice and let $\tau^*_{0^*,z^*}(p)$ denote the probability of a dual connection between the dual origin and z^*, i.e. the event $\{0^* \overset{*}{\leftrightsquigarrow} z^*\}$. Finally, let $\tau^*_n(p)$ denote the maximum of such connection probabilities with $\|z^*\|$ $(= \|z^*\|_\infty)$ within a lattice spacing of n. Then, the correlation length $\xi(p)$ is defined by

$$\lim_{n\to\infty} [\tau^*_n(p)]^{\frac{1}{n}} = e^{-\frac{1}{\xi(p)}} \tag{2.7}$$

with $\xi = 0$ if $p = 1$. As is well known, the function ξ is continuous, monotone and divergent at $p = p_c$; the power of $p - p_c$ with which this function purportedly diverges "defines" the exponent ν. Further, for future reference, the functions $\tau^*_{0^*,z^*}$ obey the *a priori* bounds

$$\tau^*_{0^*,z^*}(p) \le e^{-\frac{\|z^*\|}{\xi(p)}}. \tag{2.8}$$

Another frequently used correlation length is the (quadratic) mean radius $\tilde{\xi}(p)$ of a finite cluster, defined by

$$\tilde{\xi}(p) = \left[\frac{1}{\mathbb{E}_p\big[|C(0)|; |C(0)| < \infty\big]} \sum_z \|z\|^2 \mathbb{P}_p\big(\{0 \rightsquigarrow z\} \text{ and } |C(0)| < \infty\big)\right]^{1/2}. \tag{2.9}$$

We shall also have use for an auxiliary correlation length – often called finite–size correlation length – which we will denote by $L(p)$; technically this depends on an additional parameter δ which will be notationally suppressed. In this note, the length $L(p)$ will be defined as the smallest 3×1 parallelogram – with the short angle being the angle of the rotation symmetry – such that the probability of an occupied crossing exceeds $1 - c\delta$. Here c is a particular constant of order unity and δ may be chosen arbitrarily in $(0, 1)$[7]. The key item is that if the above mentioned estimate on the crossing probability is satisfied then, upon tripling the length scale, the improved estimate becomes $1 - c\delta^3$, so that on further rescalings, crossing probabilities tend to one exponentially fast. In particular, for all n, all $p > p_c$, the probability of a dual crossing of an $n \times 3n$ parallelogram is bounded above by a constant times $e^{-n/L(p)}$, which implies that

$$P_\infty(p) \ge c_0\, \mathbb{P}_p(\{0 \rightsquigarrow \partial S_{L(p)}\}) \tag{2.10}$$

for some universal constant $0 < c_0 < \infty$.

It is noted that for length scales smaller than $L(p)$, crossing probabilities of these shorter and longer parallelograms are bounded above and below by strictly positive constants that depend only on the aspect ratio, as this is the situation at $p = p_c$ on *all* length scales. This is proved by a variant of the Russo–Seymour–Welsh theorem, see e.g. the relevant lemmas in [39] Ch. 6. Obviously, the same kind of bounds hold for dual crossings.

It was shown in [43] that $L(p)$ and $\tilde{\xi}(p)$ are uniformly bounded above and below by (δ dependent) multiples of one another, that is, in the notation of Kesten,

$$L(p) \asymp \tilde{\xi}(p).$$

It was mentioned in [12] that the relation $\xi(p) \asymp L(p)$ was known; however to the authors' knowledge, there is no published proof. In any case, at least for 2D percolation problems, this is

[7] Kesten proved in fact the following in [43]: for any (fixed) $\delta_1, \delta_2 \in (0, 1)$, we have $L(p, \delta_1) \asymp L(p, \delta_2)$.

not hard to show – we will provide the details in a short appendix – and thus all these correlation lengths are equivalent. To define the model we have a slight preference for ξ, which is continuous and monotone, but for proofs the length L will most often be more practical.

Finally, as alluded to above, concerning asymptotic issues, we will use Kesten's notations: for two positive functions f and g, $f \asymp g$ means that there exist two positive and finite constants C_a and C_b such that $C_a g \leq f \leq C_b g$ (so that their ratio is bounded away from 0 and $+\infty$), whereas $f \approx g$ means that $\log f / \log g \to 1$ ("logarithmic equivalence"). These items will refer to $p \to p_c$ or $N \to \infty$ depending on the context.

Kesten proved in [43] than the one–arm probability stays of the same order of magnitude if we do not go beyond the characteristic scale: more precisely,

$$\mathbb{P}_p(0 \rightsquigarrow \partial S_n) \asymp \mathbb{P}_{p_c}(0 \rightsquigarrow \partial S_n) \tag{2.11}$$

uniformly in p and $n \leq L(p)$. In particular, we can combine it with Eq.(2.10):

$$P_\infty(p) \asymp \mathbb{P}_p(\{0 \rightsquigarrow \partial S_{L(p)}\}) \asymp \mathbb{P}_{p_c}(\{0 \rightsquigarrow \partial S_{L(p)}\}). \tag{2.12}$$

This result (stated also in [43]) will prove to be very useful when dealing with small boxes throughout which the density parameter does not vary too much.

2.2.3 Description of the model

We let $\alpha : [0, +\infty) \to (0, 1 - p_c]$ denote the inverse function of ξ with argument of the increment above threshold:

$$\xi(p_c + \alpha(r)) = r. \tag{2.13}$$

Letting $w \in (0, 1)$, our inhomogeneous density will be defined by

$$p(z) := p_c + \varepsilon(r) = p_c + \alpha(r^w), \tag{2.14}$$

with $r = r(z) = \|z\|$. It is noted that this gives $\xi(p_c + \varepsilon(r)) = r^w$ which will be the starting point of our analyses. We will denote the corresponding measure by $\tilde{\mathbb{P}}_w$ and expectations therein by $\tilde{\mathbb{E}}_w$.

Remark 46. *It is noted that the formulation in Eq.(2.14) has the slight advantage over the informal description featured in the introduction that it is well-defined at* all *points of the lattice. Moreover, in cases such as the triangular site percolation model where a logarithmic form of scaling can be established, i.e.*

$$\xi(p) \approx |p - p_c|^{-\nu} \quad (p \downarrow p_c) \tag{2.15}$$

we make direct contact with the more informal description. Indeed using Eqs.(2.15), (2.14) and (2.13) we get

$$\nu = \lim_{r \to \infty} \frac{\log(\xi(p_c + \alpha(r^w))}{|\log \alpha(r^w)|} \frac{\log r}{\log r} = w \lim_{r \to \infty} \frac{\log r}{|\log \varepsilon(r)|} \tag{2.16}$$

i.e. $\log \varepsilon(r) / \log r \to -w/\nu = -\lambda$, *that is to say* $\varepsilon(r) \approx r^{-\lambda}$.

We will now consider the inhomogeneous model as described, and we will denote by Ψ_N the number of sites in S_N that belong to the infinite cluster, and by Φ_N the number of sites in S_N that are connected to the origin by a path lying entirely in S_N. We are ready for the statement of our main theorem:

Theorem 47. *Consider the quantity*

$$I_N := \sum_{z \in S_N} P_\infty\big(p_c + \varepsilon(r(z))\big). \tag{2.17}$$

Then

(i) We have $I_N \asymp N^2 \pi(N^w)$, and this quantity measures the size of Ψ_N and Φ_N: As $N \to \infty$,

$$\tilde{\mathbb{E}}_w(\Psi_N) \asymp \tilde{\mathbb{E}}_w(\Phi_N) \asymp I_N. \tag{2.18}$$

(ii) Furthermore, we have the variance estimate: for any $\epsilon > 0$,

$$\tilde{\mathbb{V}}_w(\Psi_N) \leq C_{2\epsilon} N^{2\epsilon} N^{2+2w} \pi^2(N^w), \tag{2.19}$$

so that $\tilde{\mathbb{V}}_w(\Psi_N) = o(I_N^2)$ and

$$\frac{\Psi_N}{\tilde{\mathbb{E}}_w(\Psi_N)} \longrightarrow 1 \quad \text{in } L^2. \tag{2.20}$$

Finally, conditionally on $\{0 \rightsquigarrow \infty\}$, these results hold for Φ_N as well.

Remark 48. *Under the assumption of scaling, if we write*

$$P_\infty(p) \approx (p - p_c)^\beta \tag{2.21}$$

then

$$I_N \asymp \int_{S_N} d^2 r \frac{1}{r^{\lambda\beta}} \asymp N^{2-\lambda\beta}. \tag{2.22}$$

A result along these lines can be stated for the triangular site model.

Corollary 49. *For the triangular site model (or any model where logarithmic scaling can be established), when $N \to \infty$,*

$$\tilde{\mathbb{E}}_w(\Psi_N) \approx N^{2-\lambda\beta}. \tag{2.23}$$

Remark 50. *We have*

$$\tilde{\mathbb{E}}_w(\Phi_N \mid \{0 \rightsquigarrow \partial S_N\}) \asymp \tilde{\mathbb{E}}_w(\Phi_N \mid \{0 \rightsquigarrow \infty\}) \asymp N^2 \pi(N^w). \tag{2.24}$$

For the usual Incipient Infinite Cluster (which is the analog of the limiting case as $w \to 1$), Kesten proved in [40] that

$$\tilde{\mathbb{E}}(\Phi_N \mid \{0 \rightsquigarrow \partial S_N\}) \asymp N^2 \pi(N) \quad \text{and} \quad \tilde{\mathbb{V}}(\Phi_N \mid \{0 \rightsquigarrow \partial S_N\}) \asymp N^4 \pi^2(N). \tag{2.25}$$

Our estimates thus imply in particular that the IIIC measures for $w \in (0,1)$ are all singular with respect to the IIC measure – and to each other as well – which was proved in [36]. Actually this result can be obtained more directly by noticing that for the IIC, there is a.s. only one arm going to infinity – two infinite paths have to intersect infinitely many times as can be shown from the BK inequality – while for the IIIC we can find as many disjoint arms as we want (constructed with overlapping parallelograms, as in the proofs below).

In the last section we will prove that if $\varepsilon(r) \approx \alpha(r/[\kappa \log\log r])$ there is a κ_c above which there is percolation and below which there is not. We will defer to Section 4 a precise statement of this result.

2.3 Proofs

The following, our key lemma, is an adaptation of the typical sorts of derivations to be found in [23], [41] and [43].

Lemma 51. *Let $\ell(r)$ be standing notation for $L(p_c + \varepsilon(r))$ and $S_\ell(z) = S_{\ell(r(z))}(z) = S_{\ell(\|z\|)}(z)$. Then for any z,*

$$\tilde{\mathbb{P}}_w(\{z \rightsquigarrow \infty\}) \geq c_1 \tilde{\mathbb{P}}_w(\{z \rightsquigarrow \partial S_\ell(z)\}). \tag{2.1}$$

Similarly, if $r(z) < N - \ell(r(z))$ then

$$\tilde{\mathbb{P}}_w(\{z \overset{S_N}{\rightsquigarrow} 0\}) \geq c_2 \tilde{\mathbb{P}}_w(\{z \rightsquigarrow \partial S_\ell(z)\}). \tag{2.2}$$

In the above, c_1 and c_2 are constants of order unity independent of z.

Remark 52. *Since the above are supplemented with the obvious complementary bounds, the event $\{z \rightsquigarrow \partial S_\ell(z)\}$ is, essentially, necessary and sufficient for z to join the relevant large scale IIIC. This is the sort of result that Kesten established in the uniform system and, in fact, analogous statements are anticipated for all low–dimensional critical systems. Note also that*

$$\ell(r) = L(p_c + \varepsilon(r)) \asymp \xi(p_c + \varepsilon(r)) = r^w \ (\ll r). \tag{2.3}$$

Proof. We will establish the above for all r sufficiently large but it is remarked that just how large is sufficient may depend on w. Let us start with the first case; here, for various reasons, it is worthwhile to know that the connection to infinity can be achieved by moving *outward* from the immediate vicinity of the point z. Consider the event $\mathcal{A}_\ell(z)$ that an occupied ring separates $\partial S_\ell(z)$ from $S_{\frac{1}{3}\ell(r(z))}(z)$. Once $\mathcal{A}_\ell(z)$ has occurred, with a few more parallelogram crossings, the separating circuit can be attached to a crossing of a $3\ell(r) \times \ell(r)$ parallelogram that is heading, more or less, in a direction away from the origin. We further intersect this with a few more crossings on a few more scales – each scale 3 times the previous one. The number of times we must do this, which is on the order of just a few and not dependent on r will be made precise momentarily; the relevant crossings are depicted in Figure 1.

Denoting the intersection of the annular event and the crossing events alluded to by $\mathcal{B}_\ell$ we have, by FKG,

$$\tilde{\mathbb{P}}_w(\{z \rightsquigarrow \partial S_\ell(z)\} \cap \mathcal{B}_\ell) \geq \tilde{\mathbb{P}}_w(\{z \rightsquigarrow \partial S_\ell(z)\}) \tilde{\mathbb{P}}_w(\mathcal{B}_\ell) \geq B \tilde{\mathbb{P}}_w(\{z \rightsquigarrow \partial S_\ell(z)\}) \tag{2.4}$$

where B is the probability of $\mathcal{B}_\ell$ at $p = p_c$. We remind the reader that this is a uniformly positive constant (obtained by "Russo–Seymour–Welsh theory" and a few more applications of the FKG inequality) that does not depend on the particular scale where the action is taking place.

Now, consider the situation at a distance $2r$ from the origin. Here, by Eq.(2.14) (the definition of $p(z)$), the local correlation length has grown to 2^w its size at the distance r. Let us estimate the finite–size correlation length. First we let c_3 and c_4 denote the constants by which the two correlation lengths may be compared:

$$c_3 L(p) \leq \xi(p) \leq c_4 L(p). \tag{2.5}$$

Then, it is seen that $L(p_c + \varepsilon(2r)) \leq 2^w c_4 c_3^{-1} L(p_c + \varepsilon(r)) = 2^w c_4 c_3^{-1} \ell(r)$, and it is clear that everywhere in the annular region $S_{2r}(0) \setminus S_r(0)$, the effective finite–size scaling correlation length is going to be uniformly smaller. The constant $2^w c_4 c_3^{-1}$ determines the scale of our initial cluster (which, we recall, is attached to the annular ring which in turn is connected to z). Having

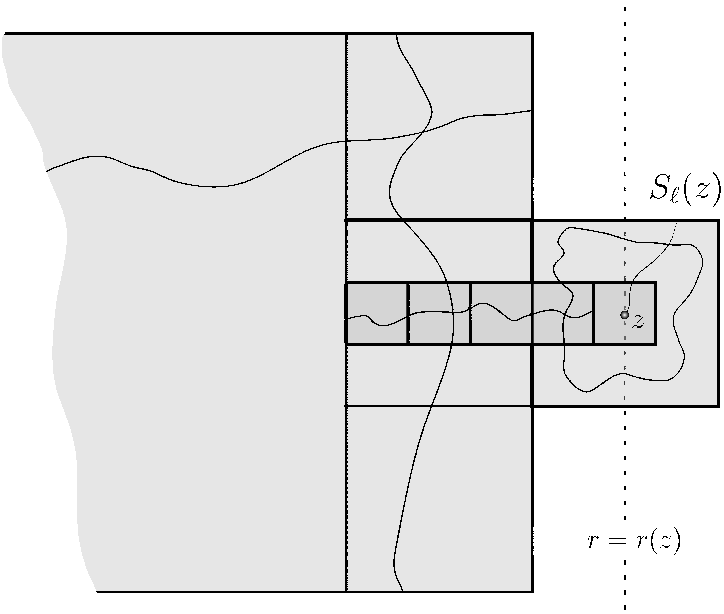


Figure 2.1: The event $\mathcal{A}_\ell(z)$ and a few subsidiary crossings which serve to attach z to a cluster with diameter moderately larger than the local correlation length. This cluster is, in turn, easily attached to points twice as far from the origin as z and, ultimately, to infinity.

achieved this scale we are beyond the correlation length as defined at the distance $2r$. Using $p(2r)$ as a bound for the density in the annular region, it is not of much cost to connect this cluster out to $\partial S_{2r}(0)$. This may be done, e.g. by a standard "rectangle rescaling program" – constructing overlapping crossings the kth of which has probability in excess of $1 - c\delta^{3^k}$ and whose scale is 3^k times that of the original aggregation. Note however, that we have to have taken r large enough so that $2^w c_4 c_3^{-1} \ell(r) \leq r$.

We have thus hooked the point z to a cluster that connects $\partial S_r(0)$ to $\partial S_{2r}(0)$ at an additional probabilistic cost, beyond what is in Eq.(2.4), of no more than $\prod_k (1 - c\delta^{3^k}) > 0$ – again using repeatedly the FKG inequality. The scale r cluster can now be directed to infinity by straightforward arguments (of a similar nature) which may be directly taken from [25] Theorem 2.

The second bound, Eq.(2.2), is proved in a similar fashion – actually easier because, æsthetics aside, we are *forced* to work inwards. The first few steps are identical: assuming that $\{z \rightsquigarrow \partial S_\ell(z)\}$ has occurred, we use the event $\mathcal{A}_\ell(z)$ and some more crossings to hook z up to a $3\ell(r) \times \ell(r)$ crossing – this time headed in the general direction of coordinate decrease. But now, agreeing to always head inwards, we may do a $\times 3$ rescaling program without apology since $p(z)$ is only getting bigger. Thus, we continue till we reach the boundary of $S_{r/2}(0)$, again at a cost of no more than $\prod_k (1 - c\delta^{3^k}) > 0$. With probability that is (stretched exponentially) close to one there is an occupied ring in $S_r(0) \setminus S_{r/2}(0)$; this may be obtained by summing Eq. (2.8) over both boundaries. Finally, with non–zero probability, the event $\{0 \rightsquigarrow \partial S_r(0)\}$ occurs and it is clear that the intersection of all these events produces the event $\{z \overset{S_N}{\rightsquigarrow} 0\}$. As before, we have made repeated use of FKG and it is noted that the probabilities of all the relevant events save $\{z \rightsquigarrow \partial S_\ell(z)\}$ are of order unity independent of z. □

Proof of Theorem 47. The above lemma proves almost completely the portion of Theorem 47

which concerns expectations. Indeed, for a uniform system,

$$P_\infty(p) \asymp \pi(L(p)) \tag{2.6}$$

by Eq.(2.12), so the summand in Eq.(2.17) may be replaced by $\pi(r^w(z))$ (as $\ell(r) \asymp r^w$ by Eq.(2.3)), and it is easily seen that $\pi(r^w(z)) \asymp \tilde{\mathbb{P}}_w(\{z \rightsquigarrow \partial S_\ell(z)\})$: indeed, as z' varies throughout $S_\ell(z)$, the local correlation length varies by a fractional amount which is only of the order $[r(z)]^{-(1-w)}$. So, we may as well estimate by the largest value of p within $S_\ell(z)$ and use the associated slightly smaller L. But then, using bounds as in Eq.(2.4) and Eq.(2.12), we get that $I_N \asymp \sum_{z\in S_N} \tilde{\mathbb{P}}_w(\{z \rightsquigarrow \partial S_\ell(z)\})$, which is our asymptotic expression for $\tilde{\mathbb{E}}_w(\Psi_N)$.

Before we dispense with $\tilde{\mathbb{E}}_w(\Phi_N)$ let us first verify the (asymptotic) evaluation of the quantity I_N. We already have that

$$I_N \asymp \sum_{z\in S_N} \pi(r^w(z)). \tag{2.7}$$

Let us take a logarithmic division of S_N: define $k = k(N)$ so that $2^k < N \le 2^{k+1}$, then

$$I_N \asymp \sum_{j\le k} (2^j)^2 \pi(2^{jw}) + \mathcal{E}(k) \tag{2.8}$$

where $\mathcal{E}(k)$ is no more than the order of $N^2\pi(N^w)$. In the above, we have used Eq.(2.4) on more than one occasion. Obviously, the purported principal term is at least of this order so there is no further need to consider $\mathcal{E}$. We pull out the leading term in the sum:

$$\sum_{j\le k} (2^j)^2 \pi(2^{jw}) \asymp 2^{2k}\pi(2^{kw}) \sum_{j\le k} 2^{2(j-k)} \frac{\pi(2^{jw})}{\pi(2^{kw})} \tag{2.9}$$

Now we use the fact that $\pi(2^{wk})/\pi(2^{wj}) \asymp \pi(2^{wj}|2^{wk})$ (using Eq.(2.6)) so that the coefficient of $2^{2k}\pi(2^{kw})$ (which is also at least as *large* as the order of unity because of the last term in the sum) is no more than

$$\tilde{c}_5 = \sum_{q=0}^{\infty} 2^{-q(2-w\mu)} < \infty \tag{2.10}$$

since μ is certainly less than two. It is obvious given Eq.(2.7) for I_N that $\tilde{\mathbb{E}}(\Phi_N)$ is (asymptotically) bounded above by I_N and below by $I_{\frac{N}{2}}$ which by now, are seen to be comparable to each other.

Let us turn now to the variance bound. We first note that we can write $\Psi_N = \sum_{x\in S_N} \mathbb{I}_{\{x\rightsquigarrow\infty\}}$, so that

$$\begin{aligned}\tilde{\mathbb{V}}_w(\Psi_N) &= \sum_{x,y\in S_N} \left[\tilde{\mathbb{P}}_w(\{x \rightsquigarrow \infty\}, \{y \rightsquigarrow \infty\}) - \tilde{\mathbb{P}}_w(\{x \rightsquigarrow \infty\})\tilde{\mathbb{P}}_w(\{y \rightsquigarrow \infty\})\right] \\ &= \sum_{x,y\in S_N} \left[\tilde{\mathbb{P}}_w(F_x \cap F_y) - \tilde{\mathbb{P}}_w(F_x)\tilde{\mathbb{P}}_w(F_y)\right]\end{aligned}$$

where we have used the notation $F_x = \{x \rightsquigarrow \infty\}$. Now recall that $\ell(r) \asymp r^w$ (Eq. (2.3)). As $w < 1$, we can find some $\epsilon > 0$ such that $w + \epsilon < 1$. We introduce the enhanced length $l(r) = \ell(r)r^\epsilon$ (which is still $\ll r$) and as above, we abbreviate $S_{l(\|x\|)}(x)$ by $S_l(x)$. We denote by

F'_x the event $\{x \rightsquigarrow \partial S_l(x)\}$. It is not hard to check that there is a $b > 0$ (independent of x) such that for $\|x\|$ sufficiently large,

$$\tilde{\mathbb{P}}_w(F_x \Delta F'_x) \leq e^{-\|x\|^b}. \tag{2.11}$$

We deduce, for n some small power of N, that

$$\tilde{\mathbb{V}}_w(\Psi_N) \leq 17C_0^2 n^4 + C_1 N^4 e^{-n^b} + 2 \sum_{x \in S_n, y \in S_N \setminus S_{3n}} \left[\tilde{\mathbb{P}}_w(F_x \cap F'_y) - \tilde{\mathbb{P}}_w(F_x)\tilde{\mathbb{P}}_w(F'_y) \right]$$
$$+ \sum_{x,y \in S_N \setminus S_n} \left[\tilde{\mathbb{P}}_w(F'_x \cap F'_y) - \tilde{\mathbb{P}}_w(F'_x)\tilde{\mathbb{P}}_w(F'_y) \right].$$

The first term serves to estimate the terms in which $\{x \in S_n, y \in S_{3n}\}$ or $\{x \in S_{3n}, y \in S_n\}$ where, as we recall, C_0 is the constant that figures into the volume of a box, and the reader is invited to verify the factor of 17. Whenever x is in S_n^c, we replace F_x with F'_x and similarly for y; the error incurred is accounted for in the second term (and we have assumed that n is large enough so that the bound in Eq.(2.11) is safely in effect). The last two terms are self–explanatory and will be dispensed with below.

Let us start with the first sum. For $y \in S_{3n}^c$ and $x \in S_n$, it is observed that, for n large enough, $S_l(y)$ is disjoint from S_n. Suppose that an occupied circuit surrounding $S_l(y)$ separates it from S_n. Now the event F'_y depends only on the configuration inside $S_l(y)$ while (conditioning on the innermost such ring) the event F_x depends only on the configuration outside and, perhaps, including the ring. I.e. given such a ring, the events F_x and F'_y are conditionally independent. The probability of F'_y is unchanged while the probability of F_x *and* the ring event is bounded above by $\tilde{\mathbb{P}}_w(F_x)$ alone. Thus we learn for $y \in S_{3n}^c$ and $x \in S_n$ that

$$\tilde{\mathbb{P}}_w(F_x \cap F'_y) - \tilde{\mathbb{P}}_w(F_x)\tilde{\mathbb{P}}_w(F'_y)$$
$$\leq \tilde{\mathbb{P}}_w(\{\text{no occupied circuit separates } S_n \text{ from } S_l(y)\}). \tag{2.12}$$

The right–hand side of Eq.(2.12) is bounded by another term of the order e^{-n^b} and we may thus absorb the entire first sum into the second error term at the expense of shifting the index of the constant.

We turn attention to the final term in the above written bound on the variance. If x and y are distant enough, $S_l(x)$ and $S_l(y)$ are disjoint, and the events F'_x and F'_y are independent. Now note that $l(\|x\|) \geq l(\|y\|)$ if $\|x\| \geq \|y\|$, so that $S_l(x) \cap S_l(y) = \varnothing$ if $\|x\| \geq \|y\|$ and $y \in S_{3l}^c(x)$. Hence,

$$\sum_{x,y \in S_N \setminus S_n} \left[\tilde{\mathbb{P}}_w(F'_x \cap F'_y) - \tilde{\mathbb{P}}_w(F'_x)\tilde{\mathbb{P}}_w(F'_y) \right]$$
$$\leq 2 \sum_{x,y \in S_N \setminus S_n, \|x\| \geq \|y\|} \left[\tilde{\mathbb{P}}_w(F'_x \cap F'_y) - \tilde{\mathbb{P}}_w(F'_x)\tilde{\mathbb{P}}_w(F'_y) \right]$$
$$\leq 2 \sum_{x \in S_N \setminus S_n} \sum_{y \in S_{3l}(x) \cap S_n^c} \left[\tilde{\mathbb{P}}_w(F'_x \cap F'_y) - \tilde{\mathbb{P}}_w(F'_x)\tilde{\mathbb{P}}_w(F'_y) \right]$$

We now have to estimate, for a site $x \in S_N \setminus S_n$, the sum

$$\Sigma_w(x) = \sum_{y \in S_{3l}(x) \cap S_n^c} \tilde{\mathbb{P}}_w(x \rightsquigarrow \partial S_l(x), y \rightsquigarrow \partial S_l(y)) \tag{2.13}$$

Note that for y inside $S_{3l}(x) \cap S_n^c$, the size $l(\|y\|)$ of the associated box does not vary too much and certainly (since $\|y\|$ is already larger than n) always satisfies $l(\|y\|) \geq \ell(\|x\|)$. Thus we have

$$\Sigma_w(x) \leq \sum_{y \in S_{3l}(x)} \tilde{\mathbb{P}}_w(x \rightsquigarrow \partial S_{\ell(\|x\|)}(x), y \rightsquigarrow \partial S_{\ell(\|x\|)}(y)) \tag{2.14}$$

We can then proceed by summing over concentric annuli centered on x cutting down even further on what we require in accord with $\|x-y\|$: take $k = k(x)$ such that $2^k < \ell(\|x\|) \leq 2^{k+1}$. If y is outside of $S_{2^{k+1}}(x)$, the two boxes $S_{2^k}(x)$ and $S_{2^k}(y)$ are disjoint. Hence, for these cases,

$$\begin{aligned}\tilde{\mathbb{P}}_w(x \rightsquigarrow \partial S_{\ell(\|x\|)}(x), y \rightsquigarrow \partial S_{\ell(\|x\|)}(y)) \\ \leq \tilde{\mathbb{P}}_w(x \rightsquigarrow \partial S_{2^k}(x))\tilde{\mathbb{P}}_w(y \rightsquigarrow \partial S_{2^k}(y)) \asymp \pi^2(2^k)\end{aligned} \tag{2.15}$$

and the number of such terms does not exceed the volume of $S_{3l}(x)$. Thus, the total contribution from these well–separated terms is bounded by $C_3 l^2(\|x\|)\pi^2(\ell(\|x\|))$ where C_3 is a constant not dissimilar from C_0.

Now if $y \in S_{2^{j+1}}(x) \setminus S_{2^j}(x)$ with $j \leq k-3$, we have by independence

$$\begin{aligned}&\tilde{\mathbb{P}}_w(x \rightsquigarrow \partial S_{\ell(\|x\|)}(x), y \rightsquigarrow \partial S_{\ell(\|x\|)}(y)) \\ &\qquad \leq \tilde{\mathbb{P}}_w(x \rightsquigarrow \partial S_{2^{j-1}}(x))\tilde{\mathbb{P}}_w(y \rightsquigarrow \partial S_{2^{j-1}}(y))\tilde{\mathbb{P}}_w(\partial S_{2^{j+2}}(x) \rightsquigarrow \partial S_{2^k}(x)) \\ &\qquad \leq C_4 \pi^2(2^{j-1})\pi(2^{j+2}|2^k) \\ &\qquad \leq C_5 \pi(2^j)\pi(2^k).\end{aligned}$$

using once again Eqs.(2.4) and (2.6). If $j \geq k-2$, we just drop the last term $\tilde{\mathbb{P}}_w(\partial S_{2^{j+2}}(x) \rightsquigarrow \partial S_{2^k}(x))$ in the first inequality: since in this case $\pi(2^j) \asymp \pi(2^k)$, the final inequality still holds. Hence, we must sum $\sum_{j \leq k}(2^j)^2\pi(2^j)\pi(2^k)$. This is identical to the previous argument: pulling out an overall factor of $[2^k\pi(2^k)]^2$, the resulting summand may be expressed as $[\pi(2^j \mid 2^k)2^{2(k-j)}]^{-1}$, and if we use the bound in Eq.(2.5) with $\mu < 2$, we see

$$\sum_{j \leq k} 2^{2j}\pi(2^j)\pi(2^k) \leq C_6[2^k\pi(2^k)]^2. \tag{2.16}$$

This is somewhat smaller than the contribution from the well–separated terms (Eq.(2.15)) so, overall,

$$\Sigma_w(x) \leq C_7 l^2(\|x\|)\pi^2(\ell(\|x\|)). \tag{2.17}$$

We finally sum on x to conclude

$$\begin{aligned}\tilde{\mathbb{V}}_w(\Psi_N) &\leq 17C_0^2 n^4 + C_2 N^4 e^{-n^b} + 2\sum_{x \in S_N \setminus S_n} \Sigma_w(x) \\ &\leq 17C_0^2 n^4 + C_2 N^4 e^{-n^b} + C_7 N^{2\epsilon} \sum_{j=1}^{k(N)} 2^{2j} 2^{2wj} \pi^2(2^{jw}).\end{aligned}$$

In the above, all indexed constants are numbers which are uniformly of order unity. As in previous arguments, we may bound the sum by a constant times $N^{2+2w}\pi^2(N^w)$ and, finally, we choose n a small enough power of N so that n^4 is relatively negligible – which will still easily diminishes the other "error term". Recalling that $I_N^2 \approx N^4\pi^2(N^w)$ – and that $w + \epsilon < 1$ – we have obtained the desired statement about $\tilde{\mathbb{V}}_w(\Psi_N)$.

Concerning Φ_N, although we have to be a bit more cautious, the proof remains essentially the same. Here we can write

$$\tilde{\mathbb{V}}_w(\Phi_N \mid \{0 \rightsquigarrow \infty\}) =$$
$$\sum_{x,y\in S_N}\left[\tilde{\mathbb{P}}_w(\{x \rightsquigarrow 0, y \rightsquigarrow 0\}|\{0 \rightsquigarrow \infty\}) - \tilde{\mathbb{P}}_w(\{x \rightsquigarrow 0\}|\{0 \rightsquigarrow \infty\})\tilde{\mathbb{P}}_w(\{y \rightsquigarrow 0\}|\{0 \rightsquigarrow \infty\})\right]$$
$$= \sum_{x,y\in S_N}\left[\frac{\tilde{\mathbb{P}}_w(F_x \cap F_y \cap \{0 \rightsquigarrow \infty\})}{\tilde{\mathbb{P}}_w(\{0 \rightsquigarrow \infty\})} - \frac{\tilde{\mathbb{P}}_w(F_x \cap \{0 \rightsquigarrow \infty\})\tilde{\mathbb{P}}_w(F_y \cap \{0 \rightsquigarrow \infty\})}{\tilde{\mathbb{P}}_w(\{0 \rightsquigarrow \infty\})^2}\right]$$

We again cut out a central portion at the cost of the order n^4 and we are left with two principal contributors the first of which is given by (twice) the sum with $x \in S_n$ and $y \in S_{3n}^c$. Here, using another argument involving a separating ring, the positive term is bounded as follows:

$$\tilde{\mathbb{P}}_w(F_x \cap F_y \mid \{0 \rightsquigarrow \infty\}) \leq \tilde{\mathbb{P}}_w(F_x \mid \{0 \rightsquigarrow \infty\})\tilde{\mathbb{P}}_w(F_y') + \mathcal{N}_R \tag{2.18}$$

where $\mathcal{N}_R$ is the "no ring" event described in Eq.(2.12). Meanwhile,

$$\tilde{\mathbb{P}}_w(F_y \mid \{0 \rightsquigarrow \infty\}) \geq \tilde{\mathbb{P}}_w(F_y)$$

so we are left with $\tilde{\mathbb{P}}_w(F_x \mid \{0 \rightsquigarrow \infty\})\tilde{\mathbb{P}}_w(F_y'\Delta F_y)$ plus the $\mathcal{N}_R$ term both of which are of the order e^{-n^b}.

We are left with the principal term and first off (at small cost) we replace the events F_x and F_y by the events F_x' and F_y'. Here in addition we will replace $\{0 \rightsquigarrow \infty\}$ by the event

$$F_0^{(x,y)} = \{0 \rightsquigarrow \infty \text{ outside of } S_l(x) \cup S_l(y)\} \quad (= \{0 \overset{[S_l(x)\cup S_l(y)]^c}{\rightsquigarrow} \infty\}). \tag{2.19}$$

It is not hard to see that the two events are very close. Indeed while ostensibly $F_0^{(x,y)} \supset \{0 \rightsquigarrow \infty\}$, in the event that $S_l(x)$ and $S_l(y)$ are both surrounded by occupied circuits which separate these boxes from the origin, the conditional probability is larger. But since we are well away from the origin these sorts of separating rings occur with probability close to one and we get an upper bound similar to that of Eq.(2.11) for $\tilde{\mathbb{P}}_w(F_0^{(x,y)}\Delta\{0 \rightsquigarrow \infty\})$. The remainder of the proof is essentially identical. □

2.4 A sharp transition

To treat the marginal case, we take

$$p(z) := p_c + \varepsilon(r) = p_c + \alpha(r/\kappa \log\log(r)) \tag{2.1}$$

with κ a constant and it is assumed that r is large enough so that all quantities are positive (otherwise, we set $p = 1$). We denote by $\tilde{\mathbb{P}}_{1,\kappa}$ the associated inhomogeneous probability measure. Note that this gives $\xi(p_c + \varepsilon(r)) = r/\kappa \log\log(r)$ which is hardly distinguishable from linear in r. Nevertheless, we will prove

Theorem 53. *For the 2D inhomogeneous percolation models defined via Eq.(2.1), there is a critical value $\kappa_c \in (0, +\infty)$ such that for $\kappa > \kappa_c$ there exists $\tilde{\mathbb{P}}_{1,\kappa}$– a.s. an infinite cluster, while for $\kappa < \kappa_c$ there is $\tilde{\mathbb{P}}_{1,\kappa}$– a.s. no infinite cluster.*

Proof. By monotonicity, it suffices to prove that there exists a value of κ for which the system percolates, and another value for which it does not percolate. We start with the percolative part.

Consider the crossing of any $3r \times r$ parallelogram that is situated so that the maximum distance form the origin is no more than Mr with M a (uniform) constant of order unity. Within this parallelogram, the lowest value of p estimates a uniform value for the density. This in turn provides a finite–size scaling correlation length which is smaller than $r/[q_1 \log\log r]$ for some constant, q_1, which is large if κ is large. By starting at *this* length scale, and instituting an $\times 3$ rescaling program till the scale of the $3r \times r$ is reached, it is seen that the probability of a crossing at the larger scale is at least $1 - Q_1 \delta^{q_1 \log\log r}$. Here Q_1 is a constant of order unity – perhaps small – but independent of r and κ. Writing δ as an exponential this bounds the probability of crossing the parallelogram at scale r below by $1 - Q_2/(\log r)^{q_2}$ where Q_2 is of order unity independent of r and κ and q_2 is large if κ is large.

We now consider a sequence of overlapping 3×1 parallelograms at a sequence of scales with each scale thrice the previous one. Here the sequence is such that the smallest scale is in the vicinity of the origin and the event of simultaneous crossings of all of them (or all but a finite number of them) implies the existence of an infinite cluster.[8] If the scale of the kth rectangle is simply a constant times 3^k, the $\tilde{\mathbb{P}}_{1,\kappa}$ probability of seeing all the crossings is bounded below by

$$g(\kappa) = \prod_k \left[1 - \frac{Q_3}{k^{q_2}}\right]$$

(with Q_3 another uniform constant) which is positive for all κ large enough. The quantity $g(\kappa)$ bounds the probability that the origin belongs to an infinite cluster, the a.s. existence of an infinite cluster follows from an application of the Borel–Cantelli lemma. It is remarked that by the consideration of large scale circuits – which are present even at $p = p_c$ – the infinite cluster is a.s. unique.

For the non–percolative result, when κ is small, we shall consider events in the annular regions $S_{3r} \setminus S_r$. Within this region, $L(p(z))$ is now uniformly larger than $r/[a_1 \log\log r]$ where a_1 is small if κ is small. This implies that the long–way crossings of 4×1 parallelograms occur with probability of order unity. These crossings may be stitched together, e.g. in a square–wave fashion, to construct a dual circuit in the annulus; see Figure 2. Using FKG, the probability of such a ring can be bounded below by $A_1/(\log r)^{a_2}$, where a_2 is small if κ is small. Once more looking at interlocking annuli at scales $\propto 3^k$, this translates into a probability $\propto k^{-a_3}$ where a_3 is small if κ is small. Divergence of $\sum_k k^{-a_3}$ implies the a.s. presence of ∞-ly many of these dual circuits and, therefore, $\tilde{\mathbb{P}}_{1,\kappa}$–a.s. no infinite cluster. □

2.5 Appendix

Here we provide the promised derivation that, in context of 2D percolation models of the sort described in Section 2, all correlation lengths are asymptotically equivalent. As the reader will note, the key is already in the proof of Theorem 53.

Proof that $L(p) \asymp \xi(p)$. Let us start by defining $R_{3,N}(p)$ to be the probability of a long–way crossing of a $3N \times N$ parallelogram and $D_{3,N} = 1 - R_{3,N}$ the probability of the complimentary

[8] E.g. in the "T" construction in [25], which the reader may wish to check, there are two rectangles at each scale; although one of them was 4×1 this was only for æsthetic reasons and, in any case, the above mentioned bounds on crossing probabilities are easily extended to parallelograms with any finite aspect ratio.

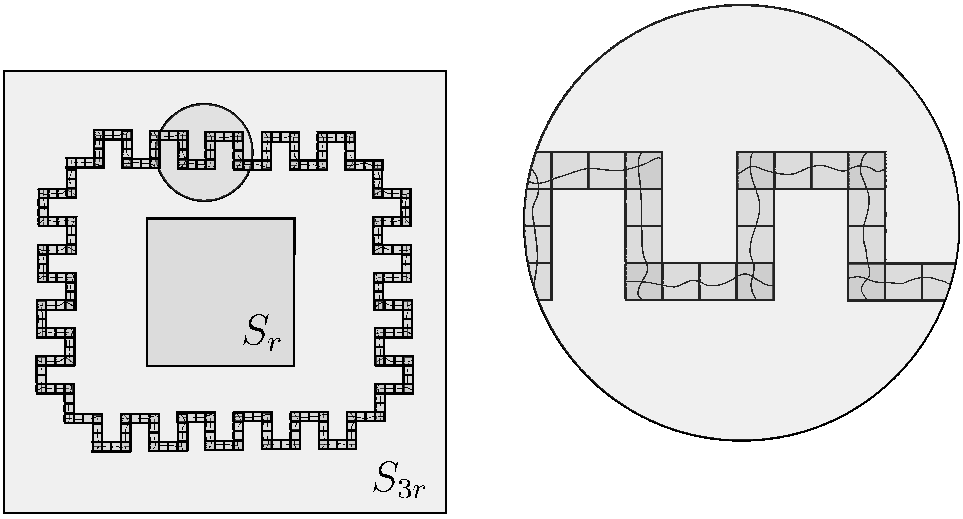


Figure 2.2: Construction of a dual circuit in the annulus $S_{3r} \setminus S_r$ for the borderline case. If κ is small, the probability of these circuits tends to zero with a small power of $\log r$ and percolation is prevented.

event, namely a short–way dual–crossing of this shape. First, it is claimed that

$$\lim_{N\to\infty} D_{3,N}^{\frac{1}{N}} = e^{-\frac{1}{\xi}}. \tag{2.1}$$

Indeed, $D_{3,N} \leq V_1 N^2 e^{-\frac{N}{\xi}}$ by the *a priori* bounds discussed in Eq.(2.8) where V_1 is a uniform constant (equal to 9 on the square lattice). On the other hand, we may obtain a lower bound for $D_{3,N}$ by just allowing the site at the center of the base to connect to its counterpart across the way. While, ostensibly, this would allow for paths to "leak out the ends", it is not hard to show that the probability of such a huge lateral excursion is as small as $e^{-\frac{3}{2}\frac{N}{\xi}}$ so, for all intents and purposes, $D_{3,N} \gtrsim \tau_N^*$ which establishes the limit. Using the $\times 3$ construction discussed at several points earlier in the text and using e.g. $\delta = e^{-1}$, we get that

$$R_{3,3^k L} \geq 1 - ce^{-3^k}. \tag{2.2}$$

Thus, for some sequence of N's, $D_{3,N} \leq ce^{-\frac{N}{L}}$ which implies $e^{-1/\xi} \leq e^{-1/L}$. Now consider the probability of a hard–way dual crossing of a 4×1 parallelogram of scale L' which is less than L but, say, larger than $\frac{1}{2}L$. This occurs with a probability of order unity independent of p (by Russo-Seymour-Welsh theory) and, as was just done in the last proof, by stitching together the order of N/L such rectangles the desired event is produced. Thus we have $D_{3,N} \geq e^{-\sigma\frac{N}{L}}$ for some constant σ (which is uniform in p) and hence $e^{-1/\xi} \geq e^{-\sigma/L}$. $\square$

Chapter 3

Asymmetry of near-critical percolation interfaces

Abstract: We study the possible scaling limits of percolation interfaces in two dimensions on the triangular lattice. When one lets the percolation parameter $p(N)$ vary with the size N of the box that one is considering, three possibilities arise in the large-scale limit. It is known that when $p(N)$ does not converge to $1/2$ fast enough, then the scaling limits are degenerate, whereas if $p(N) - 1/2$ goes to zero quickly, the scaling limits are SLE(6) as when $p = 1/2$. We study some properties of the (non-void) intermediate regime where the large scale behavior is neither SLE(6) nor degenerate. We prove that in this case, the law of any scaling limit is singular with respect to that of SLE(6), even if it is still supported on the set of curves with Hausdorff dimension equal to $7/4$.

This chapter is joint work with Wendelin Werner.

3.1 Introduction

A short general overview. The goal of the present paper is to study some aspects of two-dimensional percolation near its critical point. The mathematical understanding and description of the percolation phase transition in the plane has improved a lot during the last decade, thanks to two new main ingredients: the derivation of conformal invariance of critical percolation on the triangular lattice by Smirnov [69] and the description of the scaling limits of interfaces of critical percolation via Schramm's SLE(6) process introduced in [67]. This has allowed to identify the critical exponents that had been predicted in the physics literature via conformal field theory, Coulomb gas methods or quantum gravity. One important feature of the SLE approach compared to the methods developed by physicists is to view interfaces as global random geometric objects and to go beyond the "correlation functions". This is a little bit like defining the infinitesimal generator of a Markov process and studying its trajectories instead of just looking at its finite-dimensional distributions. One consequence of this new feature is the universality of certain random curves. For instance, one can prove that the outer boundaries of the scaling limits of percolation clusters, the outer boundaries of planar Brownian loops or the conjectured scaling limits of self-avoiding random walks/loops all have exactly the same distribution (see e.g. [51, 77]). Such a strong statement had not been formulated in the physics literature, where the corresponding models are just said to belong to the same "universality classes".

The derivation of the exponents of critical percolation enabled also to describe "near-critical" models, thanks to Kesten's scaling relations [43]. This allowed to prove various features of percolation interfaces on the large scale, when the percolation parameter is close (but not equal) to its critical value. This is roughly speaking due to the fact that in the regime under consideration, the "finite-dimensional distributions" of the interfaces are comparable to those of critical percolation. In the physics language, these near-critical models are also in the same universality class as critical percolation.

The goal of the present paper is to study the global geometry of these near-critical percolation interfaces. Our main result is that – despite the fact that their "finite-dimensional marginals" are comparable – the scaling limits of near-critical percolation interfaces do intrinsically differ from SLE(6) curves. By looking at any portion of a sample of such a curve, one can almost surely detect the difference from an SLE curve. The asymmetry of a near-critical model remains visible in the scaling limit. This shows the existence of other "universal" random curves than SLE. It is worth noticing that in most real-life occurrences of critical phenomena, the interfaces that one observes are these near-critical ones and not the "exactly critical" ones.

Background: critical percolation and SLE. We shall study site percolation on the triangular planar lattice. Recall that this can be viewed as a random coloring of the hexagonal cells of a honeycomb lattice, where the color (black or white) of each cell is chosen independently of the others: each of these cells has a probability p to be black and $1-p$ to be white, for some parameter p between 0 and 1. In percolation theory, one is interested in the connectivity properties of the set of black hexagons (or the set of white ones). They can be regrouped into maximal connected components called clusters. The phase transition for percolation on this lattice occurs at $p = 1/2$. Often, it is described mathematically as follows, in terms of almost sure properties of percolation in the infinite lattice: when $p < 1/2$, there exists with probability 1 no infinite cluster of black sites (subcritical regime) and an infinite cluster of white sites, and conversely when $p > 1/2$, there is an infinite cluster of black sites (supercritical regime) but no infinite cluster of white sites. In the critical case where $p = 1/2$, there exists neither an infinite white cluster, nor an infinite black cluster – but if one takes a finite large piece Λ of the lattice, one will see white and

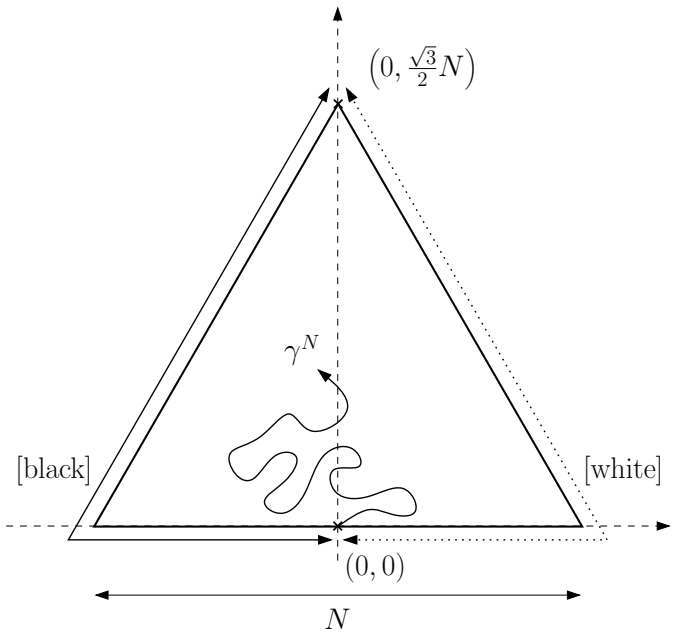


Figure 3.1: The triangle T_N and the interface γ^N (sketch).

black clusters of size comparable to that of Λ. See e.g. [35, 39] for an introduction to percolation.

A lot of progress has been made recently in the understanding of the large-scale behavior of critical percolation: in particular, Smirnov [69] proved conformal invariance of the connection probabilities, which allowed to make the link [69, 16, 70] with the Schramm-Loewner Evolution (SLE) with parameter 6 introduced in [67], and to use the SLE technology and computations [47, 48] to derive further properties of critical percolation, such as the value of some critical exponents, describing the asymptotic behavior of the probabilities of certain exceptional events (arm exponents) [50, 71]. We refer to [78] for a survey.

One precise relation to SLE goes as follows: we consider the large equilateral triangle T_N with even side length N on the triangular grid such that the middle of the bottom part is the origin and the top point is the point at distance $\sqrt{3}N/2$ above the origin. We decide to color all cells on the boundary of the triangle, in white if their x-coordinate is positive and in black if their x-coordinate is negative, and we perform critical percolation in the inside of T_N. Then, we consider the interface γ^N (viewed as a path on the hexagonal lattice dual to the triangular lattice) between the set of black sites attached to the left part of the triangle and the set of white sites connected to the right part of the triangle (see Figure 3.1). When $N \to \infty$, the law of the rescaled interface $\Gamma_N := \gamma^N/N$ converges (in an appropriate topology) to that of the SLE(6) process from $(0,0)$ to $(0, \sqrt{3}/2)$ in the equilateral triangle with unit side length. See [17, 70, 78] for details, and e.g. [46] for an introduction to SLE. Thanks to this convergence result, one is able to deduce properties of critical percolation from the properties of SLE. For instance (and we shall come back to this later), one can prove that the typical number of steps of the path γ^N is of the order $N^{7/4}$ (more precisely, for each $\epsilon > 0$, the probability that the number of steps is between $N^{7/4-\epsilon}$ and $N^{7/4+\epsilon}$ goes to 1 as $N \to \infty$).

Background: near-critical percolation. Understanding the behavior of critical percolation allows also to derive some properties of percolation when the parameter p is very close to 1/2. These are the scaling (and hyperscaling) relations that were first developed in the physics liter-

ature (the names of Fischer, Kadanoff and Wilson are often associated to these ideas), and later rigorously derived in the case of percolation by Kesten [43] (see also [55, 78]). Before explaining these relations in a little more detail, let us first make a trivial observation to help the newcomer to get a feeling of what goes on.

Recall that it is easy to couple realizations of percolation for all values of p on the same probability space in an increasing manner, in such a way that if a site is black for the realization w_p (that follows the law P_p of percolation with parameter p), then it is also black for all $w_{p'}$ with $p' > p$. To do this, one can for instance introduce an i.i.d. family of uniform random variables U_x on $[0,1]$, where x spans the set of all sites on the lattice, and declare that x is black for w_p if $U_x \leq p$ and white otherwise. For each given x, the probability that the color of x is not the same for w_p as for $w_{p'}$ is thus $|p - p'|$.

Suppose now that we consider the percolation interface in the large equilateral triangle as before, but choose $p = p(N)$ in such a way that $p - 1/2 = o(N^{-7/4-\epsilon})$. Then, one can couple it with a critical percolation interface (i.e. with the case when $p = 1/2$ exactly) in the same triangle in such a way that the two paths are identical except on an event of probability that goes to 0 as $N \to \infty$ (this is just because the probability that the color of one of the neighbors of the interface differs between w_p and $w_{1/2}$ is bounded by $|p - 1/2|$ times the number of neighbors of the interface – and we have just seen that this number is $o(N^{7/4+\epsilon})$). Hence, if one lets N go to infinity, the scaling limit of the interface is still SLE(6). In particular, the probability that the interface Γ_N hits the right side of the triangle before the left side of the triangle – let us call $R(p, N)$ this probability – goes to $1/2$ as $N \to \infty$.

On the other hand, for any fixed $\epsilon \in (0, 1/2)$, one can define

$$p^*(N) = p^*(N, \epsilon) := \inf\{p : R(p, N) > 1/2 + \epsilon\}.$$

For this choice of $p = p^*(N)$, if one looks at the possible limiting behavior of Γ_N, it is clear that the law can not be exactly SLE(6) anymore because it will hit the right side of the triangle before the left one with probability at least $1/2 + \epsilon$ (whereas this probability is $1/2$ for SLE(6)). It is therefore natural to ask what can happen to the scaling limit of this curve when $N \to \infty$ in this regime, and to see how it is related (or not) to SLE(6).

One can equivalently define the so-called correlation length $L(p) = L(p, \epsilon)$ in such a way that $p^*(L(p), \epsilon) \simeq p$. In other words, for $p > 1/2$,

$$L(p) = L(p, \epsilon) := \inf\{N : R(p, N) > 1/2 + \epsilon\}$$

(note that for $p > 1/2$ fixed, $R(p, N) \to 1$ as $N \to \infty$ so that $L(p) < \infty$). Kesten [43] has shown that it is possible to deduce from the arm exponents of critical percolation the behavior of $L(p)$ as $p \to 1/2$. This derivation relies on the "four-arm exponent": In the standard coupling, the event corresponding to $R(p, N)$ appears exactly when a "pivotal" site is flipped (we will discuss this is more detail in the next section). Combining Kesten's results with the exponents computed using SLE, one gets [71] that

$$L(p) = (p - 1/2)^{-4/3+o(1)}$$

when $p \to 1/2^+$, for any fixed choice of $\epsilon \in (0, 1/2)$. This is (see [43, 55, 78]) a crucial step in the rigorous proof of the fact that the "density" $\theta(p)$ of the infinite cluster for percolation with parameter p decays like $(p - 1/2)^{5/36+o(1)}$ as p decays to $1/2^+$.

We shall see (and this is quite easy) that in order to get a non-trivial limit for Γ_N (i.e. neither SLE(6) nor a path that just sticks to the boundary of T), one has to take $p(N)$ in such a way that N is of the order of the correlation length $L(p)$ i.e. that $p(N) \in [p^*(N, \epsilon), p^*(N, \epsilon')]$ for

some $\epsilon < \epsilon'$. The terminology to describe this regime depends on the circumstances and the papers: near-critical, off-critical, massive scaling. It is also very closely related to the scaling and hyperscaling relations. Anyway, it has been the subject of numerous and interesting works, see e.g. [1, 22, 13, 24] and the references therein.

One general rule behind the derivation of these results is that as long as $n \leq L(p)$, estimates remains roughly the same as at criticality. For example, the probabilities of existence of crossings of annuli for near-critical percolation are bounded by a constant times those at criticality (see more details in the next section). This implies that the exponents describing critical percolation also describe off-critical percolation and lead to Kesten's scaling relations. In [18, 19], Camia, Fontes and Newman have suggested that the scaling limit of the near-critical picture could be obtained from that of the critical picture by just opening a Poissonian family of "pivotal" points. We understand that [33] have succeeded in proving this fact. A consequence would be that the subsequential limits that we are discussing in the present paper are in fact limits, and that they are all related to each other by scaling. Let us also mention that near-critical percolation is closely related to the questions concerning the noise-sensitivity of percolation, such as studied in [68, 32].

Before describing the results of the present paper, let us just emphasize the relevance of such near-critical models for physical applications. Suppose for instance that one is considering a percolation model in a large box, and that one "heats" progressively the system i.e. that cells become progressively black one after the other and independently (this corresponds to increase p in the standard coupling described above). Then, we know that the percolation parameter at which a left-to-right crossing of the big box will appear is (with a large probability) close to the critical one. But, if one looks at it more precisely, it is easy to see that the value at which the crossing occurs will be $p^*(N, u)$ where u is a uniform random variable in $[-1/2, 1/2]$. Hence, at this moment, the picture and its fine properties are that of near-critical percolation. Another example where this near-critical picture appears spontaneously is that of gradient percolation, an inhomogeneous percolation model introduced in [63] and studied mathematically in [54].

Contributions of this paper. Our focus here will be on the scaling limits of near-critical interfaces. Here is a list of results that we shall derive in the present paper. Choose a sequence $(p(N), N \geq 1)$ and study the behavior of the law of Γ_N as $N \to \infty$:

- For any $\epsilon \in (0, 1/2)$, if we choose $p(N)$ in such a way that $p(N)$ is close to $p^*(N, \epsilon)$, then there exist subsequential limits for the law of Γ_N.
- These limiting laws are all singular with respect to the law of SLE(6).
- The only other possible subsequential limits for the laws of such curves Γ_N are SLE(6) itself (this is when $p(N)$ is closer to $1/2$) or a degenerate case where γ follows (the right part of) the boundary of the triangle (this is when $|p(N) - 1/2|$ is larger).
- The critical exponents associated to these non-degenerate scaling limits are the same as those of SLE(6). In particular, the Hausdorff dimension of the curves is almost surely $7/4$.

The first result is a rather direct application of the tightness arguments of Aizenman and Burchard [2] and the definition of the correlation length. The last two statements follow from the ideas of Kesten's paper [43]. Our main result is probably the second one, the fact that the laws of non-trivial scaling limits are singular with respect to SLE(6). One way to explain it is the following. In the off-critical regime, one sees on a macroscopic scale a difference between the law of the interface and that of the critical percolation interface (i.e. the non-critical interface is

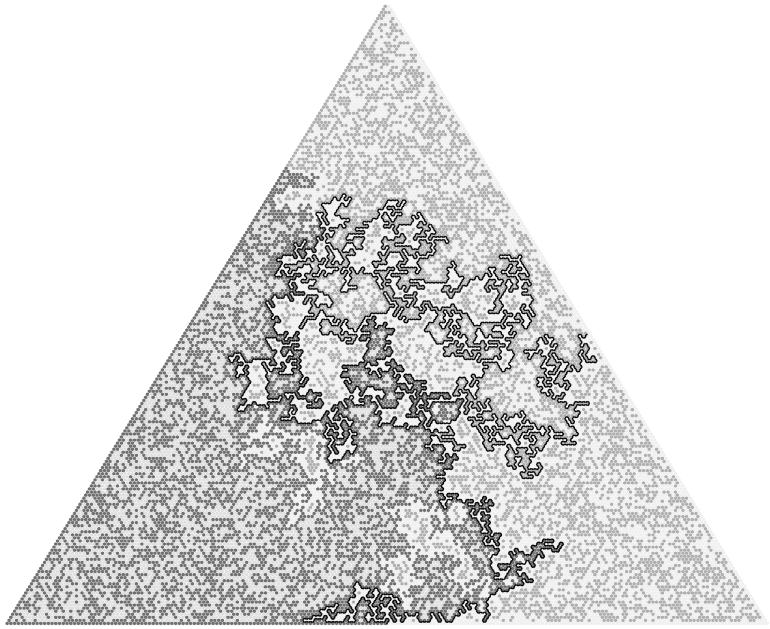


Figure 3.2: A coupling of a near-critical interface with critical percolation.

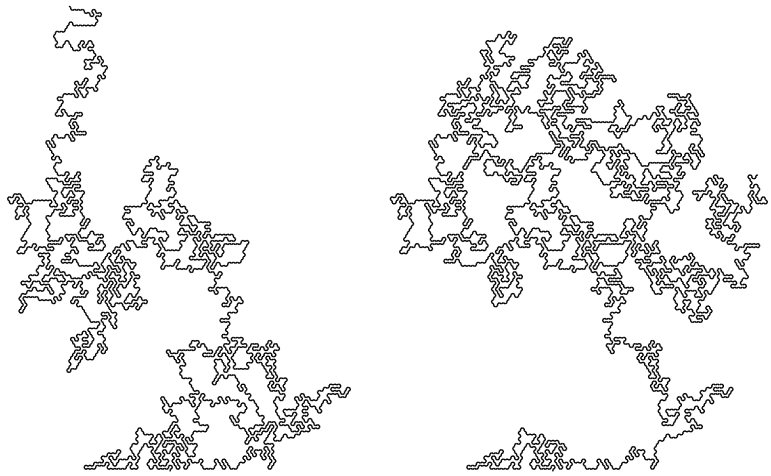


Figure 3.3: The interfaces at p_c and at $p^*(N, \epsilon)$ of Figure 3.2 depicted separately.

more to the "right" for instance). If one zooms in by a factor λ, one still sees a difference, but this difference tends to disappear, because one is not looking at a picture of size ca. $L(p)$ any more, but at a picture of size $L(p)/\lambda$. The question is whether this difference disappears sufficiently fast when $\lambda \to \infty$ or not. Note that one can show that (just as for the critical interface) the number of boxes of size N/λ visited by the path is of order $\lambda^{7/4}$ when λ is large (and N very large). Either this difference vanishes fast with λ and one is not able to almost surely detect a difference between the two macroscopic interfaces, or the difference between these two behaviors can be detected by averaging them out over the $\lambda^{7/4}$ parts of the path. In the end, one has to compare certain critical exponents to decide which scenario is correct and it turns out that it is the second one. A "flavor" of supercritical percolation is therefore still present in the scaling limit.

We would like to emphasize that we do not really use any complex analytic SLE technology to derive our results. Our considerations are based on percolation techniques, Kesten's scaling ideas and the knowledge of the exact value of the critical exponents (the derivation of which however used SLE).

3.2 Preliminaries

This paper will build on earlier results on near-critical percolation, and in particular on Kesten's paper [43]. In this section, we recall some results and ideas that we shall use. All these results are stated and derived in [55]. See also the last section of [78] for a rough survey. Throughout the paper, we will use the notation $\asymp$ to say that the ratio between two quantities remains bounded away from 0 and ∞. For each $p \in [0,1]$, P_p will denote the law of percolation with parameter p (i.e. where each cell/site is black with probability p).

Recall our definition of the correlation length

$$L(p) = L(p,\epsilon) = \inf\{N : R(p,N) > 1/2 + \epsilon\}.$$

Note that L is non-increasing with respect to p and non-decreasing with respect to ϵ.

We define the event $A^4(n)$ that there exist four disjoint paths of alternating colors (when ordered clockwise around the origin), each of them connecting a neighbor of the origin to the circle of radius n. Similarly, we define the event $A^2(n)$ (resp. $A^1(n)$) that there exist two paths of different colors (resp. one black path) from a neighbor of the origin to the circle of radius n. We will use the following properties, that hold for any fixed $\epsilon \in (0,1/2)$:

1. The Russo-Seymour-Welsh estimates remain valid below $L(p,\epsilon)$: for all $k \geq 1$, there exists $\delta_k(\epsilon) > 0$ such that for all p, for all $N \leq L(p,\epsilon)$, the probability (for P_p) that there is a black (resp. white) horizontal crossing of a $kN \times N$ parallelogram is at least δ_k.

2. Let $A^2(n_1,n_2)$ denote the event that there exist two arms of different colors joining the circles of radii n_1 and n_2. Then (this follows from the "arm-separation lemmas" that we will briefly describe at the end of this section),

$$P_p(A^2(n_1/2)) \times P_p(A^2(2n_1,n_2)) \asymp P_p(A^2(n_2))$$

uniformly for $p \geq 1/2$ and $2n_1 \leq n_2 \leq L(p,\epsilon)$ (this is known as the *quasi-multiplicativity* property). The same is true for four arms and one arm, *i.e.* with A^4 or A^1 instead of A^2.

3. It can be used to prove that

$$\sum_{j=1}^{n} jP_p(A^2(j)) \asymp n^2 P_p(A^2(n)),$$

uniformly for $p \geq 1/2$ and $n \leq L(p,\epsilon)$.

4. We have

$$P_p(A^2(n)) \asymp P_{1/2}(A^2(n)) \text{ and } P_p(A^4(n)) \asymp P_{1/2}(A^4(n))$$

uniformly for $p \geq 1/2$ and $n \leq L(p,\epsilon)$.

5. Finally,

$$(p-1/2)L(p,\epsilon)^2 P_{1/2}(A^4(L(p,\epsilon))) \asymp 1$$

as $p \to 1/2^+$.

The last property, which holds for each $\epsilon \in (0,1/2)$, implies in particular that for any fixed $\epsilon, \epsilon' \in (0,1/2)$,

$$p^*(N,\epsilon) - 1/2 \asymp p^*(N,\epsilon') - 1/2$$

as $N \to \infty$, and that

$$L(p,\epsilon) \asymp L(p,\epsilon')$$

as $p \to 1/2^+$. Note that the combination of items 2. and 4. shows that

$$P_p(A^2(n_1,n_2)) \asymp P_{1/2}(A^2(n_1,n_2))$$

uniformly for $p \geq 1/2$, $2n_1 \leq n_2 \leq L(p,\epsilon)$ (and similarly for A^4).

Recall also [71] (see also [78]) that the convergence of critical interfaces to SLE(6) imply that for any $\eta \in (0,1)$,

$$P_{1/2}(A^2(\eta n, n)) \to f_2(\eta) \text{ and } P_{1/2}(A^4(\eta n, n)) \to f_4(\eta)$$

as $n \to \infty$, where $f_2(\eta) \sim \eta^{1/4+o(1)}$ and $f_4(\eta) \sim \eta^{5/4+o(1)}$ as $\eta \to 0^+$, which in turn (using the properties of SLE(6)) implies that

$$P_{1/2}(A^2(n)) = n^{-1/4+o(1)} \text{ and } P_{1/2}(A^4(n)) = n^{-5/4+o(1)}$$

as $n \to \infty$. Analogous statements hold for any number of arms (see [55]).

Note that the correlation length introduced here differs slightly from the usual one – let us denote it here by $L^*(p,\epsilon)$ – that is defined in terms of crossing probabilities of rectangles or rhombi (the one introduced in [23], and used for instance in [43]). Using Russo-Seymour-Welsh considerations (and also the fact that for any $\epsilon, \epsilon' \in (0,1/2)$, $L^*(p,\epsilon) \asymp L^*(p,\epsilon')$ – see [43, 55]), these two definitions can easily be shown to be equivalent: for any $\epsilon \in (0,1/2)$, $L(p,\epsilon) \asymp L^*(p,\epsilon)$. We are thus allowed to use for L results established for L^*. Alternatively, we could have chosen to work with rhombi instead of triangles and then, we would have used directly the usual definition of correlation length. These two definitions are also known to be equivalent to other "natural" correlation lengths, describing for example the mean radius of a finite cluster or the rate of decay of connectivity properties (see e.g. the discussion in section 2.2 of [28]). Note also for future reference that L can be increased by at least any constant factor by choosing a larger ϵ' (still by Russo-Seymour-Welsh).

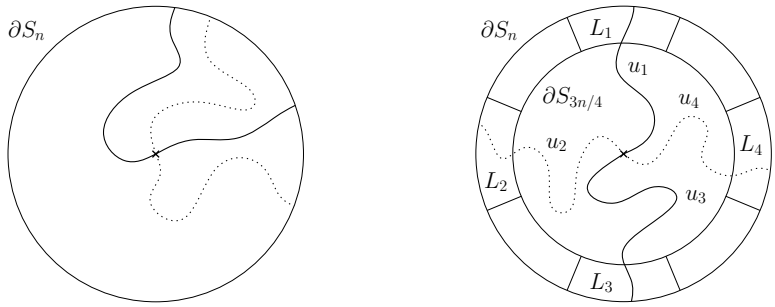


Figure 3.4: The events $A^4(n)$ and $\hat{A}^4(n)$ (sketch).

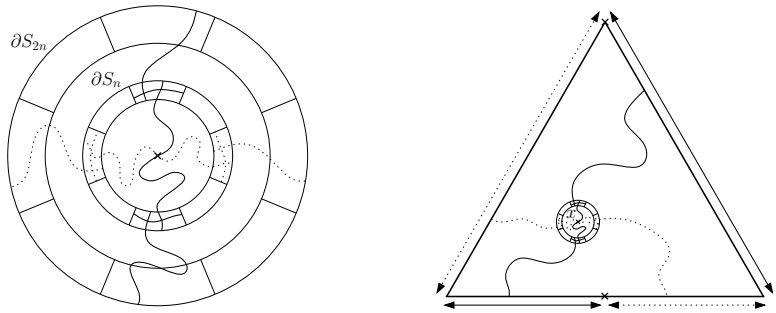


Figure 3.5: Extension of $\hat{A}^4(n)$ into $\hat{A}^4(2n)$, and the pivotal site for the crossing event.

An instrumental technique in the derivation of the quasi-multiplicativity result (and that we shall use in this paper at least three times) are Kesten's "separation lemmas". The precise statements are derived in detail in [55] (see also [78] for a survey); here is a hand-waving description of these ideas in the case of four arms (this is the case that is most relevant for us in the present paper). Suppose for instance that we divide the annulus $\mathcal{A}_n$ between the circles of radii $3n/4$ and n into 8 slices as depicted in Figure 3.4, and let L_1, L_2, L_3 and L_4 be four of them that are all at positive distance from each other. Consider percolation on the triangular grid restricted to the disc of radius n. Recall that $A^4(n)$ is the event that four disjoint arms $u_1, \dots, u_4$ of alternating color originate from the neighbors of the origin and reach the unit disc. We define the sub-event $\hat{A}^4(n)$ of $A^4(n)$ where we add the constraint that for $j = 1, \dots, 4$, $u_j \cap \mathcal{A}_n \subset L_j$ (see Figure 3.4). The separation statement states that (uniformly with respect to $n \leq L(p)$),

$$P_p(A^4(n)) \asymp P_p(\hat{A}^4(n)).$$

The events $\hat{A}^4$ have the nice feature that it is possible (with some care) to "extend" the arms further via Russo-Seymour-Welsh estimates: the conditional probability is positive and bounded from below. For instance (see Figure 3.5), one gets that

$$P_p(\hat{A}^4(2n)) \asymp P_p(\hat{A}^4(n)).$$

More generally, the same argument shows that with positive conditional probability bounded from below, it is possible to extend these four arms into disjoint paths that land on some prescribed boundary parts of a larger domain than the disc (if one considers the percolation in this larger domain).

Since we will be using this type of argument several times, let us briefly describe how this allows to deduce that

$$R(p,N) - R(1/2,N) \geq c(p-1/2) \times N^2 \times P_{1/2}(A^4(N))$$

uniformly for $N \leq L(p)$ and $p > 1/2$. Note that $R(p,N) - R(1/2,N)$ is exactly the probability that for some $p' \in [1/2,p]$, the color of some site x is changed exactly at p', and that at that moment, the configuration around x is that of four arms of alternating colors originating at x that join x to prescribed parts of the boundary of the triangle (as in Figure 3.5). This can happen for only one x and one p for each realization of the coupling. Hence, if we sum only over the sites x that are at distance at least $N/10$ of the boundary of T_N (and there are at least a constant times N^2 of them), use the separation lemmas and the fact that the four-arm probabilities are uniformly controlled for $n \leq L(p)$, we obtain the desired lower bound.

3.3 Tightness

For simplicity of presentation, we will stick to the setup that we described in the introduction, even if the shape of the considered domain (in our case, it is the triangle) could be chosen arbitrarily.

Let us describe this setup more precisely. We follow here the definitions of [2], and refer the reader to this paper for more details. We consider T the (filled) equilateral triangle of unit side length, with corners $(1/2,0)$, $(-1/2,0)$ and $(0,\sqrt{3}/2)$. The rescaled interfaces will be elements of $\mathcal{S}_T$, the space of curves in T: these are equivalence classes of continuous functions from $[0,1]$ to T, where two functions f_1 and f_2 represent the same curve if and only if there exists a continuous increasing bijection $\phi : [0,1] \to [0,1]$ such that $f_1 = f_2 \circ \phi$. We endow this space with the quotient metric:

$$\Delta(f_1,f_2) := \inf_{\phi} \left(\max_{[0,1]} |f_1 - f_2| \right)$$

where the infimum is over the set of increasing bijections ϕ from $[0,1]$ onto itself.

We call $P_{p,N}$ the law of the rescaled interface $\Gamma_N := \gamma^N/N$ of percolation with parameter p in our triangle (with mesh size $1/N$): this is a probability measure on $\mathcal{S}_T$. Endowed with the previous metric, $\mathcal{S}_T$ is a complete separable space, so that tightness and relative compactness are equivalent (by Prohorov's theorem).

Proposition 54. *The family $(P_{p,N}, p \in [0,1], N \geq 1)$ is relatively compact in the set of probability measures on $\mathcal{S}_T$.*

Proof. Let us consider a sequence (P_{p_k,N_k}) in this family. Our goal is to find a converging subsequence:

- By symmetry, we can assume that $p_k \geq 1/2$ for all $k \geq 1$.
- If N_k remains bounded along a subsequence, proving convergence of a subsequence of (P_{p_k,N_k}) is trivial.
- We can therefore restrict ourselves to the case where $N_k \to \infty$. Suppose that for all $\epsilon \in (0,1/2)$, $N_k/L(p_k,\epsilon) \to \infty$ along a subsequence; then it is easy to check (using Russo-Seymour-Welsh arguments) that along that subsequence, $\Gamma^k = \Gamma_{N_k}$ converges in law to the concatenation of the two parametrized segments $[0,1/2] \cup [1/2,\sqrt{3}i/2]$.

- We now suppose that for some $\epsilon \in (0, 1/2)$, $N_k/L(p_k, \epsilon)$ remains bounded. In particular, we get that for some fixed $\epsilon' > 0$, $N_k \leq L(p_k, \epsilon')$ for all large enough k (since one can increase L by at least any constant factor, by considering ϵ' close enough to $1/2$). We can then use the machinery developed by Aizenman and Burchard (Theorem 1.2 in [2]): Russo-Seymour-Welsh estimates hold uniformly for all p and $N \leq L(p, \epsilon')$, so that for any annulus $\mathcal{A}(x; r, R) = \{z \in T : r < |z - x| < R\}$,

$$P_{p_k}(\mathcal{A}(x; r, R) \text{ is traversed by } \Gamma_{N_k}) \leq P_{p_k}(A^1(rN_k, RN_k)) \leq C(r/R)^\alpha$$

  for two universal constants α, C (traversed means here that it visits the inner and the outer boundary of the annulus). The BK inequality (see e.g. [35]) then leads to the fact that

$$P_{p_k}(\mathcal{A}(x; r, R) \text{ is traversed } K \text{ times by } \Gamma_{N_k}) \leq C_K(r/R)^{\alpha K}$$

  which is exactly the hypothesis (H1) of [2] (uniform power bounds on the probability of multiple crossings in annuli).

  For the sake of completeness, let us just sketch why tightness can be derived from this. First, the hypothesis (H1) implies regularity properties (Theorem 1.1 in [2]) for the random curves Γ_{N_k}: for any $\beta > 0$, there is some Hölder continuity bound such that for each k, there is a probability at least $1-\beta$ that Γ_{N_k} can be parametrized and satisfy this bound. In particular, we can exhibit some equicontinuous set such that for each k, Γ_{N_k} belongs to this set with probability at least $1 - \beta$. Tightness follows (Theorem 1.2 in [2]) using (a slight adaptation of) Arzela-Ascoli's characterization of compactness for continuous functions on a compact set.

Hence, in all cases, one can find a converging subsequence of (P_{p_k,N_k}). □

3.4 Length and dimension of near-critical interfaces

In this section, we will study the non-degenerate case where $p(N) - 1/2$ is not too large: we assume that $p(N) \in [1/2, p^*(N, \epsilon)]$ for some fixed $\epsilon \in (0, 1/2)$ (note that this includes the critical case). We exhibit properties that near-critical interfaces share with critical ones. We will first focus on an estimate for discrete interfaces and then study the scaling limits.

3.4.1 Length of discrete interfaces

The length (number of edges) $\ell(\gamma^N)$ of the discrete interface γ^N in the near-critical regime is roughly the same as that of the critical interface:

Proposition 55. *Assume that $p(N) \in [1/2, p^*(N, \epsilon)]$. Then for any fixed $\beta > 0$,*

$$\lim_{N\to\infty} P_{p(N)}(\ell(\gamma^N) \notin [N^{7/4-\beta}, N^{7/4+\beta}]) = 0.$$

Proof. Note that the hypothesis on $p(N)$ implies in particular that $N \leq L(p(N), \epsilon)$, so that we can use the results on near-critical percolation that we have recalled in the preliminaries.

Let us first derive the upper bound for $\ell(\gamma^N)$: we estimate its expectation. We note that for any edge $x \in T_N$ on the dual hexagonal lattice, $x \in \gamma^N$ if and only if there exist two arms as depicted on Figure 3.6, one black joining a neighbor of x to the negative real half-axis, and one white joining a neighbor of x to the positive real half-axis.

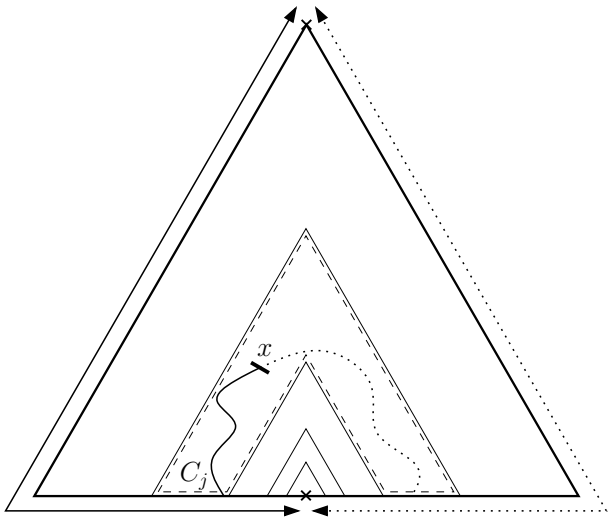


Figure 3.6: For an edge x, being on γ^N corresponds to the existence of two arms. For the lower bound, we consider semi-annuli $C_j = T_{3^j} \setminus T_{3^{j-1}}$ and we restrict to the edges at distance at least 3^{j-3} from the boundary of C_j (dashed).

In particular,

$$P_p(x \in \gamma^N) \leq P_p(A^2(d(x, \partial T_N))).$$

It allows us to get an upper bound on $E_p(\ell(\gamma^N))$: for each $\beta > 0$, and for some constants $c, c' \ldots$ (using the estimates on near-critical percolation and the fact that there are never more than $3N$ points at distance j from ∂T_N),

$$\begin{aligned} E_p(\ell(\gamma^N)) &\leq \sum_{x \in T_N} P_p(A^2(d(x, \partial T_N))) \\ &\leq c \sum_{x \in T_N} P_{1/2}(A^2(d(x, \partial T_N))) \\ &\leq c' \sum_{j=1}^{N} N P_{1/2}(A^2(j)) \\ &\leq c'' N \sum_{j=1}^{N} j^{-1/4+\beta/2} \\ &\leq c''' N^{7/4+\beta/2}. \end{aligned}$$

The upper bound for $\ell(\gamma^N)$ follows immediately by Markov's inequality.

For the lower bound, we use the standard second moment method. For that purpose, take l such that $3^l \leq N < 3^{l+1}$ and decompose the triangle T_N into concentric "semi-annuli" $C_j := T_{3^j} \setminus T_{3^{j-1}}$, $j \leq l$. Note that the curve γ^N has to cross each of these semi-annuli.

For any edge x in C_j at distance at least 3^{j-3} from the boundary of C_j, the results that we have recalled together with the separation lemmas (for two arms) imply that the probability $Q(x)$ that there exist two paths as before, from ∂x to the positive and negative real half-axes, that stay in C_j, satisfies

$$Q(x) \asymp P_{1/2}(A^2(3^j))$$

uniformly for such x and $3^j \leq L(p)$. Such an edge x (see Figure 3.6) is necessarily on γ^N by the previous remark. Let $\mathcal{N}_j$ denote the number of these edges: we have

$$E_p(\mathcal{N}_j) \asymp (3^j)^2 P_{1/2}(A^2(3^j)) \geq (3^j)^{7/4-\beta}$$

for large j (and any fixed $\beta > 0$).

Now, using the quasi-multiplicativity property and then item 3. we get

$$\begin{aligned} E_p(\mathcal{N}_j^2) &\leq \sum_{x,x'} P_p(A^2(d(x,x')/2))^2 P_p(A^2(2d(x,x'), 3^{j-3})) \\ &\leq cE_p(\mathcal{N}_j) \times \sum_{k=1}^{3^j} kP_p(A^2(k)) \\ &\leq c' E_p(\mathcal{N}_j)^2. \end{aligned}$$

It follows that (this is the standard "second-moment method")

$$P_p(\mathcal{N}_j \geq E_p(\mathcal{N}_j)/2) \geq \frac{E_p(\mathcal{N}_j 1_{\mathcal{N}_j \geq E_p(\mathcal{N}_j)/2})^2}{E_p((\mathcal{N}_j)^2)} \geq \frac{E_p(\mathcal{N}_j)^2/4}{E_p((\mathcal{N}_j)^2)} \geq \frac{1}{4c'}.$$

Consider some j_0: since the events $\{\mathcal{N}_{l-j} \geq E_p(\mathcal{N}_{l-j})/2\}$ for $j \in \{0, \ldots, j_0\}$ are independent, we get that

$$P_p(\mathcal{N}_{l-j} < E_p(\mathcal{N}_{l-j})/2 \text{ for all } j \leq j_0) \leq (1 - 1/4c')^{j_0+1}.$$

The lower bound follows readily because

$$\ell(\gamma^N) \geq \max(\mathcal{N}_l, \mathcal{N}_{l-1}, \ldots)$$

and our lower bound for $E_p(\mathcal{N}_j)$. □

3.4.2 Dimension of scaling limits

We now show that scaling limits of near-critical interfaces have the same Hausdorff dimension as in the critical regime:

Proposition 56. *Assume that the law of the curve γ is the weak limit of (P_{p_k,N_k}), with $N_k \to \infty$ and $p_k \in [1/2, p^*(N_k, \epsilon)]$. Then the Hausdorff dimension of γ is almost surely $7/4$.*

Similar statements for n-point correlation functions, multiple crossings, multifractral spectrum etc. could be derived in the same way. We leave this to the interested reader (see the informal discussion at the end of the paper) and we focus here only on this fractal dimension.

We might remark that the a priori estimates for the existence of two or three arms near a half-plane, or for the existence of five arms in the interior of the domain still hold (uniformly) for $n \leq L(p, \epsilon)$, as these are consequences of the Russo-Seymour-Welsh estimates only. Hence, exactly as in the convergence to SLE(6) case (e.g. [70, 78]), the discrete hitting probabilities converge to the continuous hitting probabilities.

Proof. The argument goes along similar lines as for the discrete length. For the upper bound, it suffices to prove that for all $\beta > 0$, the expected number of balls of radius η needed to cover γ is bounded by $\eta^{-7/4-\beta}$. But, we know that this expected number is bounded uniformly by this quantity for the discrete paths Γ_{N_k}, and so this also holds in the scaling limit.

For the lower bound, we decompose the triangle into concentric semi-annuli, and we exhibit a family $(\mathcal{C}_j, j \geq 1)$ of independent events of probability at least c_0, such that on each $\mathcal{C}_j$, the curve γ has dimension 7/4. Again, we follow exactly the same idea (with second moment estimates) as in the discrete case, using the fact (noticed in Section 3.2) that the discrete estimates for $P_p(A^2(n_1, n_2))$ hold uniformly. □

3.5 The alternative

We now prove that for the scaling limits of percolation interfaces, there are only three main options.

Proposition 57. *Suppose that P is the weak limit of a sequence (P_{p_k,N_k}) when $k \to \infty$, with $N_k \to \infty$ and $p_k \geq 1/2$. Then:*

- *either for each $\epsilon > 0$, $p_k \geq p^*(N_k, \epsilon)$ for all large k, and P is the Dirac mass on the concatenation of the two boundary segments $[0, 1/2]$ and $[1/2, i\sqrt{3}/2]$,*
- *or for each $\epsilon > 0$, $p_k \leq p^*(N_k, \epsilon)$ for all large k, and P is the law of SLE(6),*
- *or for some $0 < \epsilon' < \epsilon < 1/2$, $p^*(N_k, \epsilon') \leq p_k \leq p^*(N_k, \epsilon)$ for all large k.*

Note that these three cases are disjoint. In the last case, it is easy to check that under P, the law of the first point on $[1/2, i\sqrt{3}/2] \cup [-1/2, i\sqrt{3}/2]$ that the curve γ hits is neither supported on the bottom corners (this is because of $p_k \leq p^*(N_k, \epsilon)$) nor symmetric with repect to the imaginary axis (because $p_k \geq p^*(N_k, \epsilon')$).

Proof. Suppose now that P is the limiting law of a sequence $\Gamma^k := \Gamma_{N_k}$ defined under P_{p_k,N_k}, where $N_k \to \infty$ and $p_k \geq 1/2$.

- Let us first suppose that for each $\epsilon \in (0, 1/2)$, $N_k \geq L(p_k, \epsilon)$ for all sufficiently large k. This implies that for each fixed ϵ', $N_k/L(p_k, \epsilon') \to \infty$ (since we know that the ratio $L(p, \epsilon)/L(p, \epsilon')$ can be made arbitrarily large, by choosing ϵ very close to 1/2). It follows readily that the probability that there exists a white cluster of radius greater than any fixed positive u for P_{p_k,N_k} in the rescaled triangle goes to 0 as $k \to \infty$ and the P must be supported on the path that goes along the right boundary of the triangle.

- Let us now suppose that for each $\epsilon > 0$, $N_k \leq L(p_k, \epsilon)$ for all sufficiently large k. We assume that the law P is not the law of an SLE(6). The idea is to first translate this asumption in a lower bound for a certain four-arm event, and then using the facts recalled in section 3.2, to show that this leads to a contradiction.

  In the discrete setting, it is always possible to couple Γ^k (corresponding to P_{p_k}) with a realization of the interface $\tilde{\Gamma}^k$ for $P_{1/2}$ in such a way that the sites that are black for the realization of $P_{1/2}$ are also black for the realization of P_{p_k}. In this way, the path Γ^k is more "to the right" than $\tilde{\Gamma}^k$. Letting $k \to \infty$, we see that we can couple a realization Γ of P with an SLE(6) $\tilde{\Gamma}$ in the triangle T with the same property. We will denote by $\tilde{P}$ the probability measure under which these couplings are defined.

  It follows from our results on Hausdorff dimensions (in fact the uniform Russo-Seymour-Welsh bounds would suffice) that Γ is almost surely a continuous curve with zero Lebesgue measure. Clearly, one can divide the set of connected components of $T \setminus \Gamma$ into two parts. Intuitively speaking, those "to the right of Γ" and those "to the left of Γ". Let us call $\mathcal{R}(\Gamma)$

and $\mathcal{L}(\Gamma)$ the respective unions of these components. Similarly, we can define $\mathcal{R}(\tilde{\Gamma})$ and $\mathcal{L}(\tilde{\Gamma})$. Then, we have that $\mathcal{L}(\tilde{\Gamma}) \subset \mathcal{L}(\Gamma)$.

The fact that the laws of Γ and $\tilde{\Gamma}$ are not identical implies that with positive probability the two sets $\mathcal{L}(\tilde{\Gamma})$ and $\mathcal{L}(\Gamma)$ are not equal. It follows (using also the fact that Γ is a curve that does not backtrack on its own past) that for some $z \in T$ with rational coordinates and for some positive r, if we denote by $D(z,r)$ the disc centered at z with radius r,

$$\tilde{P}(D(z,r) \subset \mathcal{L}(\Gamma) \text{ and } D(z,r) \subset \mathcal{R}(\tilde{\Gamma})) > 0.$$

We can also choose r small enough so that the distance between z and the boundary of the triangle is at least $4r$. From now on, z and r will be fixed.

The convergence of the discrete interfaces to the continuous ones imply that for some positive constant c,

$$\tilde{P}(D(z,3r/4) \subset \mathcal{L}(\Gamma^k) \text{ and } D(z,3r/4) \subset \mathcal{R}(\tilde{\Gamma}^k)) > c$$

for any large k, with obvious notation. Note that these discrete events are independent of the state of the sites inside $N_k D(z,5r/8)$ as they imply that neither Γ^k nor $\tilde{\Gamma}^k$ intersect that disc. On the other hand, the uniform RSW estimates imply that the probability of the event $\mathcal{H}^k$ that there exist a white circuit in $N_k(D(z,5r/8) \setminus D(z,r/2))$ for P_{p_k} and a black circuit for $P_{1/2}$ in the same annulus (where both circuits disconnect z from $N_k \partial D(z,5r/8)$) is bounded from below (uniformly for all large enough k) by a positive constant. This event depends only on the state of the sites inside $N_k D(z,5r/8)$. Hence, we get that for some absolute positive constant c' and all sufficiently large k,

$$\tilde{P}(\mathcal{H}^k \text{ and } D(z,r/2) \subset \mathcal{L}(\Gamma^k) \text{ and } D(z,r/2) \subset \mathcal{R}(\tilde{\Gamma}^k)) > c'.$$

It follows that if one increases continuously p from $1/2$ to p_k (in the standard coupling), there will exist – with probability bounded from below – a value p at which the interface jumps from the left to the right of $D(z,r/2)$ without hitting it. For this value of p, there exist necessarily four arms of alternating colors originating at this site x that is flipped precisely at p. The different possibilities are depicted in Figure 3.7.

For each given site x on the grid T_{N_k}/N_k, let us evaluate the probability that this occurs. When x is not close to the boundary of the triangle, i.e. if it is at distance at least $r/2$, then we see that four arms of alternating colors must originate from x all the way to distance $r/2$ from x. The probability is then bounded by $P_p(A^4(2N_k/r)) \asymp P_{1/2}(A^4(N_k))$ (recall that r is fixed and that $p \leq p_k$).

When x is at distance $j/N_k < r/2$ from the boundary, then one must have four arms of alternating colors up to distance j/N_k, and then two arms that cross a semi-annulus (in a half-plane) from radius j/N_k to $r/2$. Combining this with quasi-multiplicativity and the known bounds (2 arms in a half-plane, 5 arms in the plane that give an upper-bound for the four-arm probability) leads to a bound of the type

$$cP_{1/2}(A^4(N_k)) \times (2j/rN_k) \times (j/N_k)^{-2+\beta}.$$

For any j, there are at most a constant times N_k points on our grid that are at distance exactly j/N_k from the boundary of the triangle. Hence, when we sum over all possible sites x in the triangle, we get that the infinitesimal probability that between p and $p+dp$ a site

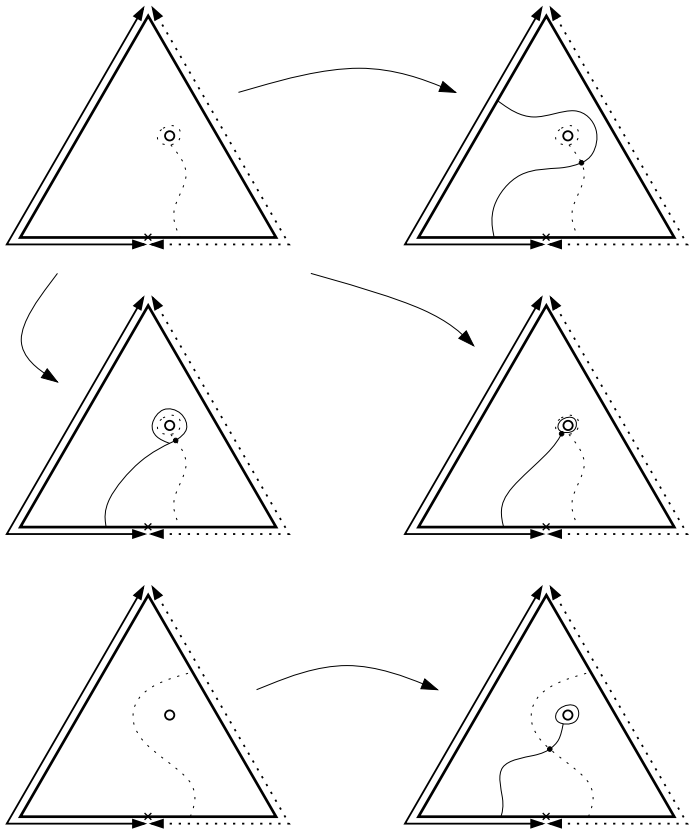


Figure 3.7: The different possibilities to jump from the left to the right of the small disc $D(z, r/2)$.

x is flipped in such a way that the curve jumps from the left to the right of the small disc is bounded by dp times

$$c' P_{1/2}(A^4(N_k))\left(N_k^2 + N_k \times N_k^{1-\beta} \sum_{j=1}^{N_k} j^{-1+\beta}\right) \leq c'' N_k^2 P_{1/2}(A^4(N_k))$$

(for some constants c', c'' that do not depend on k).

It finally follows that for some positive constant c,

$$(p_k - 1/2) \times N_k^2 \times P_{1/2}(A^4(N_k)) \geq c.$$

We now have shown that the fact that the scaling limit is not SLE(6) implies a lower bound for $p_k - 1/2$. But we know (this is the statement recalled at the end of Section 3.2) that

$$R(p_k, N_k) - R(1/2, N_k) \geq c'''(p_k - 1/2) \times N_k^2 \times P_{1/2}(A^4(N_k)).$$

for some constant c'''. Hence, we finally get that there exists a positive constant c'''' such that $R(p_k, N_k) \geq 1/2 + c''''$ for all sufficiently large k. In other words, $N_k \geq L(p_k, c'''')$ for all sufficiently large k. This contradicts our assumption.

- Finally to conclude the proof of the proposition, let us suppose that we are not in the last scenario described in its statement i.e. that it is not true that for some $0 < \epsilon' < \epsilon < 1/2$,

$p^*(N_k, \epsilon') \leq p_k \leq p^*(N_k, \epsilon)$ for all large k. Then it means that a subsequence of (p_k, N_k) falls into one of the two cases that we have just studied. Hence, the limiting probability measure is either the law of SLE(6) or the Dirac mass on the union of the two boundary segments.

□

3.6 Asymmetry of near-critical interfaces

In this section, we are going to assume that we are in the intermediate regime where the discrete percolation parameter p used at scale N satisfies

$$p(N) \in [p^*(N, \epsilon), p^*(N, \epsilon')]$$

for some fixed $0 < \epsilon < \epsilon' < 1/2$. The previous alternative tells us that these are the "non-trivial" cases in the large-scale limit. We are going to exhibit asymmetry properties that enable to detect the difference with a critical interface. We will first derive a result in the discrete setting, and then a result in the continuous setting for the scaling limit.

3.6.1 Discrete asymmetry

On the discrete level, it is possible to explore the interface dynamically: each time one discovers a new hexagon, one tosses a coin to decide if it is black or white, so that with probability p (resp. $1-p$), one makes a 60 degree turn to the right (resp. to the left). Of course, one often bumps into an already discovered hexagon, and in this case, the turn is already determined by a previous tossing. Hence, the percolation interface is the image under some map of a sequence of coin tosses. Let us call $\ell^+(\gamma^N)$ and $\ell^-(\gamma^N)$ the number of black (resp. white) cells neighboring the path γ^N.

Proposition 58. *Suppose that $p(N) \in [p^*(N, \epsilon), p^*(N, \epsilon')]$. Then for any fixed $\beta > 0$,*

$$\lim_{N\to\infty} P_{p(N)}(\ell^+(\gamma^N) - \ell^-(\gamma^N) \geq N^{1-\beta}) = 1.$$

On the other hand, for critical percolation,

$$\lim_{N\to\infty} P_{1/2}(\ell^+(\gamma^N) - \ell^-(\gamma^N) \leq N^{7/8+\beta}) = 1.$$

Proof. Define $v(N) = p(N) - 1/2$. Clearly the difference between the number of black neighbors discovered by γ^N and the number of white discovered neighbors evolves like a biased simple random walk with drift $2v$ (at each step the probability to add one is $1/2 + v$) stopped after a certain random number of steps ℓ that is roughly of the order $N^{7/4}$ (see Proposition 55). Hence, we see that this difference will be of the order

$$2v \times \ell + O(\sqrt{\ell})$$

(with high probability) i.e. more precisely that for all $\beta > 0$,

$$\lim_{N\to\infty} P_{p(N)}(\ell^+(\gamma^N) - \ell^-(\gamma^N) \geq 2v\ell - \ell^{1/2+\beta/2}) = 1.$$

In the off-critical case where $v(N) \geq N^{-3/4-\beta/2}$, we see that the drift term dominates because of Proposition 55 so that the first statement follows readily. Similarly, when $v = 0$, the result follows directly from Proposition 55. □

Hence, the discrete interface in the off-critical regime where $p(N) \geq p^*(N, \epsilon)$ will have (on a microscopical scale) anomalously more "black neighbors" than "white neighbors". Moreover, the proof shows that one can detect a good approximation of the actual value of $p(N)$ by looking at this difference because $(\ell^+(\gamma^N) - \ell^-(\gamma^N))/\ell$ will be close to $v(N)$.

But this result is not sufficient to deduce asymmetry of scaling limits. Indeed, if we had for instance taken

$$p(N) = 1/2 + N^{-b}$$

for some $b \in (3/4, 7/8)$, then an analogous discrete asymmetry result would hold i.e. for some small $\beta > 0$

$$\lim_{N\to\infty} P_{p(N)}(\ell^+(\gamma^N) - \ell^-(\gamma^N) \geq N^{7/8+\beta}) = 1,$$

but we have proved in the previous section that in this case, the interface still converges to SLE(6) as $N \to \infty$ (and we know that SLE(6) is symmetric).

3.6.2 Continuous asymmetry

The goal of this section is to prove the following result:

Proposition 59. *Suppose that γ is the limit (in law) of a sequence of Γ_{N_k}'s where $p_k \in [p^*(N_k, \epsilon), p^*(N_k, \epsilon')]$ (and $N_k \to \infty$). Then the law of γ is singular with respect to that of SLE(6).*

Proof. Let us fix a very small positive β. Define the rectangle $R = [-a, a] \times [0, 1/4]$ for a small fixed a. Define the segments $M = [-a, a] \times \{1/8\}$ and $B = [-a, a] \times \{1/4\}$, and the smaller triangle $T' = \{(x, y) \in T \ : \ y \geq 1/8\}$ (see Figure 3.8). For each small equilateral triangle t of size η (i.e. that one can obtain by translating ηT), we define analogously the sets r, m, b and t'. We say that the triangle t is *good* for γ, if at the first hitting time $\sigma(t)$ of $t \setminus r$, $\gamma(\sigma(t)) \in b$, and if γ does exit r through b after its first hitting time of m. When there is no ambiguity, we will simply write σ instead of $\sigma(t)$.

Let us consider the connected component d of $t' \setminus \gamma[0, \sigma]$ that has the top boundaries of t' on its boundary. We define the following points on the boundary of d: suppose that we go along the boundary of d clockwise starting from the bottom right corner of t' (that we call a_0). Then a_1 is the right-most point on $m \cap \gamma(0, \sigma)$ and a_2 is the left-most point on $m \cap \gamma(0, \sigma)$ (see Figure 3.9). We call ∂_1 (resp. ∂_2) the part of the boundary of d between a_0 and a_1 (resp. between a_2 and a_0).

When t is a good triangle, we define $F(\gamma, t)$ to be the probability given by Cardy's formula of the existence in d of a black crossing from ∂_1 to the part ∂_3 of the boundary between $\gamma(\sigma)$ and a_2. If after σ, γ does hit ∂_1 before ∂_2, we then say that t is *very good* for γ. If γ was an SLE(6), then $F(\gamma, t)$ would just be the conditional probability given $\gamma([0, \sigma])$ that t is very good for γ.

But here, the law of γ is the limit of the law of near-critical percolation interfaces $\Gamma^k = \Gamma_{N_k}$ defined under P_{p_k, N_k}, where $p_k \geq p^*(N_k, \epsilon)$ and $N_k \to \infty$. By Skorokhod's representation theorem, we can couple on the same probability space samples of all Γ^k's and γ in such a way that $\Gamma^k \to \gamma$ almost surely just as in the proof of convergence of critical percolation interfaces to SLE(6) (as in [78] for instance). For Γ^k, we can also check if a triangle t is good and very good. In fact, for each fixed triangle, the probability that its status for γ is not identical to its status for Γ^k goes to zero as $k \to \infty$. Indeed, if we suppose for instance that t is good for γ but not for Γ^k, then it means that a three-arm event occurs for γ and a priori bounds do exclude this.

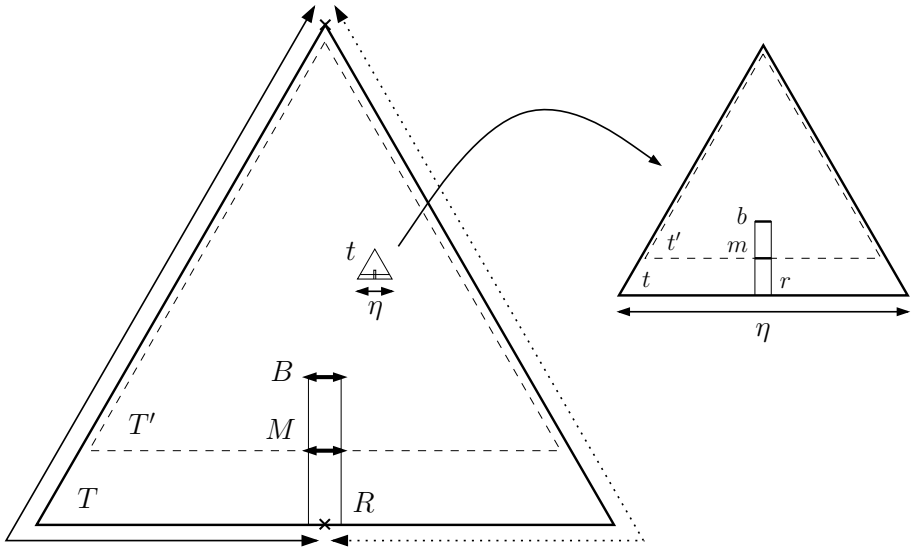


Figure 3.8: In the triangle T, we consider the subsets R, M, B and T'. In any small equilateral triangle t of size η, we define similarly the corresponding sets r, m, b and t'.

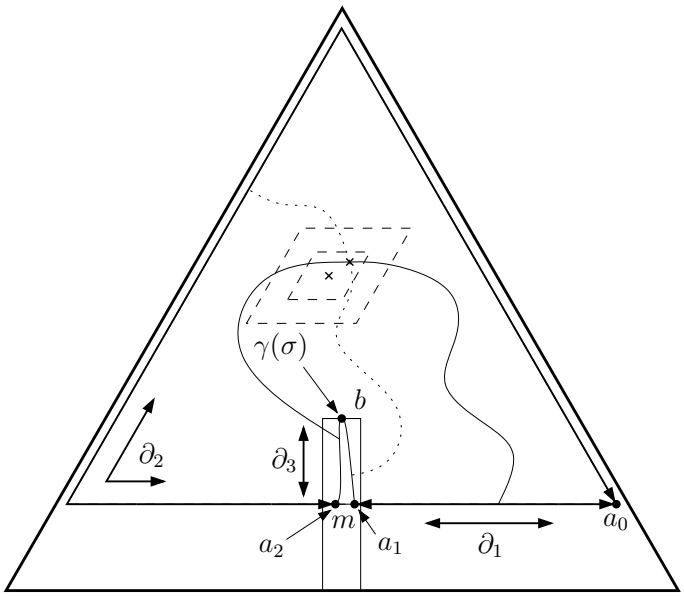


Figure 3.9: Using "pivotal" sites like the one depicted here provides a lower bound on the conditional probability that the small triangle t is very good.

If we define the stopping time σ^k associated to the triangle t and the path Γ^k, this means that when t is good, then $\Gamma^k[0,\sigma^k]$ converges almost surely to $\gamma[0,\sigma]$. So, the domains d^k (with obvious notation) converge in Carathéodory topology towards d almost surely.

We now couple each realization of Γ^k with a realization w^k of critical percolation on the same lattice, in such a way that the realization used to define Γ^k dominates w^k (*i.e.* its set of black sites is larger). Then (see e.g. [78]), it follows that the probability $F^k(\Gamma^k,t)$ that there exists a crossing from ∂_1^k to ∂_3^k in d^k for w^k (with natural notation) converges almost surely to $F(\gamma,t)$.

Suppose now that the triangle t is good for Γ^k and consider the configurations as depicted on Figure 3.9: we restrict ourselves to the case where there exist sites in a smaller rhombus of size comparable to ηN_k from which four arms of alternating colors originate. For each such site x, and conditionally on $\Gamma^k[0,\sigma^k]$, the probability that this four-arm configuration exists for percolation of parameter $p\in[1/2,p_k]$ in the domain not yet explored by Γ^k is bounded from below by a constant c times $P_{1/2}(A^4(\eta N_k))$, uniformly over p, k and the choice of p (this follows again from the uniform estimates for four arms, the separation lemmas and Russo-Seymour-Welsh). The probability that this site is flipped from white to black when one increases p from $1/2$ to p_k, and that at this value p the configuration is as described above, is therefore larger than

$$(p_k-1/2)\times c\times P_{1/2}(A^4(\eta N_k)).$$

Finally, note that this event can happen for only one site (when p increases, since the crossing event is increasing, there can exist only one value of p and therefore one site x such that the state of x is flipped exactly at p and x at that moment is pivotal). Hence, we can sum over all sites x in the small rhombus, and deduce that the conditional probability given $\Gamma^k[0,\sigma^k]$ that t is very good is bounded from below by

$$\begin{aligned}
&F^k(\Gamma^k,t)+(p_k-1/2)\times c'(\eta N_k)^2\times P_{1/2}(A^4(\eta N_k))\\
&\quad\geq\ F^k(\Gamma^k,t)+c''\eta^2\frac{P_{1/2}(A^4(\eta N_k))}{P_{1/2}(A^4(N_k))}\times(p_k-1/2)N_k^2P_{1/2}(A^4(N_k))\\
&\quad\geq\ F^k(\Gamma^k,t)+\frac{c'''\eta^2}{P_{1/2}(A^4(\eta N_k,N_k))}\\
&\quad\geq\ F^k(\Gamma^k,t)+\eta^{3/4+\beta}
\end{aligned}$$

(we have used the quasi-multiplicativity to estimate the ratio on the second line) for all small η and uniformly with respect to k and to the good configurations $\Gamma^k[0,\sigma^k]$. Letting $k\to\infty$, we conclude that for any good triangle t, the conditional probability given $\gamma[0,\sigma]$ that t is very good for γ is at least

$$F(\gamma,t)+\eta^{3/4+\beta}.$$

We now want to estimate the number of good triangles t. We divide the triangle T into $M(\eta)\asymp\eta^{-2}$ small deterministic triangles. Using the same method as when studying the dimension of γ (and the arm-separation ideas that allow to estimate the probability that a triangle is good i.e. to say that the probability for a triangle t to be good is comparable to the probability to be hit), we see that with probability going to 1 as $\eta\to 0$, the number of good triangles is at least $\eta^{-7/4+\beta}$.

We now count how many triangles are very good among the first $\eta^{-7/4+\beta}$ good ones, and we subtract the sum of the $F(t,\gamma)$ for these good triangles. If γ was an SLE(6), the obtained quantity $Z(\eta)$ would satisfy $E(Z)=0$ and $\mathrm{var}(Z)\leq\eta^{-7/4+\beta}$. But, in our case, we see that

$$E(Z)\geq\eta^{-7/4+\beta}\times\eta^{3/4+\beta}\geq\eta^{-1+2\beta}$$

and $\mathrm{var}(Z) \leq 4\eta^{-7/4+\beta}$.

If the sequence (η_l) has been chosen to converge sufficiently fast to zero, we see that almost surely (using Borel-Cantelli), for all sufficiently large l,

$$Z(\eta_l) \geq \eta_l^{-1+3\beta},$$

whereas if γ had been an SLE(6), we would almost surely have had

$$Z(\eta_l) \leq \eta_l^{-7/8-\beta}$$

for infinitely many l's.

We thus found an event that holds with probability one for γ, but with probability zero for SLE(6). □

3.7 Informal discussion

Seemingly, the fact that the laws of near-critical interfaces are singular with respect to the law of SLE(6) surprises some theoretical physicists who work on this topic. Recall that one important aspect of the SLE approach to critical systems was precisely to show that critical conformally invariant models in the same "universality class" give rise to exactly the same curves in the scaling limit. For near-critical models that are not strictly conformally invariant such as near-critical percolation here, this strong "universality" can fail to be true.

Suppose that a random curve γ is the scaling limit of a near-critical percolation interface. The tightness arguments, together with the estimates for existence of multiple arms indicate that the curve γ is almost surely a Loewner chain, and that it can be defined via its random driving function $(w(t), t \geq 0)$. It is of course a natural question to ask what this driving function could be (recall that in the case of SLE(6), $w(t/6)$ is a standard Brownian motion). Note that after each (fixed or stopping) time, we expect the curve γ to turn "more right" as what a SLE(6) would do in the same situation. This leads naturally to the conjecture that (if one extends properly the probability space) one can couple $(w(t), t \geq 0)$ with a standard Brownian motion $(\beta(t), t \geq 0)$ in such a way that $w(t) - \beta(6t)$ is an increasing continuous process adapted to the filtration generated by the processes $(\beta(6t))$ and $(w(t))$.

One may think that this contradicts the fact that the law of $(w(t/6), t \geq 0)$ is singular with respect to that of β (since the curves are generated by the driving functions, this is equivalent to say that the law of γ is singular with respect to that of SLE(6)) because the law of Brownian motion with drift is known to be absolutely continuous with respect to that of Brownian motion. But this last fact is only valid under some regularity properties for the drift, and in the present case, one can expect this drift to be quite complicated (its derivative measure might be supported on a set of exceptional times i.e. a fractal-type set).

We have seen that important properties (such as the dimension of the curve) are shared by the critical interfaces and the near-critical interfaces in the scaling limit, so that the technology based on conformal invariance of the critical model still provides the correct description of the near-critical interfaces in terms of exponents (this was used in [54] to describe the "percolation front"). In fact, the near-critical percolation technology allows to prove the following result (this is a direct consequence of the quasi-multiplicativity property and the fact that arm probabilities are comparable uniformly to the arm probabilities at $p = 1/2$): for any k, there exists a constant c_k, such that for any $z_1, \cdots, z_k$ in T, and for any sufficiently small ϵ, the ratio between the probability that γ visits the ϵ-neighborhood of these k points and the probability of the same

event for SLE(6) stays in $(c_k, 1/c_k)$ for all small ϵ and uniformly over the choices of $z_1, \ldots, z_k$. Loosely speaking, the "finite-dimensional marginals" of the law of γ are uniformly comparable to those of SLE(6) (even when the z_i's are very close to each other). Note that this is the type of functions that is computable via the Conformal Field Theory technology. But this result does not contradict the fact that the law of γ is singular with respect to that of SLE(6). For instance, the constant c_k does depend on the number of considered points. Our proof used the fact that we did not zoom on the behavior of the curve at just one point, but that we detected the asymmetry by looking at the mean of the behavior of the curve near to more and more points.

Chapter 4

Critical exponents of planar gradient percolation

Abstract: We study gradient percolation for site percolation on the triangular lattice. This is a percolation model where the percolation probability depends linearly on the location of the site. We prove the results predicted by physicists for this model. More precisely, we describe the fluctuations of the interfaces around their (straight) scaling limits, the expected and typical lengths of these interfaces. These results build on the recent results for critical percolation on this lattice by Smirnov, Lawler, Schramm and Werner, and on the scaling ideas developed by Kesten.

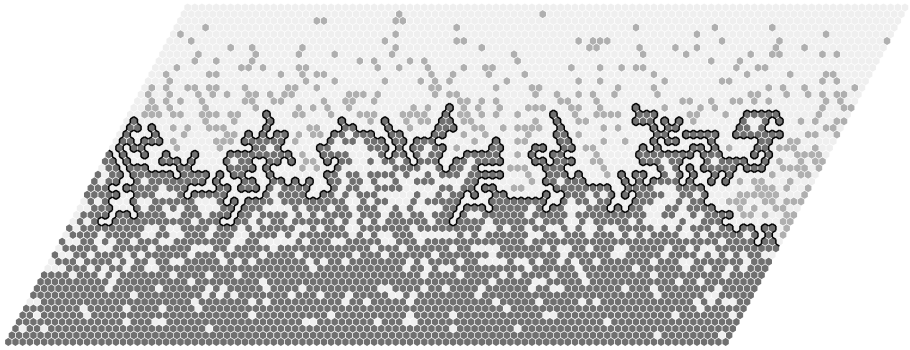


Figure 4.1: A simulation for $N = 50$ and $\ell_N = 100$.

4.1 Introduction

We study some aspects of "gradient percolation". This is a model of inhomogeneous site percolation where the probability for each site to be occupied varies along some fixed direction, for instance the y-axis in the plane. This model is described in the percolation survey paper [9]. It has been introduced by physicists (see [63, 29]) to model phenomena like diffusion or chemical etching, for example to describe the interface created by the welding of two pieces of metal. They argued that this is a case where some aspects of critical percolation can be empirically observed, without fine-tuning any parameter (this is a self-critical system – the critical phenomenon appears spontaneously).

Since we will restrict our study to the triangular planar lattice anyway, let us briefly describe the model in this particular case, even if it makes sense for other lattices and dimensions.

Recall that if one colors each cell of a honeycomb lattice independently in black or white with respective probability p and $1-p$, then when $p > 1/2$ there is an infinite connected component of black cells, and when $p < 1/2$ there is an infinite connected component of white cells. The value $1/2$ is called the critical probability of this homogeneous percolation model (and percolation at $p = 1/2$ is called critical percolation). This model has recently received a lot of attention, which has led among other things to the computation of "critical exponents" (see [71]).

Suppose now that a large integer N is given. We consider an inhomogeneous percolation model, where each cell z is colored in black and white independently, but with a probability that depends on z. More precisely, a cell z with y-coordinate equal to $y(z) \in [0, N]$ will be colored in black with probability $p(z) = y/N$ (when $y < 0$, we take $p(z) = 0$ and when $y > N$, we take $p(z) = 1$). It is then easy to see that there almost surely is a (unique) infinite black connected component (that contains the half space $\{y > N\}$) and a unique white connected component (containing the half-space $\{y < 0\}$). Furthermore, the upper boundary of this white cluster and the lower boundary of the black cluster coincide. This separating curve is called the "percolation front".

Intuitively, it is quite clear what happens when N is large. The percolation front will tend to be localized near the line $\{y/N = 1/2\}$. Furthermore, since at this level, the percolation is close to critical, the fine structure of the percolation front will be described in terms of critical percolation and its critical exponents.

Let us now briefly describe the main results of the present paper. Suppose that we restrict ourselves to a strip of height N and of length ℓ_N. Then, as $N \to \infty$ (provided ℓ_N goes to infinity too – not too fast but not too slowly – for instance $\ell_N = N$ is OK for what follows), one can

with high probability still define "the" percolation front separating the two "giant components". We call $\mathcal{R}_N$ the front (a precise definition will be given in Section 4.3), and T_N its length (*i.e.* number of steps). We shall see that for each positive δ, when $N \to \infty$, and with high probability:

- The front will remain in the strip of width $N^{4/7+\delta}$ near the line $\{y = N/2\}$.
- The front will not remain in the strip of width $N^{4/7-\delta}$ near the line $\{y = N/2\}$.
- The expected length $t_N = \mathbb{E}[T_N]$ of the front satisfies $N^{3/7-\delta}\ell_N \leq t_N \leq N^{3/7+\delta}\ell_N$.
- T_N is close to its expected value, *i.e.* T_N/t_N is close to 1.

The localization of the front near the line $\{y = N/2\}$ and the value of the two exponents (4/7 for the width and 3/7 for the length) had been predicted by Sapoval et al. in [63, 29].

Our proofs build on the following mathematical results and ideas: Kesten's scaling relations [43], Smirnov's conformal invariance result [69], and the computation of the critical exponents for SLE_6 by Lawler, Schramm and Werner [48]. More precisely, we shall develop further some ideas introduced in the first paper [43], and use directly results of [71], where Smirnov and Werner showed how to derive the critical exponents for percolation from those for SLE_6. We will not mention SLE_6 in our proofs but we would like to stress that it plays a crucial role in the derivation of the results of [71] that we will use.

In fact, gradient percolation is rather easy to simulate. It has turned out to be an efficient practical tool to obtain numerical estimates for the critical probability of percolation on various lattices (by using the mean height of the front in order to approximate p_c, see e.g. [59, 79]), for instance the square lattice, and it has also been one of the first ways to get numerical evidence for values of the critical exponents of standard percolation (that then supported the conjectures based on Coulomb gas and conformal field theory).

Let us stress the fact that the anisotropy of this model yields that despite the fact that the front converges to a straight line in the fine mesh limit (*i.e.* a curve of dimension one), its length for a lattice approximation of mesh-size δ (in a rhombus) behaves roughly like $\delta^{-3/7}$ (*i.e.* it has $\delta^{-10/7}$ steps). One can expect to observe a non-trivial limit – of fractal dimension 7/4 – with an appropriate scaling (in $N^{4/7}$) of the axes, but the critical exponents obtained do not correspond directly to a fractal dimension for the limiting object.

4.2 Homogeneous percolation preliminaries

We recall in this section some known facts concerning percolation (not gradient percolation) that we will use later on.

4.2.1 Setting

The setting in this paper will be site percolation in two dimensions on the triangular lattice. We will represent it as usual as a random (black or white) coloring of the faces of the hexagonal lattice. It has been proved by Harry Kesten [38] and it is by now a classical fact that for this model, the critical probability is $p_c = 1/2$ and that there is almost surely no infinite cluster when the percolation parameter is taken to be 1/2. The reason why we focus here on this lattice is that it is (at present) the only one for which conformal invariance in the scaling limit has been proved (Smirnov [69]). Conformal invariance will not be used directly in the following, but this property combined with the study of SLE by Lawler, Schramm and Werner [47, 48] allows the rigorous computation of the so-called "critical exponents" that will be instrumental in our

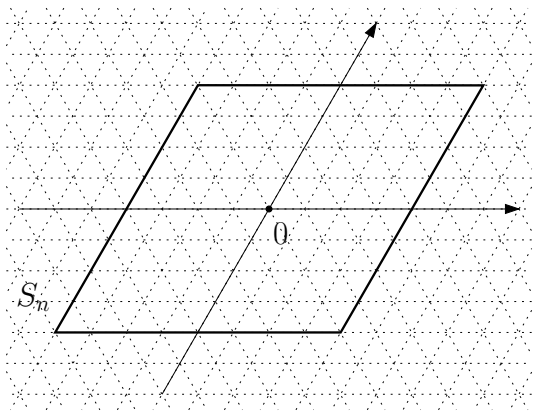


Figure 4.2: The triangular lattice, its associated basis and S_n.

considerations. On other regular lattices like the square lattice $\mathbb{Z}^2$, these exponents are expected to be the same, but no rigorous derivation exists yet. However, some inequalities can be proved, that would imply weaker but nevertheless interesting statements. We will not develop this aspect in the present paper.

The percolation parameter will be denoted by p: each site is *occupied* (or black) with probability p, and *vacant* (white) otherwise, independently of each other. The corresponding probability measure on the set of configurations will be referred to as $\mathbb{P}_p$, and $\mathbb{E}_p$ will be the expectation.

We will use oblique coordinates, with the origin in 0 and the basis given by 1 and $e^{i\pi/3}$ (in complex notation). The parallelogram R of corners $a_j + b_k e^{i\pi/3}$ $(j, k = 1, 2)$ will thus be denoted by $[a_1, a_2] \times [b_1, b_2]$. Its interior and its boundary will be denoted by $\mathring{R}$ and ∂R, respectively. We will consider $\|z\|$ the infinity norm of a site z as measured with respect to the chosen coordinates, and $d(z, z') := \|z - z'\|$ the associated distance. For a site $z = (z_1, z_2)$, we will often use the rhombus $S_n(z) := [z_1 - n, z_1 + n] \times [z_2 - n, z_2 + n]$, which is the set of sites at distance at most n from z (its interior $\mathring{S}_n(z)$ and its boundary $\partial S_n(z)$ consisting of the sites at distance strictly less than n and exactly n, respectively). We will refer to $S_n(0)$ simply as S_n, and call it the "box of size n".

For two positive functions f and g, $f \asymp g$ means that there exist two positive and finite constants C_1 and C_2 such that $C_1 g \leq f \leq C_2 g$ (so that their ratio is bounded away from 0 and $+\infty$), and $f \approx g$ means that $\log f / \log g \to 1$ (when $p \to 1/2$ or when $n \to \infty$, which will be clear from the context).

We now recall some relevant results on critical percolation that we will use in the present paper.

4.2.2 Arm exponents

We first briefly recall some facts concerning critical exponents for the existence of a certain number of "arms". These exponents describe the asymptotic behavior of the probability of certain exceptional events.

Let us consider a fixed integer $j \geq 2$. For each positive integers $m \leq n$, define the event $A^j(m, n)$ that there exist j disjoint monochromatic paths from ∂S_m to ∂S_n that are not all of the same color (each path is either completely black/occupied or completely white/vacant, and there is at least one vacant path, and one occupied path). As noticed in [3, 71], we could also

prescribe the cyclic order of the paths without changing the results that we state below.

Combining the property of conformal invariance in the scaling limit (see [69, 16]) with the study of SLE made by Lawler, Schramm and Werner [48], it has been proved that:

Proposition 60 ([71]). *For all fixed $j \geq 2$, $m \geq j$,*

$$\mathbb{P}_{1/2}(A^j(m,n)) \approx n^{-(j^2-1)/12} \tag{4.1}$$

when $n \to \infty$.

In fact, we will use this result only for $j = 2, 3$ and 4. Let us just remark that to derive these three exponents, it might be possible to bypass the use of the rather involved results of Camia and Newman [16] as these are exponents related to outer boundaries of clusters, so that "universality"-based ideas might just be enough.

The value of the related "one-arm" exponent is 5/48 and has been derived in [50], but we shall not need it in the present paper.

4.2.3 Behavior near criticality

In the seminal paper [43], Kesten showed how the behavior of percolation at its critical point was related to the asymptotic behavior of percolation near its critical point, and derived the so-called scaling relations that link some of the previous arm exponents to other critical exponents describing the behavior of connectivity probabilities near $p = p_c$.

A key idea in his article is to use a certain "characteristic length" $L(p)$ defined in terms of crossing probabilities (sometimes referred to as "sponge-crossing probabilities"). This idea (or adaptations of it) was also used in papers concerning finite-size scaling, e.g. [23, 24, 12, 13].

Let us introduce some more notation. We will denote by $\mathcal{C}_H([a_1,a_2]\times[b_1,b_2])$ (resp. $\mathcal{C}_V([a_1,a_2]\times[b_1,b_2])$) the event that there exists a horizontal (resp. vertical) occupied crossing of the parallelogram $[a_1,a_2]\times[b_1,b_2]$, and by $\mathcal{C}_H^*$, $\mathcal{C}_V^*$ the same events with vacant crossings. However, we will use for convenience a definition slightly different from the usual one. First, we decide to relax the condition on the boundary sites: a horizontal occupied crossing of a parallelogram R will be a path connecting its left and right sides, all the sites of which *except its two extremities* are in $\mathring{R}$ and occupied. The events $\mathcal{C}_H(R)$ and $\mathcal{C}_H(R')$ will thus be independent if R and R' share one side. We also allow such a crossing to incorporate "dangling ends": it is not necessarily a self-avoiding path.

We have for example

$$\mathbb{P}_p(\mathcal{C}_H([a_1,a_2]\times[b_1,b_2])) = 1 - \mathbb{P}_p(\mathcal{C}_V^*([a_1,a_2]\times[b_1,b_2])). \tag{4.2}$$

Consider now rhombi $[0,n]\times[0,n]$. At $p = 1/2$, $\mathbb{P}_p(\mathcal{C}_H([0,n]\times[0,n])) = 1/2$. When $p < 1/2$ (sub-critical regime), this probability tends to 0 when n goes to infinity, and it tends to 1 when $p > 1/2$ (super-critical regime).

We define a quantity that will roughly measure the scale up to which these crossing probabilities remain bounded away from 0 and 1. For each fixed $\epsilon_0 > 0$, we define

$$L(p,\epsilon_0) = \begin{cases} \min\{n \text{ s.t. } \mathbb{P}_p(\mathcal{C}_H([0,n]\times[0,n])) \leq \epsilon_0\} & \text{when } p < 1/2, \\ \min\{n \text{ s.t. } \mathbb{P}_p(\mathcal{C}_H^*([0,n]\times[0,n])) \leq \epsilon_0\} & \text{when } p > 1/2. \end{cases} \tag{4.3}$$

If we use the Russo-Seymour-Welsh theory (see e.g. [35, 39]), we see that for each $k \geq 1$, there exists some $\delta_k > 0$ (depending only on ϵ_0) such that

$$\forall n \leq L(p), \quad \mathbb{P}_p(\mathcal{C}_H([0,kn]\times[0,n])) \geq \delta_k. \tag{4.4}$$

For symmetry reasons, this bound is also valid for horizontal vacant crossings.

These estimates for crossing probabilities are then the basic building blocks on which many further considerations are built. One of the main results of Kesten's paper [43] is for instance the following one-arm probability estimate:

$$\mathbb{P}_p[0 \rightsquigarrow \partial S_n] \asymp \mathbb{P}_{1/2}[0 \rightsquigarrow \partial S_n] \tag{4.5}$$

for all $n \leq L(p)$ (uniformly in p). In Section 4.4.1 of the present paper, we will derive the analogous result for "two-arm probabilities".

This result is basically saying that when n is not larger than $L(p)$, things can be compared to critical percolation. On the other hand, the definition of $L(p)$ shows that when $n > L(p)$, the picture starts to look like super/sub-critical percolation. For instance:

Lemma 61 (exponential decay with respect to $n/L(p)$)**.** *If ϵ_0 has been chosen sufficiently small, there exists a constant $C > 0$ such that for all n, all $p < 1/2$,*

$$\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \leq Ce^{-n/L(p)}. \tag{4.6}$$

Variants of this result are implicitly used or mentioned in Kesten's paper [43] or in other papers on finite-size scaling. We now give its proof, as it just takes a couple of lines and clarifies things.

Proof. Observe first that for all integer n,

$$\mathbb{P}_p(\mathcal{C}_H([0,2n] \times [0,4n])) \leq C'[\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,2n]))]^2 \tag{4.7}$$

with $C' = 10^2$ some universal constant. It suffices for that (see figure) to divide the parallelogram $[0,2n]\times[0,4n]$ into 4 horizontal sub-parallelograms $[0,2n]\times[in,(i+1)n]$ $(i = 0\dots3)$ and 6 vertical ones $[in,(i+1)n] \times [jn,(j+2)n]$ $(i = 0,1, j = 0\dots2)$. Indeed, consider a horizontal crossing of the big parallelogram: we can extract from it two pieces, one between its extremity on the left side and its first intersection with the vertical median $x = n$, and in the same way another one starting from the right side. These two sub-paths both cross one of the sub-parallelograms "in the easy way": as they are disjoint by construction, the claim follows by using the BK inequality ([10, 35]).

We then obtain by iterating:

$$C'\mathbb{P}_p(\mathcal{C}_H([0,2^kL(p)] \times [0,2^{k+1}L(p)])) \leq (C'\epsilon_1)^{2^k} \tag{4.8}$$

as soon as $\epsilon_1 \geq \mathbb{P}_p(\mathcal{C}_H([0,L(p)] \times [0,2L(p)]))$.

Recall that by definition, $\mathbb{P}_p(\mathcal{C}_H([0,L(p)] \times [0,L(p)])) \leq \epsilon_0$. Consequently, the RSW theorem entails that for all fixed $\epsilon_1 > 0$, if we take ϵ_0 sufficiently small, we get automatically (and independently of p) that

$$\mathbb{P}_p(\mathcal{C}_H([0,L(p)] \times [0,2L(p)])) \leq \epsilon_1. \tag{4.9}$$

We now choose $\epsilon_1 = 1/(e^2C')$. For each integer $n \geq L(p)$, we can define $k = k(n)$ such that $2^k \leq n/L(p) < 2^{k+1}$, and then,

$$\begin{aligned}\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) &\leq \mathbb{P}_p(\mathcal{C}_H([0,2^kL(p)] \times [0,2^{k+1}L(p)])) \\ &\leq e^{-2^{k+1}}/C' \\ &\leq e \times e^{-n/L(p)},\end{aligned}$$

which is also valid for $n < L(p)$, thanks to the extra factor e.

□

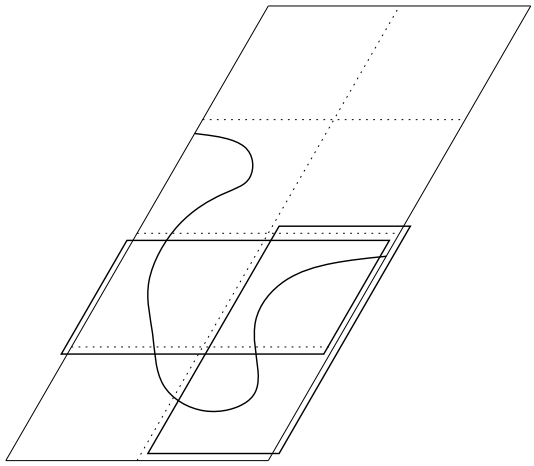


Figure 4.3: A horizontal crossing of the parallelogram $[0, 2n] \times [0, 4n]$.

By counting the sites which are pivotal for the existence of a crossing, as sketched in Kesten's paper [43] (see the remark following Lemma 8 p.144-145 – and also [55] for more details), we can prove that

$$|p - 1/2|(L(p))^2 \mathbb{P}_{1/2}(A^4(1, L(p))) \asymp 1. \tag{4.10}$$

Combining it with the estimate of Proposition 60 for 4 arms leads to the following:

Proposition 62 ([43, 71])**.** *When $p \to 1/2$,*

$$L(p) \approx |p - 1/2|^{-4/3}. \tag{4.11}$$

Remark 63. *It has been shown in [43] that for any fixed ϵ_1 and ϵ_2 with $0 < \epsilon_1, \epsilon_2 \leq \epsilon_0$,*

$$L(p, \epsilon_1) \asymp L(p, \epsilon_2). \tag{4.12}$$

Thus, the particular choice of ϵ_0 is not really important here (as long as it is sufficiently small).

4.3 Localization of the front

4.3.1 Framework of gradient percolation

We will now define the model itself and fix some notation. The starting point is still site percolation on the triangular lattice. We first consider a strip $\mathcal{S}_N$ of finite width $2N$ (we assume it to be even for simplicity), centered around the x-axis, such that the y-coordinate varies between $-N$ and N. This strip may for the moment be unbounded in one or both directions, and we will denote its length by ℓ_N.

In this strip, we assume the parameter to decrease linearly according to y, that is we choose it to be

$$p(y) = 1/2 - y/2N. \tag{4.1}$$

With this choice, all the sites on the bottom edge $\mathcal{B}_N$ will be occupied ($p = 1$), all the sites on the top edge $\mathcal{T}_N$ vacant ($p = 0$). The corresponding probability measure will be denoted by $\mathbb{P}$.

When we perform such a percolation, two opposite regions appear. At the bottom of $\mathcal{S}_N$, the parameter is close to 1, we are in a super-critical region and most occupied sites are connected

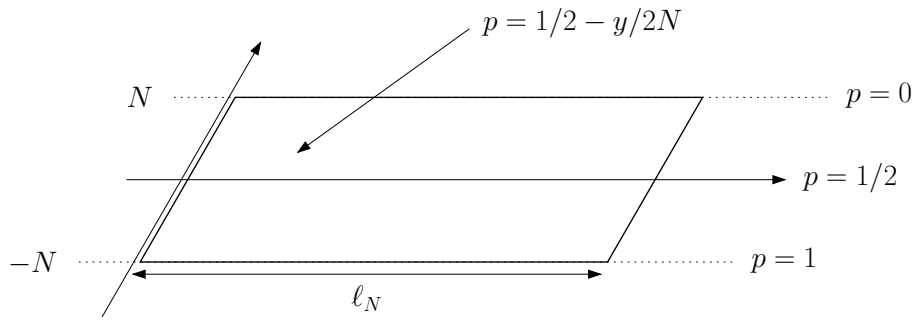


Figure 4.4: The strip where the percolation parameter p varies.

to the bottom edge. On the contrary, we observe on the top a big cluster of vacant sites. The characteristic phenomenon of this model is the existence of a unique "front", a continuous interface touching in the same time the occupied sites connected (by an occupied path) to the bottom of the strip, and the vacant sites connected to the top (by a vacant path). Temporarily, we adopt the following definition:

Definition 64. *A* front *will be any path γ_N on the dual hexagonal lattice that is bordered on one side by an occupied (horizontal) crossing of $\mathcal{S}_N$, and by a vacant (horizontal) crossing on the other side.*

We will prove at the end of this section that under suitable hypotheses, with high probability there is a unique such interface. Note that when the length ℓ_N of the strip is finite, there is not necessarily a *unique* front. For instance, there is a positive probability to observe two horizontal crossings, one vacant and one occupied above it. If the strip is infinite, by independence of the different columns there exists almost surely a column on which all the sites, except the highest one, are occupied. In that case, the front is unique and can be determined "dynamically" by starting two exploration processes from the top of this random column in the two possible directions (these processes follow the interface between black and white sites – making at each step a 60 degree turn, to the left or to the right according to the color of the hexagon they meet).

However, for practical purposes, it will be more natural to consider strips of finite length. The argument above shows that on an infinite strip, exceptional events occur (the front touches the top and the bottom of the strip), and we may expect them not to happen when the length ℓ_N is not too large compared to N.

We now use the strip $\mathcal{S}_N = [0, \ell_N] \times [-N, N]$. We will often consider sub-strips of this big strip. For the sake of simplicity, the strip $[0, \ell_N] \times [-\lfloor N^\alpha \rfloor, \lfloor N^\alpha \rfloor]$ will be written $[\pm N^\alpha]$.

4.3.2 Localization

We study now the convergence of the front when it is unique. Temporarily, we will consider, instead of "the" front, the highest horizontal crossing, and we will denote it by $\mathcal{R}_N$. Note that the sites just above $\mathcal{R}_N$ form a vacant horizontal crossing of $\mathcal{S}_N$, so that the upper boundary ρ_N of $\mathcal{R}_N$ (the path on the dual hexagonal lattice bordering it above) is *a* front. We shall see a little bit later that it is indeed *the* front (*i.e.* the only interface) with high probability.

We a priori expect it to be close to the line $\{y = 0\}$ corresponding to the sites where p is critical. A hand-waiving argument goes as follows: if we are at a distance approximately $N^{4/7}$ from the line, say for instance above it, the corresponding percolation parameter will be about

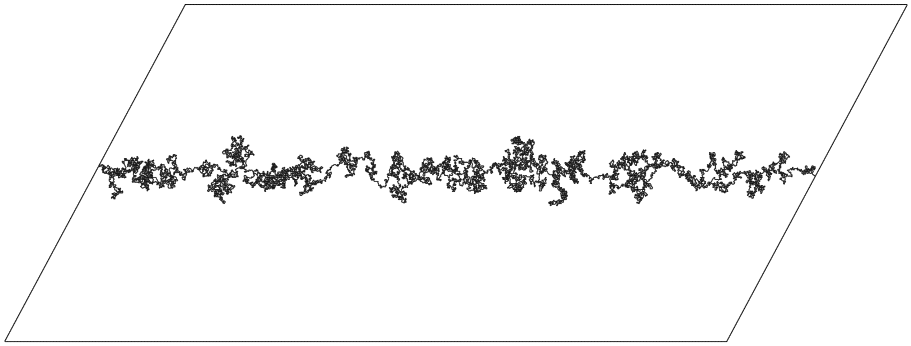


Figure 4.5: Localization of the front ($2N = 500$, $\ell_N = 1000$).

$1/2 - N^{-3/7}$ and the associated characteristic length of order $N^{4/7}$ too (by using (4.11)). Hence, the strip on which everything looks close to critical percolation has width of order $N^{4/7}$, and outside of this strip, one is in the super- or sub-critical regime. This explains intuitively the following result.

Theorem 65 (Localization)**.**

- *For all $\delta > 0$, there exists a $\delta' > 0$ such that for all sufficiently large N and all $\ell_N \geq N^{4/7}$,*

$$\mathbb{P}(\mathcal{R}_N \subseteq [\pm N^{4/7-\delta}]) \leq e^{-N^{\delta'}}. \tag{4.2}$$

- *For all $\delta > 0$, for all $\gamma \geq 1$, there exists a $\delta' > 0$ such that for all sufficiently large N, and all $N^{4/7} \leq \ell_N \leq N^{\gamma}$,*

$$\mathbb{P}(\mathcal{R}_N \nsubseteq [\pm N^{4/7+\delta}]) \leq e^{-N^{\delta'}}. \tag{4.3}$$

Proof. Let us first derive the statement concerning $\mathbb{P}(\mathcal{R}_N \subseteq [\pm N^{4/7-\delta}])$. For that purpose, consider disjoint rhombi of the form

$$[i, i + 2N^{4/7-\delta}] \times [-N^{4/7-\delta}, N^{4/7-\delta}] \quad (i = 0,\ 2N^{4/7-\delta} + 1,\ 4N^{4/7-\delta} + 2 \ldots).$$

We can take at least $\ell_N/3N^{4/7-\delta}$ such rhombi, and each of them possesses (independently of the other ones) a vertical vacant crossing with probability larger than

$$\mathbb{P}_{1/2+N^{-3/7-\delta}/2}\big(\mathcal{C}^*_H([0, 2N^{4/7-\delta}]^2)\big).$$

But (using (4.11), the critical exponent for L)

$$L(1/2 + N^{-3/7-\delta}/2) \approx (N^{-3/7-\delta}/2)^{-4/3} \approx N^{4/7+4\delta/3},$$

hence

$$L(1/2 + N^{-3/7-\delta}/2) \gg 2N^{4/7-\delta},$$

so that there exists a vertical vacant crossing with probability larger than ϵ_0. By independence, a "block" entailing that $\mathcal{R}_N \nsubseteq [\pm N^{4/7-\delta}]$, will occur with probability larger than

$$1 - (1 - \epsilon_0)^{\ell_N/3N^{4/7-\delta}},$$

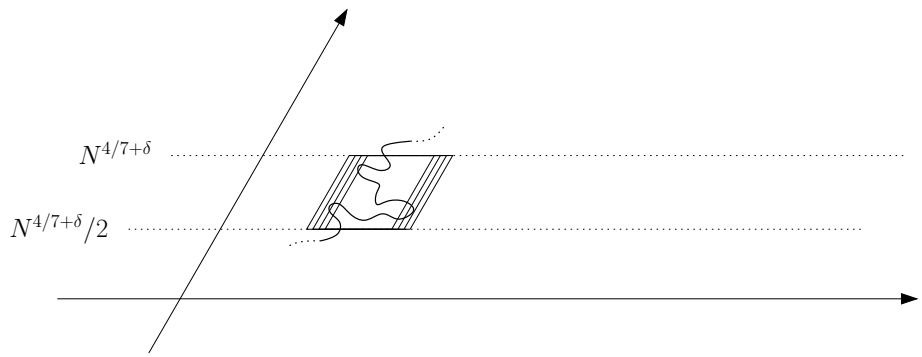


Figure 4.6: $\mathcal{R}_N$ crosses horizontally or vertically one of the small rhombi.

which proves the claim as $\ell_N \geq N^{4/7}$ by assumption.

Let us now turn our attention to the quantity $\mathbb{P}(\mathcal{R}_N \not\subseteq [\pm N^{4/7+\delta}])$. Assume that $\mathcal{R}_N$ is not entirely contained in the strip $[\pm N^{4/7+\delta}]$.

If at some point it is for example above this strip, we face the following alternative:

- Either $\mathcal{R}_N$ is in the strip $[\pm N^{4/7+\delta}/2]$ at some point. In that case, consider the following rhombi, located between the lines $y = N^{4/7+\delta}/2$ and $y = N^{4/7+\delta}$: $[i, i + N^{4/7+\delta}/2] \times [N^{4/7+\delta}/2, N^{4/7+\delta}]$ $(i = 0, 1, 2 \ldots)$. It is easy to see that $\mathcal{R}_N$ will have to cross vertically *or horizontally* one of them.

  There are at most ℓ_N such rhombi, and they are in a zone for which $p \leq 1/2 - N^{-3/7+\delta}/4$, a crossing thus occurs with probability less than

  $$\begin{aligned}\ell_N \mathbb{P}_{1/2-N^{-3/7+\delta}/4}&(\mathcal{C}_H([0, N^{4/7+\delta}/2]^2) \cup \mathcal{C}_V([0, N^{4/7+\delta}/2]^2)) \\ &\leq 2\ell_N \mathbb{P}_{1/2-N^{-3/7+\delta}/4}(\mathcal{C}_H([0, N^{4/7+\delta}/2]^2)) \\ &\leq 2C\ell_N e^{-N^{4/7+\delta}/2L(1/2-N^{-3/7+\delta}/4)}\end{aligned}$$

  by using Lemma 61 (of sub-exponential decay).

  But (by (4.11))

  $$L(1/2 - N^{-3/7+\delta}/4) \approx (N^{-3/7+\delta}/4)^{-4/3} \approx N^{4/7-4\delta/3},$$

  and ℓ_N does not grow too fast, so that the probability of the considered event tends to 0 sub-exponentially fast.

- Either $\mathcal{R}_N$ stays constantly above the strip $[\pm N^{4/7+\delta}/2]$, where the percolation parameter remains smaller than $1/2 - N^{-3/7+\delta}/4$. In that case, $\mathcal{R}_N$ will cross vertically or horizontally one of the rhombi (forming a "column") $[0, N^{4/7+\delta}/2] \times [j + N^{4/7+\delta}/2, j + N^{4/7+\delta}]$ $(j = 0, 1, 2 \ldots)$.

  There are at most N such rhombi. Once again, Lemma 61 entails that a crossing occurs with probability less than

  $$2CNe^{-N^{4/7+\delta}/2L(1/2-N^{-3/7+\delta}/4)}.$$

As before,

$$L(1/2 - N^{-3/7+\delta}/4) \approx N^{4/7-4\delta/3},$$

so that the considered probability tends to 0 sub-exponentially fast.

If $\mathcal{R}_N$ is below the strip at some point, the argument is identical: consider instead the vacant crossing bordering $\mathcal{R}_N$. Hence, the final probability just has to be multiplied by 2.

□

4.3.3 Uniqueness of the front

In the previous subsection, we have focused on the highest horizontal crossing $\mathcal{R}_N$. Clearly, the results remain valid if we consider the lowest horizontal vacant crossing $\mathcal{R}_N^*$ instead. Recall that $\mathcal{R}_N$ is bordered above by a horizontal vacant crossing, so that its upper boundary ρ_N is a front. Similarly, $\mathcal{R}_N^*$ is bordered below by an occupied crossing, and its lower boundary ρ_N^* is a front too.

Note that ρ_N^* is always below ρ_N. It is easy to see that uniqueness of the front amounts to checking whether ρ_N and ρ_N^* coincide. It is also equivalent to verifying that $\mathcal{R}_N$ is connected to the bottom $\mathcal{B}_N$ by an occupied path (or that $\mathcal{R}_N^*$ is connected to the top $\mathcal{T}_N$ by a vacant path). We are now going to prove that this indeed occurs with a very large probability.

Note that if one starts an exploration process from the top-left corner of $\mathcal{S}_N$, one discovers the top-most occupied crossing without discovering the status of the sites below it. This RSW-type observation will be essential in our proof.

Proposition 66 (Uniqueness)**.** *Assume that $\ell_N \geq N^{4/7+\delta}$ for some $\delta > 0$. Then, there is a $\delta' > 0$ (depending on δ) such that for all sufficiently large N,*

$$\mathbb{P}(\rho_N = \rho_N^*) \geq 1 - e^{-N^{\delta'}}. \tag{4.4}$$

Proof. As announced, we will work with $\mathcal{R}_N$. Our goal is to show that the probability for $\mathcal{R}_N$ not to be connected to $\mathcal{B}_N$ is very small. For that purpose, we first divide the strip $\mathcal{S}_N$ into disjoint sub-strips $(\mathcal{S}_N^i)$ as follows. For $\epsilon := \delta/4$, we choose $N^{3\epsilon}/6$ disjoint sub-strips of length $3N^{4/7+\epsilon}$ (not necessarily covering entirely $\mathcal{S}_N$) of the type

$$\mathcal{S}_N^i = [n_N^i, n_N^i + 3N^{4/7+\epsilon}] \times [-N, N] \quad (i = 0, \ldots, N^{3\epsilon}/6 - 1).$$

Consider one of these strips $\mathcal{S}_N^i$. $\mathcal{R}_N$ crosses it horizontally, and remains "below" its highest horizontal crossing, that we denote by r_N^i. Consequently, it will be sufficient to show that one of the r_N^i's is connected to $\mathcal{B}_N$ in $\mathcal{S}_N^i$.

We now fix an i, and we try to find a lower bound for the probability that r_N^i is connected to the bottom of $\mathcal{S}_N^i$ by an occupied path that stays in that sub-strip. Let us suppose for notational convenience that $i = 0$ and $n_N^i = 0$. Note first that with probability at least $1/2$, there exists a vacant top-to-bottom crossing of the rhombus $[N^{4/7+\epsilon}, 2N^{4/7+\epsilon}] \times [0, N^{4/7+\epsilon}]$ (percolation is sub-critical in this region), so that a lowest point z on r_N^i in the middle part $[N^{4/7+\epsilon}, 2N^{4/7+\epsilon}] \times [-N, N]$ of the strip lies below the x-axis with probability at least $1/2 + o(1)$ (the localization result (Theorem 65) tells us that r_N^i remains below the height $N^{4/7+\epsilon}$ with high probability).

Now, if we have explored this highest crossing of $\mathcal{S}_N^0$ "from above", we have not yet discovered the status of the sites below it, so that we can apply the FKG inequality for events involving only the state of these remaining sites. We are now going to show that in the case where r_N^0

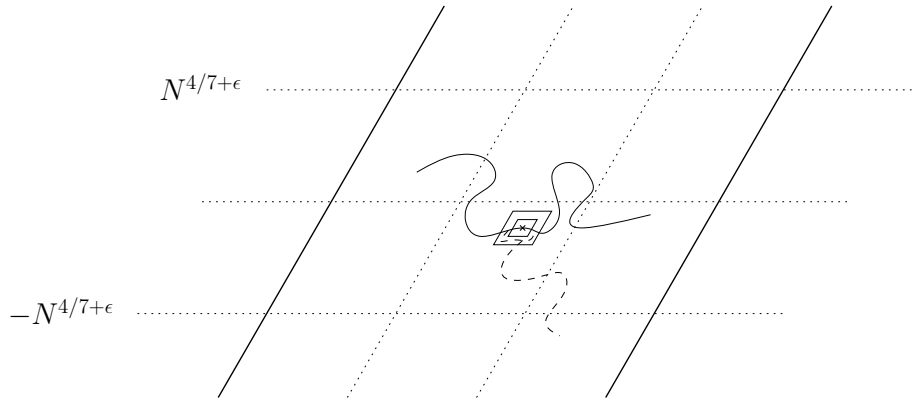


Figure 4.7: Construction of an arm going out of $[\pm N^{4/7+\epsilon}]$.

passes below the x-axis, the conditional probability that it is connected to the bottom part of the sub-strip by an occupied crossing is bounded from below by a quantity of order $N^{-2\epsilon}$.

A way to prove this goes as follows. Let us first choose z, and define the annulus $S_{2N^{4/7-\epsilon}}(z) \setminus \mathring{S}_{N^{4/7-\epsilon}}(z)$ around z. Since it is contained in the region where $p \geq 1/2 - 2N^{-3/7-\epsilon}$, and since the characteristic length corresponding to this value of the parameter is of order

$$L(1/2 - 2N^{-3/7-\epsilon}) \approx N^{4/7+4\epsilon/3},$$

there is a probability at least δ_4^4 (this is the constant coming from RSW-theory) to observe an occupied circuit in this annulus.

We now want to connect this circuit to the bottom boundary of the sub-strip. Note that the part of the circuit that is below r_N^0 together with r_N^0 contain an occupied circuit around the segment $I = z + [-N^{4/7-\epsilon}, N^{4/7-\epsilon}] \times \{-N^{4/7-\epsilon}\}$. We need the following simple lemma for critical percolation:

Lemma 67. *Consider the rhombus $[-N^{4/7+\epsilon}, N^{4/7+\epsilon}] \times [-2N^{4/7+\epsilon}, 0]$ and the sub-interval $I_N = [-N^{4/7-\epsilon}, N^{4/7-\epsilon}] \times \{0\}$ on its top edge. Then the event $\mathcal{C}_V^{I_N}([-N^{4/7+\epsilon}, N^{4/7+\epsilon}] \times [-2N^{4/7+\epsilon}, 0])$ that there exists a vertical occupied crossing connecting I_N to the bottom edge has a probability at least*

$$\mathbb{P}_{1/2}\big[\, \mathcal{C}_V^{I_N}([-N^{4/7+\epsilon}, N^{4/7+\epsilon}] \times [-2N^{4/7+\epsilon}, 0])\,\big] \geq \frac{C}{N^{2\epsilon}} \tag{4.5}$$

for some universal constant C (depending neither on N nor on ϵ).

Proof. Consider the parallelogram $[0, N^{4/7+\epsilon}] \times [-2N^{4/7+\epsilon}, 0]$, and cover its top edge by less than $N^{2\epsilon}$ intervals $I_N^j = [n_N'^j - N^{4/7-\epsilon}, n_N'^j + N^{4/7-\epsilon}] \times \{0\}$ of length $2N^{4/7-\epsilon}$. We know from the RSW theorem that there exists a vertical occupied crossing with probability at least $\delta_2 > 0$, so that

$$\delta_2 \leq \sum_j \mathbb{P}_{1/2}\big[\, \mathcal{C}_V^{I_N^j}([0, N^{4/7+\epsilon}] \times [-2N^{4/7+\epsilon}, 0])\,\big].$$

But for each j,

$$\begin{aligned}
&\mathbb{P}_{1/2}\big[\, \mathcal{C}_V^{I_N^j}([0, N^{4/7+\epsilon}] \times [-2N^{4/7+\epsilon}, 0])\,\big] \\
&\qquad \leq \mathbb{P}_{1/2}\big[\, \mathcal{C}_V^{I_N^j}([n_N'^j - N^{4/7+\epsilon}, n_N'^j + N^{4/7+\epsilon}] \times [-2N^{4/7+\epsilon}, 0])\,\big] \\
&\qquad = \mathbb{P}_{1/2}\big[\, \mathcal{C}_V^{I_N}([-N^{4/7+\epsilon}, N^{4/7+\epsilon}] \times [-2N^{4/7+\epsilon}, 0])\,\big]
\end{aligned}$$

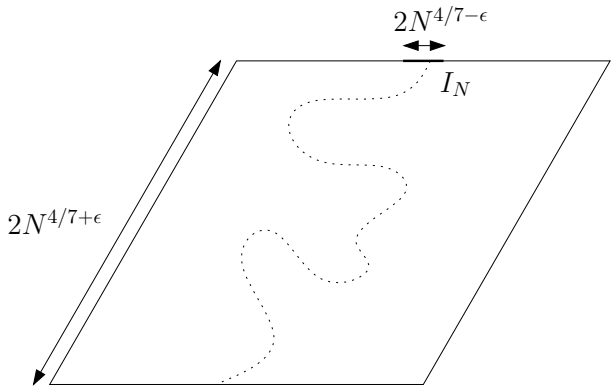


Figure 4.8: Existence of an arm from I_N to the bottom side.

by translation invariance. Hence

$$\delta_2 \leq N^{2\epsilon} \times \mathbb{P}_{1/2}\big[\, \mathcal{C}_V^{I_N}([-N^{4/7+\epsilon}, N^{4/7+\epsilon}] \times [-2N^{4/7+\epsilon}, 0])\,\big],$$

which completes the proof. Note that a repeated application of the RSW theorem ($\log(N^{2\epsilon})$ times) would have given a lower bound of the type $N^{-\kappa\epsilon}$ that would have been also sufficient for our purpose here. □

Putting the pieces together, with the help also of the FKG inequality, we get that for each $i \leq N^{3\epsilon}/6 - 1$, the probability that $\mathcal{R}_N$ is connected to the bottom part of the strip in the sub-strip $\mathcal{S}_N^i$ is bounded from below by $C'/N^{2\epsilon}$ independently for each i. The proposition then follows readily: indeed, $\mathcal{R}_N$ is connected to $\mathcal{B}_N$ with probability at least

$$1 - (1 - C'N^{-2\epsilon})^{N^{3\epsilon}/6} \geq 1 - e^{-N^{\epsilon'}} \tag{4.6}$$

for some positive ϵ'.

□

Remark 68. *On the other hand, note that if $\ell_N \leq N^{4/7-\delta}$ for some $\delta > 0$, then by RSW there exist several interfaces with probability bounded away from* 0.

The previous results suggest to make some restrictions on the length ℓ_N of the strip. In the following, we will thus assume that there exists a $\delta > 0$ such that $\ell_N \geq N^{4/7+\delta}$: this hypothesis entails that the event corresponding to uniqueness has a probability tending to 1 sub-exponentially fast. For convenience, we also assume that $\ell_N = o(N^\gamma)$ for some $\gamma \geq 1$, to ensure that the front remains localized.

From now on, we will simply refer to *the* front and denote it by $\mathcal{F}_N$, which means that we will implicitly neglect the error term in the estimates that we derive. In particular, the front will be exactly the set of edges from which two arms can be drawn, one occupied to the bottom $\mathcal{B}_N$, and one vacant to the top $\mathcal{T}_N$.

4.4 Length of the front

We would like now to study the length T_N of the front *i.e.* its number of edges. The preceding remark shows that we will need a two-arm probability estimate for that purpose. Unless otherwise stated, the expression "two arms" refers to two arms *of opposite colors.*

We will often have to count edges rather than sites. Since we are mostly interested in rough estimates, we will then just use the fact that number of edges and number of corresponding sites are comparable (i.e. up to a multiplicative factor of 6). To simplify notations, it will be convenient to associate to each edge e one of its two neighboring sites x_e, which we do arbitrarily and once for all.

4.4.1 Two-arm estimates

We are now going to derive the analog of (4.5) in the case of two arms and for non-constant p. This lemma will enable us to estimate the probability of having two arms from an edge e in the "critical strip". The goal is to show roughly that

$$\mathbb{P}_p[A^2(0,n)] \asymp \mathbb{P}_{1/2}[A^2(0,n)] \quad (n \leq L(p)).$$

In fact, for our purpose, we will have to consider, instead of $\mathbb{P}_p$, product measures P' with associated parameters $p'(v)$ which may depend on the site v but remain between p and $1-p$ (we will simply say that P' is "between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$"). The present situation is a little more complicated than for the one-arm estimate, because of the lack of monotonicity (A^2 events correspond to the combination of one path of each type, so that they are neither increasing nor decreasing).

For a parallelogram R and a site v contained in its interior, Kesten considered in [43] the event $\Gamma(v,R)$ that there exist 4 arms alternatively occupied and vacant from the set ∂v of vertices neighboring v to the boundary ∂R of R:

> $\Gamma(v,R) = \{$There exist four paths $r_1,\dots,r_4$ from ∂v to ∂R ordered clockwise, such that r_1, r_3 are occupied, and r_2, r_4 are vacant$\}$.

Here we will need this event in the course of proof, but we will be more interested in the analog for two arms:

> $\Gamma_2(v,R) = \{$There exist an occupied path r_1 and a vacant path r_2 from ∂v to $\partial R\}$.

Note that $A^2(0,n) = \Gamma_2(0,S_n)$. Let us now state and prove the result:

Lemma 69. *Uniformly in p, $\hat{P}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, $n \leq L(p)$, we have*

$$\hat{P}[\Gamma_2(0,S_n)] \asymp \mathbb{P}_{1/2}[\Gamma_2(0,S_n)]. \tag{4.1}$$

Proof. This proof is an adaptation of the proof of Theorem 1 in Kesten's paper [43]. We first recall some estimates contained in this paper, and then we adapt the original proof for 1 arm to the case of 2 arms. However, note that this result could be addressed as in [43]: Lemma 8 in this paper is the "four-arm" version of the present two-arm result.

Let us first introduce the events that we use throughout the proof. All of them are exact analogs of events defined in Kesten's paper [43]. We consider a parallelogram R and a site v such that $v \in \mathring{R}$.

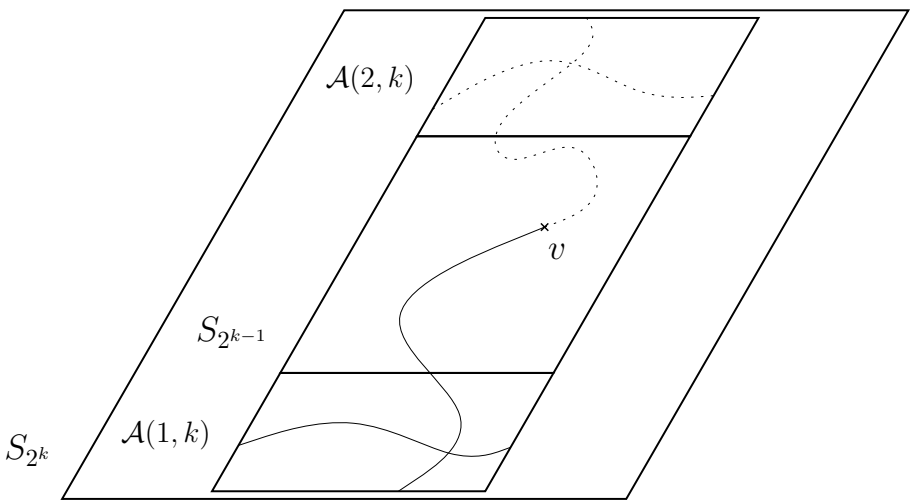


Figure 4.9: Definition of the event $\Delta_2(v, S_{2^k})$.

- In the case of 1 arm, we do not lose much (a factor 4) by imposing the extremity to be on a specified edge of R, at least if v is not too close to one of the edges. Here we impose the occupied arm to arrive on the bottom edge of R, and the vacant arm on the top edge. This event is denoted by $\Omega_2(v, R)$:

  $\Omega_2(v, R) = \{\Gamma_2(v, R)$ occurs, r_1 and r_2 having their extremities on (resp.) the bottom and the top edges of $R\}$.

- The two paths r_1 and r_2 might be a bit difficult to extend, so we may want to add furthermore "security strips". This leads to the definition of the event Δ_2, analog of the event Δ (in the case of 4 arms). Albeit more restrictive, this event has a probability that remains comparable to $\Omega_2(0, R)$.

  We consider thus the two horizontal strips

  $$\mathcal{A}(1,k) := [-2^{k-1}, 2^{k-1}] \times [-2^k, -2^{k-1}]$$
  $$\mathcal{A}(2,k) := [-2^{k-1}, 2^{k-1}] \times [2^{k-1}, 2^k],$$

  and we define for a site v in $S_{2^{k-1}}$,

  $\Delta_2(v, S_{2^k}) = \{\Gamma_2(v, S_{2^k})$ occurs, with $r_i \cap (S_{2^k} \setminus \mathring{S}_{2^{k-1}}) \subseteq \mathcal{A}(i,k)$ and there exist two horizontal crossings, one occupied of $\mathcal{A}(1,k)$ and one vacant of $\mathcal{A}(2,k)\}$.

- We define similarly for a parallelogram R' contained in the interior of S_{2^k}:

  $\tilde{\Gamma}_2(S_{2^k}, R') = \{$There exist an occupied path r_1 and a vacant path r_2 from $\partial R'$ to the bottom and top edges respectively of S_{2^k}, which are (with the exception of their extremities on $\partial R'$) contained in $S_{2^k} \setminus R'\}$.

  and for a site v in $S_{2^{k-1}}$, $j \leq k-2$, the following strips (centered on v):

  $$\tilde{\mathcal{A}}(1,j) := [v(1) - 2^j, v(1) + 2^j] \times [v(2) - 2^{j+1}, v(2) - 2^j]$$

$$\tilde{\mathcal{A}}(2,j) := [v(1) - 2^j, v(1) + 2^j] \times [v(2) + 2^j, v(2) + 2^{j+1}],$$

and then

$\tilde{\Delta}_2(S_{2^k}, S_{2^j}(v)) = \{\tilde{\Gamma}_2(S_{2^k}, S_{2^j}(v))$ occurs, with associated paths r_i satisfying $r_i \cap (S_{2^{j+1}}(v) \setminus \mathring{S}_{2^j}(v)) \subseteq \tilde{\mathcal{A}}(i,j)$, and there exist two horizontal crossings, one occupied of $\tilde{\mathcal{A}}(1,j)$ and one vacant of $\tilde{\mathcal{A}}(2,j)\}$.

Let us now state the estimates that we will use. We will not prove them, since these are exact analogs for 2 arms of results stated for 4 arms in Kesten's paper [43], except that we generalize the condition "between $\mathbb{P}_{1/2}$ and $\mathbb{P}_p$" to "between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$". This generalization can be done as the only tool used in the proofs is the Russo-Seymour-Welsh theorem (we also implicitly use the symmetry about 1/2, which implies that $L(p) = L(1-p)$). In the following, $\hat{P}$ denotes any product measure "between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$".

(1) **Extension of Δ_2**: There exists a constant $C_1 < \infty$ such that

$$\hat{P}(\Delta_2(0, S_{2^k})) \leq C_1 \hat{P}(\Delta_2(0, S_{2^{k+1}})) \tag{4.2}$$

for all p, $\hat{P}$ and $2^k \leq L(p)$.

(2) **Comparison of Γ_2 and Δ_2** (analog of Lemma 4 in [43]): There exists a constant $C_2 < \infty$ such that

$$\hat{P}(\Gamma_2(0, S_{2^k})) \leq C_2 \hat{P}(\Delta_2(0, S_{2^k})) \tag{4.3}$$

for all p, $\hat{P}$ and $2^k \leq L(p)$.

(3) **Comparison of $\tilde{\Gamma}_2$ and $\tilde{\Delta}_2$** (analog of Lemma 5 in [43]): There exists a constant $C_3 < \infty$ such that

$$\hat{P}(\tilde{\Gamma}_2(S_{2^k}, S_{2^j})) \leq C_3 \hat{P}(\tilde{\Delta}_2(S_{2^k}, S_{2^j})) \tag{4.4}$$

for all p, $\hat{P}$, $2^k \leq L(p)$ and $j \leq k-2$.

These prerequisites being recalled, we are now able to turn to the proof of Lemma 69. For that purpose, we will have to adapt Kesten's original proof. Recall that we consider a parameter p, a measure $\hat{P}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$ and an integer $n \leq L(p)$. The parameters of $\hat{P}$ will be denoted by $p(v)$. We may assume that $p < 1/2$. Note also that it suffices to prove the theorem with $\mathbb{P}_p$ instead of $\mathbb{P}_{1/2}$.

1st step: We first notice that we can replace $\Gamma_2(0, S_n)$ by $\Omega_2(0, S_{2^k})$, with k such that $2^k \leq n < 2^{k+1}$. Indeed, the estimates (4.2) and (4.3) entail

$$\begin{aligned}
\hat{P}(\Gamma_2(0, S_n)) &\geq \hat{P}(\Gamma_2(0, S_{2^{k+1}})) \geq \hat{P}(\Delta_2(0, S_{2^{k+1}})) \\
&\geq C_1^{-1} \hat{P}(\Delta_2(0, S_{2^k})) \\
&\geq C_2^{-1} C_1^{-1} \hat{P}(\Gamma_2(0, S_{2^k})) \geq C_2^{-1} C_1^{-1} \hat{P}(\Omega_2(0, S_{2^k}))
\end{aligned}$$

and

$$\begin{aligned}
\hat{P}(\Omega_2(0, S_{2^k})) &\geq \hat{P}(\Delta_2(0, S_{2^k})) \\
&\geq C_2^{-1} \hat{P}(\Gamma_2(0, S_{2^k})) \geq C_2^{-1} \hat{P}(\Gamma_2(0, S_n)).
\end{aligned}$$

2nd step: We would like now to use Russo's formula, but for technical reasons which will become clear in the next step, this will work nicely only for the points that are not too close to the boundary. We therefore first have to see how the change of $p(v)$ for the points that are close to the boundary affects the probabilities that we investigate. More precisely, we start with $\mathbb{P}_p$ and change $p \rightsquigarrow p(v)$ only in $S_{2^k} \setminus S_{2^{k-3}}$. The resulting measure $\tilde{P}$ is between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$: so,

$$\mathbb{P}_p(\Omega_2(0, S_{2^k})) \asymp \mathbb{P}_p(\Delta_2(0, S_{2^k})) \asymp \mathbb{P}_p(\Delta_2(0, S_{2^{k-3}}))$$

thanks to (4.2) and (4.3), and similarly

$$\tilde{P}(\Omega_2(0, S_{2^k})) \asymp \tilde{P}(\Delta_2(0, S_{2^{k-3}})),$$

which allows us to conclude, since $\mathbb{P}_p(\Delta_2(0, S_{2^{k-3}})) = \tilde{P}(\Delta_2(0, S_{2^{k-3}}))$ by definition.

3rd step: In order to handle the sites v located in $S_{2^{k-3}}$, we will apply a generalization of Russo's formula to the family of measures $(\hat{P}_t)_{t\in[0,1]}$ with parameters

$$p(v, t) = tp(v) + (1 - t)p,$$

which corresponds to a linear interpolation between $p \rightsquigarrow p(v)$ in $S_{2^{k-3}}$. The event $\Omega_2(0, S_{2^k})$ can be written as the intersection of $A_2(0, S_{2^k}) = \{$There exists an occupied path from $\partial 0$ to the bottom side of $S_{2^k}\}$ and $B_2(0, S_{2^k}) = \{$There exists a vacant path from $\partial 0$ to the top side of $S_{2^k}\}$. These two events are respectively increasing and decreasing, and in that case, $\frac{d}{dt}\hat{P}_t(A_2 \cap B_2)$ can be expressed as (see Lemma 1 in [43]):

$$\sum_{v\in S_{2^{k-3}}} \frac{dp(v,t)}{dt}\big[\hat{P}_t(v \text{ is pivotal for } A_2, \text{ but not for } B_2, \text{ and } B_2 \text{ occurs}) \\ - \hat{P}_t(v \text{ is pivotal for } B_2, \text{ but not for } A_2, \text{ and } A_2 \text{ occurs})\big].$$

A vertex $v \in S_{2^{k-3}}$, $v \neq 0$, is pivotal for A_2 *iff* there exist two paths both containing v such that

1. Their sites, except for v, are respectively all occupied and all vacant.
2. The first path connects $\partial 0$ to the bottom side of S_{2^k}.
3. The second path separates 0 from the bottom side of S_{2^k}.

In the case of 1 arm, Kesten used a vacant loop around the origin, here the separating path can have its extremities on the boundary. This will not change the computations, since we still have 4 arms locally, that we will sum in the same way.

Actually, we must also assume that $k \geq 7$ and put apart the vertices which are too close to the origin, for instance those in S_{16}. For these sites,

$$\begin{aligned}\hat{P}_t(v \text{ is pivotal for } A_2, \text{ but not for } B_2, \text{ and } B_2 \text{ occurs}) \\ &\leq \hat{P}_t(\tilde{\Gamma}_2(S_{2^k}, S_{16})) \\ &\leq C_4\hat{P}_t(\Omega_2(0, S_{2^k}))\end{aligned}$$

for some universal constant C_4.

We now associate to each site $v \in S_{2^{k-3}} \setminus S_{16}$ a parallelogram $R(v)$ such that $0 \notin R(v)$, so that we make appear four arms locally. We choose them to present the following property: if j is such that $2^{j+1} < d(v,0) \leq 2^{j+2}$, then $R(v)$ is included in $\mathring{S}_{2^{j+3}} \setminus S_{2^j}$. This will imply that

$$\begin{aligned}
&\hat{P}_t(v \text{ is pivotal for } A_2, \text{ but not for } B_2, \text{ and } B_2 \text{ occurs}) \\
&\qquad \leq \hat{P}_t\big[\Gamma_2(0, S_{2^j}) \cap \Gamma(v, R(v)) \cap \tilde{\Gamma}_2(S_{2^k}, S_{2^{j+3}})\big] \\
&\qquad = \hat{P}_t[\Gamma_2(0, S_{2^j})]\hat{P}_t[\Gamma(v, R(v))]\hat{P}_t[\tilde{\Gamma}_2(S_{2^k}, S_{2^{j+3}})]
\end{aligned}$$

by independence (these events depending on sites in disjoint sets).

We can take the parallelograms $R(v)$ like in Kesten's paper (see [43] p. 117). As noticed in this paper, the precise choice is not really important. On one hand, we have to ensure that the four arms are not too small. For that purpose, the distances between v and each of the sides of $R(v)$ must be comparable to the distance between 0 and v. On the other hand, if the announced property is satisfied, we will be in position to join the paths outside $R(v)$, that are respectively between 0 and ∂S_{2^j}, and between $\partial S_{2^{j+3}}$ and ∂S_{2^k}. For the sake of completeness, let us just recall briefly how Kesten chooses these $R(v)$. Consider $v = (v_1, v_2) \notin S_{16}$. If $|v_1| \leq |v_2| \leq 2^{k-3}$ and $16 \leq 2^{j+1} < v_2 \leq 2^{j+2}$, take l_1 and l_2 such that

$$l_1 2^{j-2} < v_1 \leq (l_1+1)2^{j-2} \quad \text{and} \quad l_2 2^{j-2} < v_2 \leq (l_2+1)2^{j-2},$$

and define

$$R(v) = [(l_1-2)2^{j-2}, (l_1+2)2^{j-2}] \times [l_2 2^{j-2} - 2^j, l_2 2^{j-2} + 2^j].$$

If $v_2 < 0$, take the image of $R((v_1, -v_2))$ by the symmetry with respect to the x-axis. Finally, if $|v_2| < |v_1|$, simply exchange the roles of the first and second coordinates. We can easily check that these parallelograms possess by construction the desired property.

Now, by joining the two terms $\hat{P}_t[\Gamma_2(0, S_{2^j})]$ and $\hat{P}_t[\tilde{\Gamma}_2(S_{2^k}, S_{2^{j+3}})]$, we can make appear $\hat{P}_t[\Omega_2(0, S_{2^k})]$: indeed, we get easily from the Russo-Seymour-Welsh theorem (see Lemma 6 in [43])

$$\delta_{16}^2 \hat{P}_t[\Delta_2(0, S_{2^j})]\hat{P}_t[\tilde{\Delta}_2(S_{2^k}, S_{2^{j+3}})] \leq \hat{P}_t[\Omega_2(0, S_{2^k})]$$

as, by assumption, $2^k \leq L(p)$ and $j \leq k-5$ (here a slight generalization of the FKG inequality is needed (see for instance Lemma 3 in [43]), invoking zones where "everything is monotonous").

We obtain then from

$$\hat{P}_t[\Delta_2(0, S_{2^j})] \geq C_2^{-1} \hat{P}_t[\Gamma_2(0, S_{2^j})]$$

(which follows from (4.3)), and

$$\hat{P}_t[\tilde{\Delta}_2(S_{2^k}, S_{2^{j+3}})] \geq C_3^{-1} \hat{P}_t[\tilde{\Gamma}_2(S_{2^k}, S_{2^{j+3}})]$$

(using (4.4)), that

$$\hat{P}_t[\Gamma_2(0, S_{2^j})]\hat{P}_t[\tilde{\Gamma}_2(S_{2^k}, S_{2^{j+3}})] \leq C_5 \hat{P}_t[\Omega_2(0, S_{2^k})]. \tag{4.5}$$

Hence,

$$\begin{aligned}
&\hat{P}_t(v \text{ is pivotal for } A_2, \text{ but not for } B_2, \text{ and } B_2 \text{ occurs}) \\
&\qquad \leq C_5 \hat{P}_t[\Omega_2(0, S_{2^k})]\hat{P}_t[\Gamma(v, R(v))].
\end{aligned}$$

For symmetry reasons, the same is true if we invert A_2 and B_2. Consequently, dividing by $\hat{P}_t[\Omega_2(0, S_{2^k})]$ will make appear its logarithmic derivative in the left-hand side of Russo's formula:

$$\left|\frac{d}{dt}\log\left[\hat{P}_t(\Omega_2(0,S_{2^k}))\right]\right| \\ \leq 2C_5 \sum_{v\in S_{2^{k-3}}\setminus S_{16}} \left|\frac{dp(v,t)}{dt}\right| \hat{P}_t[\Gamma(v,R(v))] + 2C_4 \sum_{v\in S_{16}} \left|\frac{dp(v,t)}{dt}\right|.$$

Finally, the first term that we obtain (the sum encoding the existence of 4 arms locally) is exactly the same as in Kesten's paper [43] (end of the proof of Theorem 1 p. 140), and its integral between 0 and 1 is bounded by some universal constant C_6:

$$\int_0^1 \left(\sum_{v\in S_{2^{k-3}}\setminus S_{16}} \left|\frac{dp(v,t)}{dt}\right| \hat{P}_t[\Gamma(v,R(v))] \right) dt \leq C_6.$$

The desired conclusion then follows. □

4.4.2 Expected length of the front

We are now able to study properties of the front in the "critical strip". Roughly speaking, only the sites in $[\pm N^{4/7}]$ must be taken into account, and each of these sites has a probability approximately $(N^{4/7})^{-1/4} = N^{-1/7}$ (the two-arm exponent being equal to 1/4) to be on the front. Starting from this idea, we will prove the following estimate on the expectation of T_N:

Proposition 70. *Recall that by assumption, $\ell_N \geq N^{4/7+\delta}$ and $\ell_N = o(N^\gamma)$. For all $\epsilon > 0$, we have for N sufficiently large:*

$$N^{3/7-\epsilon}\ell_N \leq \mathbb{E}[T_N] \leq N^{3/7+\epsilon}\ell_N. \tag{4.6}$$

Proof. Throughout the proof, we will use the fact that

$$\mathbb{E}[T_N] = \sum_{e\in\mathcal{S}_N} \mathbb{P}(e \in \mathcal{F}_N). \tag{4.7}$$

We consider first the upper bound. Take $\epsilon' = \epsilon/4$: we have

$$\mathbb{E}[T_N] \leq 6|\mathcal{S}_N| \times \mathbb{P}(\mathcal{F}_N \not\subseteq [\pm N^{4/7+\epsilon'}]) + \sum_{e\in[\pm N^{4/7+\epsilon'}]} \mathbb{P}(e \in \mathcal{F}_N),$$

and it follows from localization and the fact that $|\mathcal{S}_N| = (2N+1)\times(\ell_N+1) = o(N^{\gamma+1})$, that the first term tends to 0 sub-exponentially fast. We can thus restrict the summation to the vertices in the strip $[\pm N^{4/7+\epsilon'}]$.

But for $e \in [\pm N^{4/7+\epsilon'}]$,

$$\mathbb{P}(e \in \mathcal{F}_N) \leq \mathbb{P}[2 \text{ arms } x_e \rightsquigarrow \partial S_{N^{4/7-2\epsilon'}}(x_e)],$$

and as $S_{N^{4/7-2\epsilon'}}(x_e) \subseteq [\pm 2N^{4/7+\epsilon'}]$, the percolation parameter in this box remains in the range $[1/2 \pm 2N^{-3/7+\epsilon'}]$. The associated characteristic length being

$$L(1/2 \pm 2N^{-3/7+\epsilon'}) \approx N^{4/7-4\epsilon'/3} \gg N^{4/7-2\epsilon'},$$

we get (using Lemma 69)

$$\begin{aligned}\mathbb{P}[2 \text{ arms } x_e \rightsquigarrow \partial S_{N^{4/7-2\epsilon'}}(x_e)] \\ &\asymp \mathbb{P}_{1/2}[2 \text{ arms } x_e \rightsquigarrow \partial S_{N^{4/7-2\epsilon'}}(x_e)] \\ &\approx (N^{4/7-2\epsilon'})^{-1/4} \quad \text{(by using the 2-arm exponent)} \\ &\ll N^{-1/7+\epsilon'}.\end{aligned}$$

Now we just have to sum this inequality to get the desired result: for N large enough,

$$\begin{aligned}\sum_{e\in[\pm N^{4/7+\epsilon'}]} \mathbb{P}(e \in \mathcal{F}_N) &\leq 6(2N^{4/7+\epsilon'}+1)(\ell_N+1) \times N^{-1/7+\epsilon'} \\ &\leq N^{3/7+\epsilon}\ell_N.\end{aligned}$$

Let us turn now to the lower bound. We restrict to the edges e in the strip $[2N^{4/7+\epsilon'}, \ell_N - 2N^{4/7+\epsilon'}] \times [\pm N^{4/7-\epsilon'}]$, with $\epsilon' = \epsilon/6$. For such edges, we would like to estimate the probability of having two arms, one occupied to $\mathcal{B}_N$ and one vacant to $\mathcal{T}_N$, we will so use the event Δ_2 rather than Γ_2. Indeed, take j such that $N^{4/7-\epsilon'} < 2^j \leq 2N^{4/7-\epsilon'}$: the probability of having two arms, one occupied to the bottom of $\partial S_{2^j}(x_e)$, and one vacant to the top, is at least

$$\mathbb{P}(\Delta_2(x_e, S_{2^j}(x_e))).$$

These two paths can then be extended, so that they go out of the strip $[\pm N^{4/7+\epsilon'}]$: Lemma 67 (contained in the proof of uniqueness) implies that this can be done with probability at least

$$(CN^{-2\epsilon'})^2 = C'N^{-4\epsilon'}. \tag{4.8}$$

On the other hand, since we stay in the strip $[\pm 3N^{4/7-\epsilon'}]$, of associated characteristic length $L(1/2 \pm 3N^{-3/7-\epsilon'}) \approx N^{4/7+4\epsilon'/3}$, we get that

$$\begin{aligned}\mathbb{P}(\Delta_2(x_e, S_{2^j}(x_e))) &\asymp \mathbb{P}(\Gamma_2(x_e, S_{2^j}(x_e))) \\ &\asymp \mathbb{P}_{1/2}(\Gamma_2(x_e, S_{2^j}(x_e))) \\ &\geq \mathbb{P}_{1/2}(\Gamma_2(0, S_{2N^{4/7-\epsilon'}})),\end{aligned}$$

and the 2-arm exponent implies that

$$\mathbb{P}_{1/2}(\Gamma_2(0, S_{2N^{4/7-\epsilon'}})) \approx (2N^{4/7-\epsilon'})^{-1/4} \gg N^{-1/7}. \tag{4.9}$$

We can thus construct two arms going out of the strip $[\pm N^{4/7+\epsilon'}]$ with probability at least $N^{-1/7-4\epsilon'}$ (for N large enough).

In that case, e has a high probability to be connected to the top and to the bottom of $\mathcal{S}_N$. Indeed, the front would go out of the strip $[\pm N^{4/7+\epsilon'}]$ otherwise, which occurs with a probability less that ϵ_N, for some ϵ_N (independent of e) tending to 0 sub-exponentially fast. The conclusion follows, by summing the lower bound over all edges e in the strip $[2N^{4/7+\epsilon'}, \ell_N - 2N^{4/7+\epsilon'}] \times [\pm N^{4/7-\epsilon'}]$: for N large enough,

$$\begin{aligned}\sum_{e\in\mathcal{S}_N} \mathbb{P}(e \in \mathcal{F}_N) &\geq (2N^{4/7-\epsilon'})(\ell_N - 4N^{4/7+\epsilon'}) \times (N^{-1/7-4\epsilon'} - \epsilon_N) \\ &\geq N^{3/7-\epsilon}\ell_N.\end{aligned}$$

□

4.4.3 Convergence in L^2

As seen in Section 4.4.2, everything happens in the strip $[\pm N^{4/7}]$, and it is possible to determine whether or not an edge e is on the front on a neighborhood of size $N^{4/7}$ (with probability very close to 1). Distant points will thus be almost completely decorrelated, and a phenomenon of horizontal "averaging" therefore occurs, so that we get the following bound on $\mathrm{Var}[T_N]$:

Proposition 71. *We still assume that $\ell_N \geq N^{4/7+\delta}$ and $\ell_N = o(N^\gamma)$. For each $\epsilon > 0$, we have for N sufficiently large:*

$$Var[T_N] \leq N^{10/7+\epsilon}\ell_N. \tag{4.10}$$

By combining this result to the estimates on $\mathbb{E}[T_N]$ of the previous subsection, we immediately get that

$$\mathrm{Var}[T_N] = o(\mathbb{E}[T_N]^2) \tag{4.11}$$

and consequently

Theorem 72. *If $\ell_N \geq N^{4/7+\delta}$ and $\ell_N = o(N^\gamma)$, then*

$$\frac{T_N}{\mathbb{E}[T_N]} \longrightarrow 1 \quad \text{in } L^2, \text{ when } N \to \infty. \tag{4.12}$$

Remark 73. *On each vertical line, among the (about) $N^{4/7}$ edges which lie in the critical strip, approximately $N^{3/7}$ of them will be on the front, that is a proportion $1/N^{1/7}$. If we take ℓ_N to be equal to N, we get $\mathbb{E}[T_N] \sim N^{10/7}$, and this exponent can indeed be observed numerically.*

Proof. For an edge e, the event "$e \in \mathcal{F}_N$" will be denoted by F_e. Using the expression

$$T_N = \sum_e \mathbb{I}_{F_e} \tag{4.13}$$

we get

$$\mathrm{Var}[T_N] = \sum_{e,f} [\mathbb{P}(F_e \cap F_f) - \mathbb{P}(F_e)\mathbb{P}(F_f)]. \tag{4.14}$$

First, take $\epsilon' = \epsilon/8$ and note that we can restrict the summation to the edges $e, f \in [\pm N^{4/7+\epsilon'}]$: similarly to the argument following (4.7), the remaining term tends to 0 sub-exponentially fast.

The idea is to replace F_e by an event $\tilde{F}_e$ which depends only on sites in a box around e of size $N^{4/7+\epsilon'}$, and such that $\mathbb{P}(F_e \Delta \tilde{F}_e) \to 0$ sub-exponentially fast (uniformly in e). To construct such an event, we simply invoke the result of uniqueness in the strip of length $2N^{4/7+\epsilon'}$ centered horizontally on x_e: it implies that we can take

$$\tilde{F}_e := \{ \text{ 2 arms } e \rightsquigarrow \text{ top and bottom sides of } \partial\big(S_{N^{4/7+\epsilon'}}(x_e) \cap \mathcal{S}_N\big) \ \}.$$

We have thus

$$\mathrm{Var}[T_N] = \sum_e \sum_f \big[\mathbb{P}(\tilde{F}_e \cap \tilde{F}_f) - \mathbb{P}(\tilde{F}_e)\mathbb{P}(\tilde{F}_f)\big] + \epsilon_N$$

with an error term ϵ_N tending to 0 sub-exponentially fast.

We now fix an edge e and we estimate the corresponding sum. Only the edges f in $S_{2N^{4/7+\epsilon'}}(x_e)$ must be taken into account: otherwise the two associated boxes do not intersect and

$$\mathbb{P}(\tilde{F}_e \cap \tilde{F}_f) - \mathbb{P}(\tilde{F}_e)\mathbb{P}(\tilde{F}_f) = 0.$$

We are thus turned to estimate

$$\sum_{f\in S_{2N^{4/7+\epsilon'}}(x_e)\cap\mathcal{S}_N} \mathbb{P}(\tilde{F}_e\cap\tilde{F}_f).$$

Introduce $d=d(x_e,x_f)$. If $d>N^{4/7-2\epsilon'}/4$, we have

$$\begin{aligned}\mathbb{P}(\tilde{F}_e\cap\tilde{F}_f) &\le \mathbb{P}(\Gamma_2(x_e,S_{N^{4/7-2\epsilon'}/8}(x_e))\cap\Gamma_2(x_e,S_{N^{4/7-2\epsilon'}/8}(x_f)))\\ &= \mathbb{P}(\Gamma_2(x_e,S_{N^{4/7-2\epsilon'}/8}(x_e)))\mathbb{P}(\Gamma_2(x_f,S_{N^{4/7-2\epsilon'}/8}(x_f)))\\ &\le N^{-2/7+2\epsilon'}\end{aligned}$$

for N large enough, by using the 2-arm exponent in the boxes $S_{N^{4/7-2\epsilon'}/8}(x_e)$ and $S_{N^{4/7-2\epsilon'}/8}(x_f)$. The sum over the edges f in $S_{2N^{4/7+\epsilon'}}(x_e)\setminus S_{N^{4/7-2\epsilon'}/4}(x_e)$ is thus at most

$$6(4N^{4/7+\epsilon'}+1)^2N^{-2/7+2\epsilon'}\le N^{6/7+5\epsilon'}.$$

Now assume that $4\le d\le N^{4/7-2\epsilon'}/4$. By introducing two boxes of size $d/2$ around x_e and x_f, and another box of size $2d$ around x_e, we see that

$$\begin{aligned}&\mathbb{P}(\tilde{F}_e\cap\tilde{F}_f)\\ &\quad\le \mathbb{P}(\Gamma_2(x_e,S_{d/2}(x_e)))\mathbb{P}(\Gamma_2(x_f,S_{d/2}(x_f)))\mathbb{P}(\tilde{\Gamma}_2(S_{2N^{4/7-2\epsilon'}}(x_e),S_{2d}(x_e))).\end{aligned}$$

(here to be completely rigorous we should prove that the condition on the extremities of the arms in $\tilde{\Gamma}_2$ can be relaxed: the methods of [43] apply – see also [55]). We can then join the first and the third term as in (4.5):

$$\begin{aligned}\mathbb{P}(\Gamma_2(x_e,S_{d/2}(x_e)))&\mathbb{P}(\tilde{\Gamma}_2(S_{2N^{4/7-2\epsilon'}}(x_e),S_{2d}(x_e)))\\ &\le C_1\mathbb{P}(\Gamma_2(x_e,S_{2N^{4/7-2\epsilon'}}(x_e)))\\ &\le N^{-1/7+\epsilon'}.\end{aligned}$$

Hence, using that there are at most C_2j edges at distance j from e and the 2-arm exponent,

$$\begin{aligned}&\sum_{f\in(S_{N^{4/7-2\epsilon'}/4}(x_e)\setminus S_4(x_e))\cap\mathcal{S}_N}\mathbb{P}(\tilde{F}_e\cap\tilde{F}_f)\\ &\qquad\le \sum_{j=4}^{N^{4/7-2\epsilon'}/4}(C_2j)\times(N^{-1/7+\epsilon'}\mathbb{P}(\Gamma_2(x_f,S_{j/2}(x_f))))\\ &\qquad\le C_3N^{-1/7+\epsilon'}\sum_{j=4}^{N^{4/7-2\epsilon'}/4}j\times j^{-1/4+\epsilon'}\\ &\qquad\le C_4N^{-1/7+\epsilon'}(N^{4/7-2\epsilon'}/4)^{7/4+\epsilon'}\\ &\qquad\le N^{6/7+2\epsilon'}.\end{aligned}$$

Finally, the edges at distance $d<4$ give a contribution which is at most $C_5\mathbb{P}(\tilde{F}_e)\le N^{-1/7+\epsilon'}\ll N^{6/7+2\epsilon'}$.

Summing the different contributions, we get that

$$\sum_{f \in S_{2N^{4/7+\epsilon'}}(x_e) \cap \mathcal{S}_N} \mathbb{P}(\tilde{F}_e \cap \tilde{F}_f) \leq N^{6/7+6\epsilon'}.$$

Hence,

$$\begin{aligned}\mathrm{Var}[T_N] &\leq [6(2N^{4/7+\epsilon'}+1)(\ell_N+1)] \times [N^{6/7+6\epsilon'}] + \epsilon_N \\ &\leq N^{10/7+\epsilon}\ell_N.\end{aligned}$$

□

Remark 74. *Heuristically, this bound can be explained as follows: if we divide the strip $\mathcal{S}_N$ into ca. $(\ell_N/N^{4/7})$ disjoint (and thus more or less independent) boxes, the contribution of each of them to the variance is of order $((N^{4/7})^2)^2 N^{-2/7} = N^{14/7}$, so that the total variance is of order*

$$(\ell_N/N^{4/7}) \times N^{14/7} = N^{10/7}\ell_N.$$

4.5 Outer boundary

To further describe $\mathcal{F}_N$ (still assuming it is unique), we can also consider its outer boundary (or accessible perimeter). Actually, two curves arise: the "upper" outer boundary $\mathcal{U}_N^+$, and the "lower" outer boundary $\mathcal{U}_N^-$. The upper outer boundary can be defined for instance as the lowest *self-avoiding* vacant crossing of $\mathcal{S}_N$. It can be viewed as the upper boundary of $\mathcal{F}_N$ "without the fjords" (the sites that are connected to the top of $\mathcal{S}_N$ only by a one-site wide passage). For a site v, being on $\mathcal{U}_N^+$ is equivalent to have 3 arms: two disjoint vacant arms to the left and right sides of $\mathcal{S}_N$, and an occupied arm to the bottom.

The lengths of $\mathcal{U}_N^+$ and $\mathcal{U}_N^-$, denoted respectively by U_N^+ and U_N^-, can be computed in a similar way as the length T_N of the front, by counting the 3-arm sites (two vacant and one occupied for U_N^+, two occupied and one vacant for U_N^-). There is however an extra technical difficulty to adapt Kesten's result: being pivotal still leads to 4 arms locally, but not necessarily long enough (due to consecutive arms of the same color), and the reasoning has to be adapted (see [55] for more details). We get the following estimate on $\mathbb{E}[U_N^\pm]$:

Proposition 75. *We still assume that $\ell_N \geq N^{4/7+\delta}$ and $\ell_N = o(N^\gamma)$. For all $\epsilon > 0$, we have for N sufficiently large:*

$$N^{4/21-\epsilon}\ell_N \leq \mathbb{E}[U_N^\pm] \leq N^{4/21+\epsilon}\ell_N. \tag{4.1}$$

The upper bound can be treated in the same way as in Proposition 70, but for the lower bound, we have to be a little more careful: we have to ensure that we can extend the two arms of the same type into two *disjoint* macroscopic arms. It can be done by using the fact that we can impose an occupied crossing in a $N^{4/7+\epsilon} \times N^{4/7+\epsilon}$ rhombus to arrive in a "corner" of size $N^{4/7-\epsilon}$ with probability at least $N^{-\kappa\epsilon}$ for some constant κ (for instance applying the RSW estimate $\log N^{2\epsilon}$ times: see figure).

The decorrelation arguments that we used for the length of the front can also be applied to the outer boundary, entailing convergence in L^2.

Proposition 76. *If $\ell_N \geq N^{4/7+\delta}$ and $\ell_N = o(N^\gamma)$, then for each $\epsilon > 0$, we have for N sufficiently large:*

$$Var[U_N^\pm] \leq N^{20/21+\epsilon}\ell_N. \tag{4.2}$$

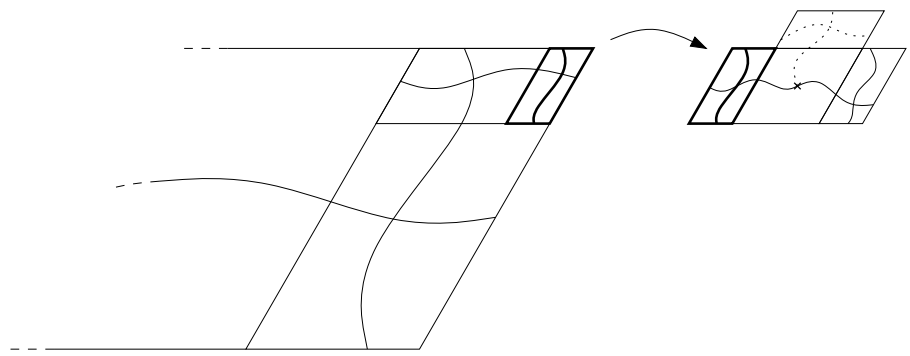


Figure 4.10: We can extend the 3 arms by successive applications of RSW.

Hence,

$$\frac{U_N^{\pm}}{\mathbb{E}[U_N^{\pm}]} \longrightarrow 1 \quad \textit{in } L^2, \textit{ when } N \to \infty. \tag{4.3}$$

Note that when $\ell_N = N$, this result implies that $U_N^{\pm}$ is of order $N^{25/21}$ with high probability.

4.6 Open questions

We have used here the logarithmic equivalence of Proposition 60, that has been proved in [71]. It is in fact conjectured that this equivalence holds up to multiplicative constants. For instance, the stronger assumption $\mathbb{P}_{1/2}(A_n^4) \asymp n^{-5/4}$ would lead to $L(p) \asymp |p - 1/2|^{-4/3}$, which would make it possible to derive a sharper description of the front.

A natural question would be to describe (if it exists) the scaling limit of the front, once properly renormalized. We could first argue as follows: in each of the sub-strips $[\pm N^{4/7-\epsilon}]$, everything "looks like criticality", so that some aspects of the front will be the same as those of critical percolation interfaces, *i.e.* SLE_6 curves.

However, we do not know at present how the front "bounces", that is to say what happens when we are at a distance approximately $N^{4/7}$ from the critical line. In fact, the front could be linked to the objects Camia, Fontes and Newman have recently introduced to describe percolation near criticality [18, 19], when we take the parameter to be $p = 1/2 + \lambda\delta^{3/4}$, with a possibly inhomogeneous λ.

Chapter 5

Scaling limits for the gradient percolation front

Abstract: We further study the interface arising in the gradient percolation model, known as the "percolation front". More precisely, we identify a characteristic length, that enables us to tighten the previous estimates and to construct non-trivial scaling limits for the front. These limiting objects are similar to off-critical percolation interfaces.

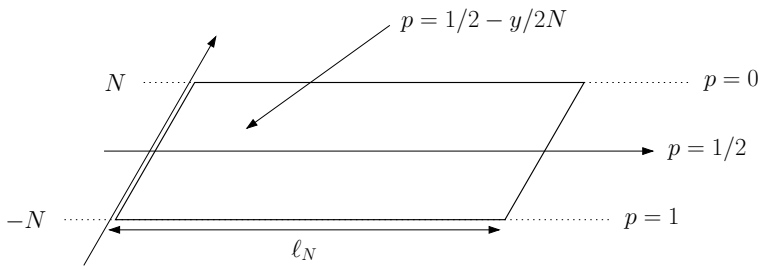


Figure 5.1: Gradient percolation in a strip.

5.1 Introduction

This paper is a sequel to [54] where we studied the gradient percolation model, an inhomogeneous percolation process where the density of black (occupied) sites depends on the location in space. We considered in that paper a strip of length ℓ_N and of finite width $2N$, with a parameter $p(y)$ decreasing linearly from 1 to 0 (see Figure 5.1).

For standard (homogeneous) percolation on the triangular lattice (we will often restrict ourselves to this lattice) at criticality, i.e. with parameter $p = 1/2$, Smirnov's proof of conformal invariance in the scaling limit [69] and the SLE computations by Lawler, Schramm and Werner (see e.g. [50]) provided a complete rigorous proof of the existence and values of the so-called "arm exponents" [71]. Combining these results with Kesten's scaling relations [43], one gets a rather precise description of percolation near criticality in two dimensions, for instance the power law for the characteristic length

$$\xi(p) = |p - 1/2|^{-\nu + o(1)}$$

as $p \to 1/2$, with $\nu = 4/3$.

We used these results and ideas to describe gradient percolation in a strip (still on the triangular lattice). We showed in particular that if the length of the strip satisfies $\ell_N \gg N^{\nu/(1+\nu)}$ $(= N^{4/7})$, then with high probability, there exists a unique "front", an interface between the cluster of black sites connected to the bottom of the strip and the cluster of white sites connected to the top, and the vertical fluctuations of this front are of order $N^{\nu/(1+\nu)}$. We also proved that various other macroscopic quantities related to it – its length for instance – can be described via critical exponents related to the exponents of standard percolation.

These results suggest that in order to construct non-trivial scaling limits for the front, one should scale it by a factor of order $N^{4/7}$ – if one scales it directly by N, then one just gets a straight line in the limit. However, the existence of such scaling limits was not proved due to possible logarithmic corrections in all estimates coming from SLE computations.

In this paper, we identify a – both horizontal and vertical – characteristic length σ_N for this model, which turns out to be the right way to scale the front. Our main result is that we can use tightness arguments due to Aizenman and Burchard to construct such non-trivial scaling limits, whose properties are reminiscent of those of interfaces in off-critical regime, studied in [56].

5.2 Characteristic length for gradient percolation

5.2.1 Percolation background

This paper uses earlier results on near-critical percolation, in particular Kesten's paper [43]. In this section, we recall some results that we shall use. All these results are stated and derived in [55], and we follow the notations of this paper. In particular, we restrict ourselves to the triangular lattice, with basis $(1, e^{i\pi/3})$, so that the parallelogram of corners $a_i + b_j e^{i\pi/3}$ $(i, j \in \{1, 2\})$ is denoted by $[a_1, a_2] \times [b_1, b_2]$. The notation $f \asymp g$ means that there exist two constants $C_1, C_2 > 0$ such that $C_1 g \leq f \leq C_2 g$, and $f \approx g$ means that $\log f / \log g \to 1$.

Recall the definition of the correlation length: for any $\epsilon \in (0, 1/2)$,

$$L_\epsilon(p) = \min\{n \text{ s.t. } \mathbb{P}_p(\mathcal{C}_H([0, n] \times [0, n])) \leq \epsilon\} \tag{5.1}$$

when $p < 1/2$, and the same with white crossings when $p > 1/2$ (so that $L_\epsilon(p) = L_\epsilon(1-p)$). We use the events

$$A_2(n_1, n_2) = A_{2,BW}(n_1, n_2) = \{\partial S_{n_1} \rightsquigarrow_{2,BW} \partial S_{n_2}\}$$

that there exist two arms, one black and one white, crossing the annulus $S_{n_1,n_2} = S_{n_2} \setminus \mathring{S}_{n_1}$ of radii n_1 and n_2 centered on the origin, and

$$A_4(n_1, n_2) = A_{4,BWBW}(n_1, n_2) = \{\partial S_{n_1} \rightsquigarrow_{4,BWBW} \partial S_{n_2}\}.$$

We also introduce the notation, for $j = 2$ or 4,

$$\pi_j(n_1, n_2) = \mathbb{P}_{1/2}(A_j(n_1, n_2)).$$

The following properties hold for any fixed $\epsilon \in (0, 1/2)$:

1. The Russo-Seymour-Welsh estimates are valid below $L_\epsilon(p)$: for all $k \geq 1$, there exists $\delta_k = \delta_k(\epsilon) > 0$ such that for all p, all $n \leq L_\epsilon(p)$, any product measure $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$,
$$\hat{\mathbb{P}}(\mathcal{C}_H([0, kn] \times [0, n])) \geq \delta_k. \tag{5.2}$$

2. The crossing probabilities decay exponentially fast with respect to $L_\epsilon(p)$: there exist constants $C_i = C_i(\epsilon) > 0$ such that for all $p < 1/2$, all n,
$$\mathbb{P}_p(\mathcal{C}_H([0, n] \times [0, 2n])) \leq C_1 e^{-C_2 n / L_\epsilon(p)}. \tag{5.3}$$

3. We have (this is known as "quasi-multiplicativity") for $j = 2$ or 4
$$\hat{\mathbb{P}}(A_j(n_1/2)) \times \hat{\mathbb{P}}(A_j(2n_1, n_2)) \asymp \hat{\mathbb{P}}(A_j(n_2)) \tag{5.4}$$
uniformly in p, $2n_1 \leq n_2 \leq L_\epsilon(p)$ and $\hat{\mathbb{P}}$ between $\mathbb{P}_p$ and $\mathbb{P}_{1-p}$, and also
$$\hat{\mathbb{P}}(A_j(n_1, n_2)) \asymp \mathbb{P}_{1/2}(A_j(n_1, n_2)). \tag{5.5}$$

4. For any $\eta \in (0, 1)$,
$$\mathbb{P}_{1/2}(A_2(\eta n, n)) \to f_2(\eta) \tag{5.6}$$
as $n \to \infty$, where $f_2(\eta) = \eta^{1/4 + o(1)}$ as $\eta \to 0^+$. This implies that
$$\mathbb{P}_{1/2}(A_2(n)) \approx n^{-1/4} \tag{5.7}$$
as $n \to \infty$.

5.2.2 Regularity and asymptotic behavior of L_ϵ

Recall [55] that for any (fixed) $\epsilon \in (0, 1/2)$,

$$|p - 1/2| L_\epsilon(p)^2 \pi_4(L_\epsilon(p)) \asymp 1 \tag{5.8}$$

as $p \to 1/2$. By combining this relation with an a-priori bound for π_4, namely that

$$\pi_4(n_1, n_2) \geq c(n_1/n_2)^{2-\alpha} \tag{5.9}$$

for any $n < N$ ($c, \alpha > 0$ being universal constants), we obtain:

- For any two $\epsilon, \epsilon' \in (0, 1/2)$,
$$L_\epsilon(p) \asymp L_{\epsilon'}(p). \tag{5.10}$$
- For any $\epsilon \in (0, 1/2)$,
$$L_\epsilon(p) \approx |p - 1/2|^{-\nu} \tag{5.11}$$
as $p \to 1/2$, with $\nu = 4/3$.

Here we will need to compare $L_\epsilon(1/2 + \delta)$ and $L_\epsilon(1/2 + \delta')$ for two small values $\delta, \delta' > 0$. We could write

$$\frac{L_\epsilon(1/2 + \delta)}{L_\epsilon(1/2 + \delta')} \approx \left(\frac{\delta}{\delta'}\right)^{-4/3},$$

but this logarithmic equivalence is not precise enough. We bypass this difficulty by deriving a weaker result that is sufficient for our purpose.

Lemma 77. *There exist some universal constants $C_1, C_2, \alpha_1, \alpha_2 > 0$ such that*

$$C_1 \left(\frac{\delta}{\delta'}\right)^{-\alpha_1} \leq \frac{L_\epsilon(1/2 + \delta)}{L_\epsilon(1/2 + \delta')} \leq C_2 \left(\frac{\delta}{\delta'}\right)^{-\alpha_2} \tag{5.12}$$

for any two $0 < \delta < \delta' < 1/2$.

Proof. The proof is essentially the same as that of Eq.(5.10). We know from Eq.(5.8) that

$$\delta\big(L_\epsilon(1/2 + \delta)\big)^2 \pi_4(L_\epsilon(1/2 + \delta)) \asymp 1 \asymp \delta'\big(L_\epsilon(1/2 + \delta')\big)^2 \pi_4(L_\epsilon(1/2 + \delta')), \tag{5.13}$$

hence,

$$\frac{\big(L_\epsilon(1/2 + \delta)\big)^2}{\big(L_\epsilon(1/2 + \delta')\big)^2} \asymp \frac{\pi_4(L_\epsilon(1/2 + \delta'))}{\pi_4(L_\epsilon(1/2 + \delta))} \times \frac{\delta'}{\delta}. \tag{5.14}$$

We deduce that

$$\frac{\big(L_\epsilon(1/2 + \delta)\big)^2}{\big(L_\epsilon(1/2 + \delta')\big)^2} \asymp \big(\pi_4(L_\epsilon(1/2 + \delta'), L_\epsilon(1/2 + \delta))\big)^{-1} \times \frac{\delta'}{\delta} \tag{5.15}$$

by quasi-multiplicativity (item 3. above, in the case of 4 arms). The estimate for 4 arms Eq.(5.9), and also the trivial bound $\pi_4(n_1, n_2) \leq 1$, now provide the desired conclusion. □

5.2.3 Gradient percolation: setup

We consider gradient percolation in the strip

$$\mathcal{S}_N^\infty = (-\infty, +\infty) \times [-N, N], \tag{5.16}$$

with parameter

$$p(y) = \frac{1}{2} - \frac{y}{2N}. \tag{5.17}$$

For our purpose, working with such a strip infinite in both directions will be more convenient: we get in this way a "stationary process", and we avoid the boundary effects.

Note that with this setting, there is a.s. a unique interface between the infinite cluster of black sites connected to $(-\infty, +\infty) \times \{-N\}$ and the infinite cluster of white sites connected to $(-\infty, +\infty) \times \{N\}$: indeed, there exists a column consisting – except for the top site – only of black sites. We refer to this (random) interface as "the front", and we denote it by $\mathcal{F}_N$. We also introduce the sub-strips

$$\mathcal{S}_N^{t_1,t_2} = [t_1, t_2] \times [-N, N] \subseteq \mathcal{S}_N^\infty \tag{5.18}$$

for $-\infty < t_1 < t_2 < +\infty$, and

$$[u_1, u_2] = (-\infty, +\infty) \times [u_1, u_2]. \tag{5.19}$$

We use in particular the notation $[\pm u] = [-u, u] = (-\infty, +\infty) \times [-u, u]$.

5.2.4 Characteristic length for gradient percolation

We now introduce a quantity σ_N^ϵ that measures the vertical fluctuations of the front. As will become clear in the following, this quantity can be seen as a – both horizontal and vertical – characteristic length for the gradient percolation model.

Definition 78. *For any $\epsilon \in (0, 1/2)$, any $N \geq 1$, we define*

$$\sigma_N^\epsilon = \sup\left\{\sigma \ s.t. \ L_\epsilon\big(p(\sigma)\big) = L_\epsilon\left(\frac{1}{2} - \frac{\sigma}{2N}\right) \geq \sigma\right\}. \tag{5.20}$$

Note that if we plug into this definition the value of the exponent ν associated to L_ϵ (i.e. Eq.(5.11)), we get that

$$\sigma_N^\epsilon \approx N^{\nu/(1+\nu)} = N^{4/7} \tag{5.21}$$

as $N \to \infty$. This implicit definition makes life easier compared to the closed value $N^{4/7}$ that we used in [54], for which we had to take care of potential logarithmic corrections.

Eq.(5.10) implies that

$$\sigma_N^\epsilon \asymp \sigma_N^{\epsilon'} \tag{5.22}$$

for any two $\epsilon, \epsilon' \in (0, 1/2)$: the scale σ_N^ϵ is unique up to multiplicative constants. We thus fix some $\epsilon \in (0, 1/2)$ (e.g. $\epsilon = 1/4$) for the rest of the paper. We will see in the next section that σ_N^ϵ is the right "scale" of this model.

Note that by definition of σ_N^ϵ, we have

$$L_\epsilon(p(\sigma_N^\epsilon)) \asymp \sigma_N^\epsilon. \tag{5.23}$$

On one hand, we know that for any fixed $u \geq 1$, the Russo-Seymour-Welsh lower bounds hold in the strip $[\pm u\sigma_N^\epsilon]$, uniformly for $N \geq 1$. On the other hand, the previous regularity lemma for L_ϵ (Lemma 77) implies that for any $u \geq 1$,

$$L_\epsilon\big(p(u\sigma_N^\epsilon)\big) = L_\epsilon\left(\frac{1}{2} - \frac{u\sigma_N^\epsilon}{2N}\right) \leq C_1^{-1}u^{-\alpha_1}L_\epsilon\left(\frac{1}{2} - \frac{\sigma_N^\epsilon}{2N}\right) = C_1^{-1}u^{-\alpha_1}L_\epsilon\big(p(\sigma_N^\epsilon)\big), \tag{5.24}$$

so that using Eq.(5.23),

$$L_\epsilon\big(p(u\sigma_N^\epsilon)\big) \leq cu^{-\alpha}\sigma_N^\epsilon \tag{5.25}$$

for some universal constants $c, \alpha > 0$ (we would expect that $\alpha = 4/3$, but this is not clear due to the logarithmic corrections that we have mentioned).

5.2.5 Macroscopic properties of the front

The definition of σ_N^ϵ enables us to tighten the estimates of [54]. The proofs are essentially the same, we recall them briefly since small adaptations are needed. Our reasonings are based on the following two main observations:

- The front $\mathcal{F}_N$ never goes far from the strip $[\pm\sigma_N^\epsilon]$, due to the exponential decay property Eq.(5.3).
- The behavior of $\mathcal{F}_N$ in any strip $[\pm u\sigma_N^\epsilon]$ $(u \geq 1)$ is roughly the same as that of critical percolation.

Localization

Proposition 79. *There exist some universal constants $\alpha, C_1, C_2 > 0$ such that*

$$\mathbb{P}(\mathcal{F}_N \cap \mathcal{S}_N^{0,t\sigma_N^\epsilon} \not\subseteq [\pm u\sigma_N^\epsilon]) \leq C_1\left(\frac{t}{u} + \frac{u}{t}\right)e^{-C_2u^\alpha} \tag{5.26}$$

for all $t, u \geq 1$.

Proof. Assume that $\mathcal{F}_N \cap \mathcal{S}_N^{0,t\sigma_N^\epsilon} \not\subseteq [\pm u\sigma_N^\epsilon]$. If for instance $\mathcal{F}_N \cap \mathcal{S}_N^{0,t\sigma_N^\epsilon}$ exits the strip $[\pm u\sigma_N^\epsilon]$ from above, then either it stays above it and crosses horizontally the strip $[0, t\sigma_N^\epsilon] \times [u\sigma_N^\epsilon, N]$, either it crosses vertically the strip $[0, t\sigma_N^\epsilon] \times [\frac{u}{2}\sigma_N^\epsilon, u\sigma_N^\epsilon]$.

Consider the first case: the event $\mathcal{C}_H([0, t\sigma_N^\epsilon] \times [u\sigma_N^\epsilon, N])$ occurs. If $t \geq u$, there is a horizontal crossing in one of the parallelograms $[0, u\sigma_N^\epsilon] \times [(u + ku)\sigma_N^\epsilon, (u + (k + 2)u)\sigma_N^\epsilon]$ $(k = 0, 1, \ldots)$, or a vertical crossing in one of the parallelograms $[0, 2u\sigma_N^\epsilon] \times [(u + k'u)\sigma_N^\epsilon, (u + (k' + 1)u)\sigma_N^\epsilon]$ $(k' = 0, 1, \ldots)$. Using the exponential decay property Eq.(5.3), we get

$$\begin{aligned}\mathbb{P}\big(\mathcal{C}_H([0, u\sigma_N^\epsilon] \times [(u + ku)\sigma_N^\epsilon, (u + (k + 2)u)\sigma_N^\epsilon])\big)& \\ &\leq C_1e^{-C_2u\sigma_N^\epsilon/L_\epsilon(p((u+ku)\sigma_N^\epsilon))} \\ &\leq C_1e^{-C_3(k+1)^\alpha u^\alpha}\end{aligned}$$

since $L_\epsilon(p((u+ku)\sigma_N^\epsilon)) \leq c((k+1)u)^{-\alpha}\sigma_N^\epsilon$ (by Eq.(5.25)). The same bound holds for $\mathbb{P}(\mathcal{C}_V([0, 2u\sigma_N^\epsilon] \times [(u + k'u)\sigma_N^\epsilon, (u + (k' + 1)u)\sigma_N^\epsilon]))$, and by summing over $k, k' \geq 0$, we get that the considered event has a probability at most

$$\begin{aligned}2\sum_{k=1}^{\infty} C_1e^{-C_3k^\alpha u^\alpha} &\leq 2C_1e^{-C_3u^\alpha}\sum_{k=1}^{\infty} e^{-C_3(k^\alpha-1)} \\ &\leq C_4\frac{t}{u}e^{-C_3u^\alpha},\end{aligned}$$

by using that $u^\alpha \geq 1$ and $t/u \geq 1$.

If $t < u$, we use parallelograms of size t instead, and the considered event has a probability at most:

$$\begin{aligned} 2\sum_{k=0}^{\infty} C_1 e^{-C_3(u+kt)^\alpha} &\leq C_4 \frac{u}{t} \sum_{l=1}^{\infty} e^{-C_3 l^\alpha u^\alpha} \\ &\leq C_5 \frac{u}{t} e^{-C_3 u^\alpha}, \end{aligned}$$

since for each l, there are of order u/t values of k for which $lu \leq u + kt < (l+1)u$.

Consider now the second case: the event $\mathcal{C}_V([0, t\sigma_N^\epsilon] \times [\frac{u}{2}\sigma_N^\epsilon, u\sigma_N^\epsilon])$ occurs. If $t \geq u$, there is a vertical crossing in one of the parallelograms $[ku\sigma_N^\epsilon, (k+1)u\sigma_N^\epsilon] \times [\frac{u}{2}\sigma_N^\epsilon, u\sigma_N^\epsilon]$ $(0 \leq ku \leq t)$, or a horizontal crossing in one of the parallelograms $[k'\frac{u}{2}\sigma_N^\epsilon, (k'+1)\frac{u}{2}\sigma_N^\epsilon] \times [\frac{u}{2}\sigma_N^\epsilon, 3\frac{u}{2}\sigma_N^\epsilon]$ $(0 \leq k'\frac{u}{2} \leq t)$. There are of order t/u such parallelograms, and for each of them there is a crossing with probability at most

$$C_1 e^{-C_2 \frac{u}{2}\sigma_N^\epsilon / L_\epsilon(p(\frac{u}{2}\sigma_N^\epsilon))} \leq C_1 e^{-C_3 u^\alpha}.$$

The considered event has thus a probability at most

$$C_4 \frac{t}{u} e^{-C_3 u^\alpha}.$$

If $t < u$, we use the parallelogram $[0, t\sigma_N^\epsilon] \times [\frac{u}{2}\sigma_N^\epsilon, u\sigma_N^\epsilon]$: it is crossed vertically with probability at most

$$C_1 \frac{u}{t} e^{-C_3 u^\alpha},$$

since $u/t \geq 1$. □

Remark 80. *In the other direction, let us fix some $u \geq 1$. For $t \geq u$, we can consider the independent parallelograms $[ku\sigma_N^\epsilon, (k+1)u\sigma_N^\epsilon] \times [-u\sigma_N^\epsilon, u\sigma_N^\epsilon]$ $(0 \leq k \leq \frac{t}{u} - 1)$: in each of them, a vertical black crossing occurs with probability at least $\delta_2 = \delta_2(u) > 0$, so that*

$$\mathbb{P}(\mathcal{F}_N \cap \mathcal{S}_N^{0,t\sigma_N^\epsilon} \subseteq [\pm u\sigma_N^\epsilon]) \leq (1-\delta_2)^{t/u-1} \leq C_1 e^{-C_2 t}, \tag{5.27}$$

for some constants $C_1, C_2 > 0$ (depending on u).

Uniqueness

Proposition 81. *There exist some universal constants $C_1, C_2 > 0$ such that*

$$\mathbb{P}(\text{ unique crossing in } \mathcal{S}_N^{0,t\sigma_N^\epsilon}) \geq 1 - C_1 e^{-C_2 t}. \tag{5.28}$$

Proof. We need the following property: we state it as a separate lemma since we will use it again later.

Lemma 82. *For each fixed $v \geq 0$, there exists some $\eta(v) > 0$ independent of N such that*

$$\mathbb{P}\big(\mathcal{C}_V([0, \sigma_N^\epsilon] \times [-N, v\sigma_N^\epsilon])\big) \geq \eta(v). \tag{5.29}$$

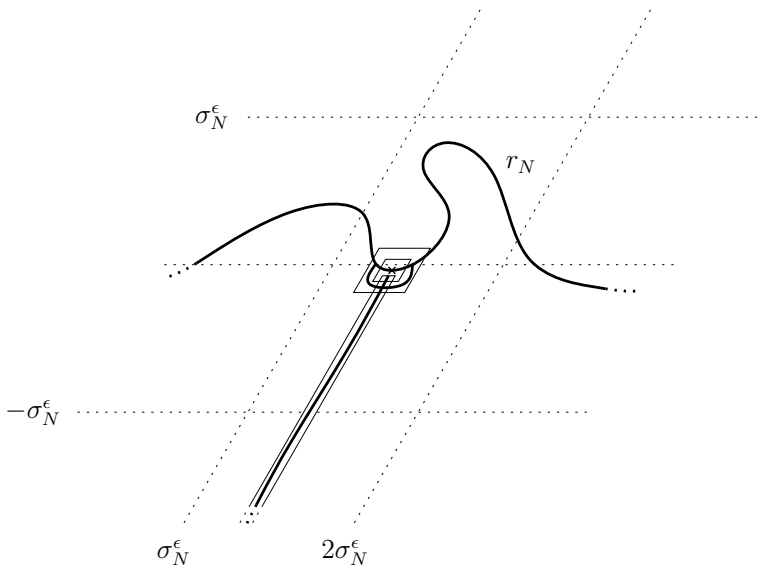


Figure 5.2: With positive probability, the front is connected to the bottom of the strip $\mathcal{S}_N^{0,3\sigma_N^\epsilon}$.

Proof. We construct a vertical crossing in $[0,\sigma_N^\epsilon]\times[-N,v\sigma_N^\epsilon]$ by combining vertical crossings in the parallelograms $[0,\sigma_N^\epsilon]\times[(v-(k+2))\sigma_N^\epsilon,(v-k)\sigma_N^\epsilon]$ $(k=0,1,\ldots)$ and horizontal crossings in the parallelograms $[0,\sigma_N^\epsilon]\times[(v-(k'+1))\sigma_N^\epsilon,(v-k')\sigma_N^\epsilon]$ $(k'=1,2,\ldots)$. It is easy to check that all these crossings exist with probability at least

$$C_1(v)\prod_{k=k_0}^{\infty}(1-C_2e^{-C_3k^\alpha})=\eta(v)>0, \tag{5.30}$$

using (as for localization) the exponential decay property Eq.(5.3) and Eq.(5.25). □

Consider now the strip $\mathcal{S}_N^{0,3\sigma_N^\epsilon}$. Let us condition on the upper-most crossing r_N in this strip. We know from Lemma 82 that

$$\mathbb{P}\big(\mathcal{C}_V^*([\sigma_N^\epsilon,2\sigma_N^\epsilon]\times[0,N])\big)\geq\eta(0)>0, \tag{5.31}$$

a lowest point on r_N thus lies below the x-axis with probability at least $\eta(0)$: let us assume that this is the case. The construction of Figure 5.2 then shows that with probability at least $\delta_4^4\eta(0)$, r_N is connected to the bottom of the strip $\mathcal{S}_N^{0,3\sigma_N^\epsilon}$, by a path staying in this strip.

Now, consider ct disjoint sub-strips $\mathcal{S}_N^{(i)}$ of length $3\sigma_N^\epsilon$ in the strip $\mathcal{S}_N^{0,t\sigma_N^\epsilon}$, for $c>0$ a small constant: for each i, $\mathcal{F}_N$ is connected to the bottom of $\mathcal{S}_N^{(i)}$ (by a path staying in $\mathcal{S}_N^{(i)}$) with probability at least $\delta'=\delta_4^4\eta(0)^2$, so that $\mathcal{F}_N$ is connected to the bottom of $\mathcal{S}_N^{0,t\sigma_N^\epsilon}$ with probability at least

$$1-(1-\delta')^{ct}\geq 1-C_1e^{-C_2t}.$$

□

Remark 83. *Note that this uniqueness property implies in particular that the front does not "bounce back" too much: we would otherwise observe multiple interfaces in some sub-strips (see*

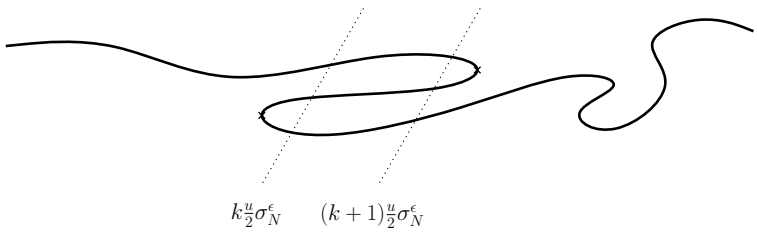


Figure 5.3: If the front bounced back of a distance larger than $u\sigma_N^\epsilon$, it would create several interfaces in one of the sub-strips $\mathcal{S}_N^{k\frac{u}{2}\sigma_N^\epsilon,(k+1)\frac{u}{2}\sigma_N^\epsilon}$.

Figure 5.3), which has a very small probability. We use this remark in Section 5.3, when constructing scaling limits. We have for instance:

$$\mathbb{P}\big(\ \mathcal{F}_N \text{ bounces back of a distance} \geq u\sigma_N^\epsilon \text{ in } \mathcal{S}_N^{0,t\sigma_N^\epsilon}\ \big) \leq C_1\frac{t}{u}e^{-C_2u}. \tag{5.32}$$

Length

Proposition 84. *The following estimate on the discrete length of $\mathcal{F}_N$ holds:*

$$\mathbb{E}\big[|\mathcal{F}_N \cap \mathcal{S}_N^{0,t\sigma_N^\epsilon}|\big] \asymp t(\sigma_N^\epsilon)^2\pi_2(\sigma_N^\epsilon) \tag{5.33}$$

uniformly for $t \geq 1$ and $N \geq 1$.

Proof. It comes from Eq.(5.5) and Lemma 82 (it also uses the "arm separation lemmas", see [55]) that

$$\mathbb{P}(e \in \mathcal{F}_N) \asymp \pi_2(\sigma_N^\epsilon)$$

uniformly for $e \in [\pm\sigma_N^\epsilon]$, which provides the lower bound.

On the other hand, the construction of Figure 5.4 implies that if $e \in [k\sigma_N^\epsilon, (k+1)\sigma_N^\epsilon]$ $(k \geq 1)$,

$$\mathbb{P}(e \in \mathcal{F}_N) \leq C_1e^{-C_2k^\alpha}\pi_2\big(L_\epsilon(p((k+1)\sigma_N^\epsilon))\big), \tag{5.34}$$

by combining Eqs.(5.3) and (5.25) as for localization. We have

$$\begin{aligned}\pi_2(L_\epsilon(p((k+1)\sigma_N^\epsilon))) &\leq C_3\pi_2\big(L_\epsilon(p((k+1)\sigma_N^\epsilon)), L_\epsilon(p(\sigma_N^\epsilon))\big)^{-1}\pi_2(L_\epsilon(p(\sigma_N^\epsilon)))\\ &\leq C_4\left(\frac{L_\epsilon(p((k+1)\sigma_N^\epsilon))}{L_\epsilon(p(\sigma_N^\epsilon))}\right)^{-\tilde\alpha}\pi_2(\sigma_N^\epsilon)\\ &\leq C_5\big(k^{-\alpha'}\big)^{-\tilde\alpha}\pi_2(\sigma_N^\epsilon),\end{aligned}$$

by quasi-multiplicativity (item 3. above) and Lemma 77 (we also used an a-priori bound on π_2, that $\pi_2(n_1,n_2) \geq \tilde c(n_1/n_2)^{\tilde\alpha}$). This provides the upper bound by summing over $k \geq 1$:

$$\begin{aligned}\sum_{k=1}^\infty t(\sigma_N^\epsilon)^2C_1e^{-C_2k^\alpha}\pi_2\big(L_\epsilon(p((k+1)\sigma_N^\epsilon))\big) &\leq C_6t(\sigma_N^\epsilon)^2\pi_2(\sigma_N^\epsilon)\left(\sum_{k=1}^\infty k^{\alpha'\tilde\alpha}e^{-C_2k^\alpha}\right)\\ &\leq C_7t(\sigma_N^\epsilon)^2\pi_2(\sigma_N^\epsilon).\end{aligned}$$

□

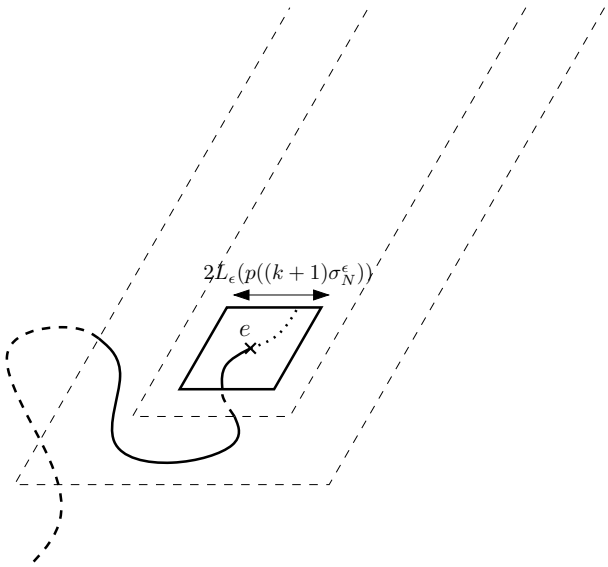


Figure 5.4: If $e \in [k\sigma_N^\epsilon, (k+1)\sigma_N^\epsilon]$ is on $\mathcal{F}_N$, then there are two arms from e going to distance $L_\epsilon(p((k+1)\sigma_N^\epsilon))$, and a black crossing in the U-shaped region.

5.3 Existence of non-trivial scaling limits

Eqs.(5.26) and (5.27) show that σ_N^ϵ is the unique way (up to multiplicative constants) to scale $\mathcal{F}_N$ in order to get non-trivial scaling limits. Indeed, using a quantity much smaller or much larger than σ_N^ϵ would not produce any interesting limit. We now consider the interface f_N, obtained by scaling $\mathcal{F}_N$ in both directions by σ_N^ϵ, and we use the classical tightness arguments of Aizenman and Burchard [2] to construct scaling limits. Eq. (5.27) will then ensure that the scaling limits so obtained are non-trivial.

Let us make the setting a bit more precise. We work with the following *space of interfaces* $\mathcal{S}$: we consider the set of continuous functions $\gamma : (-\infty, +\infty) \longrightarrow \mathbb{R}^2$ such that the x-coordinate of $\gamma(t)$ tends to $+\infty$ as $t \to +\infty$, and to $-\infty$ as $t \to -\infty$, where we identify two curves γ and $\tilde{\gamma}$ if they are the same up to reparametrization, i.e. if there exists an increasing bijection $\phi : (-\infty, +\infty) \longrightarrow (-\infty, +\infty)$ such that $\gamma = \tilde{\gamma} \circ \phi$.

Recall that we usually endow the space of curves defined on a compact, say $[0, 1]$, with the uniform distance "up to reparametrization"

$$d(\gamma_1, \gamma_2) = \inf_{\phi} \sup_{t \in [0,1]} |\gamma_1(t) - \gamma_2(\phi(t))|, \tag{5.1}$$

where the infimum is taken over the set of increasing bijections $\phi : [0, 1] \longrightarrow [0, 1]$. The same distance over $(-\infty, +\infty)$ would be too strong for our purpose, we rather use the particular structure of $\mathcal{S}$ to define a notion of convergence "on every compact".

For a curve $\gamma \in \mathcal{S}$, we consider its piece $\gamma^{(n)}$ between $\underline{t}^{(n)}$ the first time it reaches $x = -n$ and $\bar{t}^{(n)}$ the last time it hits $x = n$. We can choose the parametrization such that $\underline{t}^{(n)} = -n$ and $\bar{t}^{(n)} = n$, so that $\gamma^{(n)}$ is parametrized by $[-n, n]$. We then define

$$d^{(n)}(\gamma_1, \gamma_2) = \inf_{\phi} \sup_{t \in [0,1]} |\gamma_1(t) - \gamma_2(\phi(t))|, \tag{5.2}$$

with $\phi : [-n, n] \longrightarrow [-n, n]$, and finally the product distance

$$d(\gamma_1, \gamma_2) = \sum_{n=1}^{+\infty} \frac{1}{2^n} \big(d^{(n)}(\gamma_1, \gamma_2) \wedge 1\big). \tag{5.3}$$

One can check (see [11] for instance) that a sequence (γ_k) of interfaces converges in distribution toward γ in $(\mathcal{S}, d)$ *iff* each $\gamma_k^{(n)}$ converges toward $\gamma^{(n)}$ in $(\mathcal{S}^{(n)}, d^{(n)})$, with obvious notation for $\mathcal{S}^{(n)}$. Note also that in our setting, tightness and relative compactness are equivalent, by Prohorov's theorem.

The scaled fronts f_N are elements of $\mathcal{S}$, and we are in position to use the arguments of Aizenman and Burchard [2] in each $\mathcal{S}^{(n)}$.

Proposition 85. *Denoting by P_N the law of f_N, the sequence $(P_N)_{N\geq 1}$ is relatively compact in the set of probability measures on $(\mathcal{S}, d)$.*

Proof. We show the existence of scaling limits in each $\mathcal{S}^{(n)}$. Let us fix some $n \geq 1$. First, there exist some universal $t, u \geq 1$ such that for all N,

$$\mathbb{P}(f_N^{(n)} \subseteq [-(n+t)\sigma_N^\epsilon, (n+t)\sigma_N^\epsilon] \times [-u\sigma_N^\epsilon, u\sigma_N^\epsilon]) \geq 1 - \epsilon/2. \tag{5.4}$$

Indeed, it comes from uniqueness in the strips $\mathcal{S}_N^{-(n+t)\sigma_N^\epsilon, -n\sigma_N^\epsilon}$ and $\mathcal{S}_N^{n\sigma_N^\epsilon, (n+t)\sigma_N^\epsilon}$ (Proposition 81), and then localization for $\mathcal{F}_N \cap \mathcal{S}_N^{-(n+t)\sigma_N^\epsilon, (n+t)\sigma_N^\epsilon}$ (Proposition 79).

We can now use Theorem 1.2 in [2]. Indeed, since the Russo-Seymour-Welsh estimates hold in the strip $[\pm u\sigma_N^\epsilon]$ uniformly for all N, we have for any annulus $\mathcal{A}(x; r, R) = \{z \text{ s.t. } r < |z - x| < R\}$,

$$\mathbb{P}(\mathcal{A}(x; r, R) \text{ is crossed by } f_N^{(n)}) \leq c(r/R)^\alpha$$

for two universal constants $\alpha, c > 0$, and the BK inequality implies that

$$\mathbb{P}(\mathcal{A}(x; r, R) \text{ is crossed } k \text{ times by } f_N^{(n)}) \leq c^k (r/R)^{\alpha k}$$

which is exactly the hypothesis (H1) of [2] (uniform power bounds on the probability of multiple crossings in annuli). Hence, the sequence $(P_N^{(n)})_{N\geq 1}$, consisting of the laws of the $f_N^{(n)}$, is tight. This proves that the sequence $(P_N)_{N\geq 1}$ is relatively compact (using a diagonal argument). □

Remark 86. *Up to now, for the definition of σ_N^ϵ and the subsequent reasonings, we have made no mention of the existence of the exponent ν. The previous results are thus valid on the other "regular" lattices – for which conformal invariance has not been established – like the square lattice $\mathbb{Z}^2$. For this lattice, we know for instance that*

$$L_\epsilon(p) \leq C|p - p_c|^{-A}$$

for some $C, A > 0$ (see [55]), which implies that for some $\alpha > 0$,

$$\sigma_N^\epsilon \leq N^{1-\alpha}.$$

Hence, our proofs still show in this case that the front is unique, converges toward the line $p = p_c$, and that non-trivial scaling limits can be constructed by scaling by σ_N^ϵ.

5.4 Properties of the scaling limits

We now examine some properties of the potential scaling limits of the front. Their behavior is comparable to that of near-critical interfaces, studied in [56], since the front remains (with high probability) in a region where $L_\epsilon(p(y)) = O(y)$.

5.4.1 Similarities with critical percolation

On one hand, these scaling limits share some properties with critical percolation interfaces, i.e. SLE_6, since for instance the probabilities of arm events remain the same up to multiplicative constants. Hence the (discrete) length remains comparable (this is Proposition 84).

In the scaling limit, the Hausdorff dimension of any limit is the same as that of critical percolation interfaces: if f is the weak limit along some sub-sequence (f_{N_k}) of the sequence of interfaces, then the Haussdorff dimension of f is almost surely $7/4$.

5.4.2 Local asymmetry

On the other hand, when the front is macroscopically far from $y = 0$, it is by construction in a region where $L_\epsilon(p(y)) \asymp y \asymp \sigma_N^\epsilon$. Gradient percolation thus provides a natural setting where the off-critical regime, studied in [56], arises.

A hexagon h is on the front *iff* there exist two arms from h to the bottom and top boundaries of the strip (respectively black and white), no matter the state of h. Hence, denoting by $\partial\mathcal{F}_N$ (resp. $\partial^+\mathcal{F}_N$, $\partial^-\mathcal{F}_N$) the set of hexagons adjacent to $\mathcal{F}_N$ (resp. black hexagons, white hexagons),

$$\begin{aligned}\mathbb{P}(h \in \partial^+\mathcal{F}_N) &- \mathbb{P}(h \in \partial^-\mathcal{F}_N)\\ &= \mathbb{P}(h \in \partial\mathcal{F}_N \text{ and } h \text{ is black}) - \mathbb{P}(h \in \partial\mathcal{F}_N \text{ and } h \text{ is white})\\ &= \mathbb{P}(\text{ two arms from } h\ ) \times (2p(y_h) - 1).\end{aligned}$$

Consequently, in any sub-strip $s = [0, t\sigma_N^\epsilon] \times [u_1\sigma_N^\epsilon, u_2\sigma_N^\epsilon]$ $(0 < u_1 < u_2)$, the following asymmetry property holds:

$$\mathbb{E}\big[|\partial^+\mathcal{F}_N \cap s|\big] - \mathbb{E}\big[|\partial^-\mathcal{F}_N \cap s|\big] \asymp -\frac{\sigma_N^\epsilon}{N} \times \big(t(\sigma_N^\epsilon)^2\pi_2(\sigma_N^\epsilon)\big) \approx -N^{4/7}. \tag{5.1}$$

This quantity happens to be much larger than the statistical deviation: indeed,

$$\sqrt{\mathbb{E}\big[|\partial\mathcal{F}_N \cap s|\big]} \asymp \sqrt{t(\sigma_N^\epsilon)^2\pi_2(\sigma_N^\epsilon)} \approx \sqrt{N}. \tag{5.2}$$

If f is the weak limit along some sub-sequence (f_{N_k}) of the sequence of interfaces, the same type of *local* asymmetry as for off-critical interfaces in [56] holds for any (small) portion of f located in a strip of the form $[u_1, u_2]$ $(0 < u_1 < u_2)$: the interface turns more to the right for instance. In particular, its law is (expected to be) singular with respect to that of SLE_6.

Chapter 6

Geometry of diffusion fronts

Abstract: We study the diffusion front for a natural two-dimensional model where many particles that start at the origin diffuse independently. It turns out that this model can be described using properties of near-critical percolation, and provides a natural example where critical fractal geometries spontaneously arise.

6.1 Introduction

In this paper, we study a simple two-dimensional model where a large number of particles that start at a given site diffuse independently on a planar lattice (we will often restrict ourselves to the triangular lattice $\mathbb{T}$). As the particles evolve, a concentration gradient appears, and the random interfaces that arise can be described by comparing the model to an inhomogeneous percolation model (where the probability that a site is occupied or vacant depends on its location). We exhibit in particular a regime where the (properly rescaled) interfaces are fractal with dimension 7/4: this model thus provides a natural setting where *fractal geometry* spontaneously appears, as predicted by physicists.

To our knowledge, the study of the geometry of such "diffusion fronts" has been initiated by the physicists Gouyet, Rosso and Sapoval in [63]. In this 1985 paper, they showed numerical evidence that such interfaces are fractal, and they measured the dimension $D_f = 1.76 \pm 0.02$. To carry on simulations, they used the approximation that the status of the different sites (occupied or not) are independent of each other: they computed the probability $p(z)$ of presence of a particle at site z, and introduced the gradient percolation model, an inhomogeneous percolation process with occupation parameter $p(z)$. Gradient percolation provides a model for phenomena like diffusion or chemical etching – the interface created by welding two pieces of metal for instance – and it has been since then applied to a wide variety of situations (for more details, see the recent account [29] and the references therein).

Based on the observations that the front remains localized where $p(z) \simeq p_c$ and that the dimension D_f was close to 7/4, a value already observed for critical (homogeneous) percolation interfaces, they argued that there should be a close similarity between these diffusion fronts and the boundary of a large percolation cluster. For critical percolation, this conjectured dimension 7/4 was justified later via theoretical physics arguments (see for instance [61, 20, 3]). In 2001, thanks to Smirnov's proof of conformal invariance [69] and to SLE computations by Lawler, Schramm and Werner (see e.g. [50]), a complete rigorous proof appeared [71] in the case of the triangular lattice.

In the paper [54], we have studied the gradient percolation model from a mathematical perspective, building on the works of Lawler, Schramm, Smirnov and Werner. Combining their results with Kesten's scaling relations [43], one first gets a rather precise description of homogeneous percolation near criticality in two dimensions, for instance the power law for the so-called characteristic length

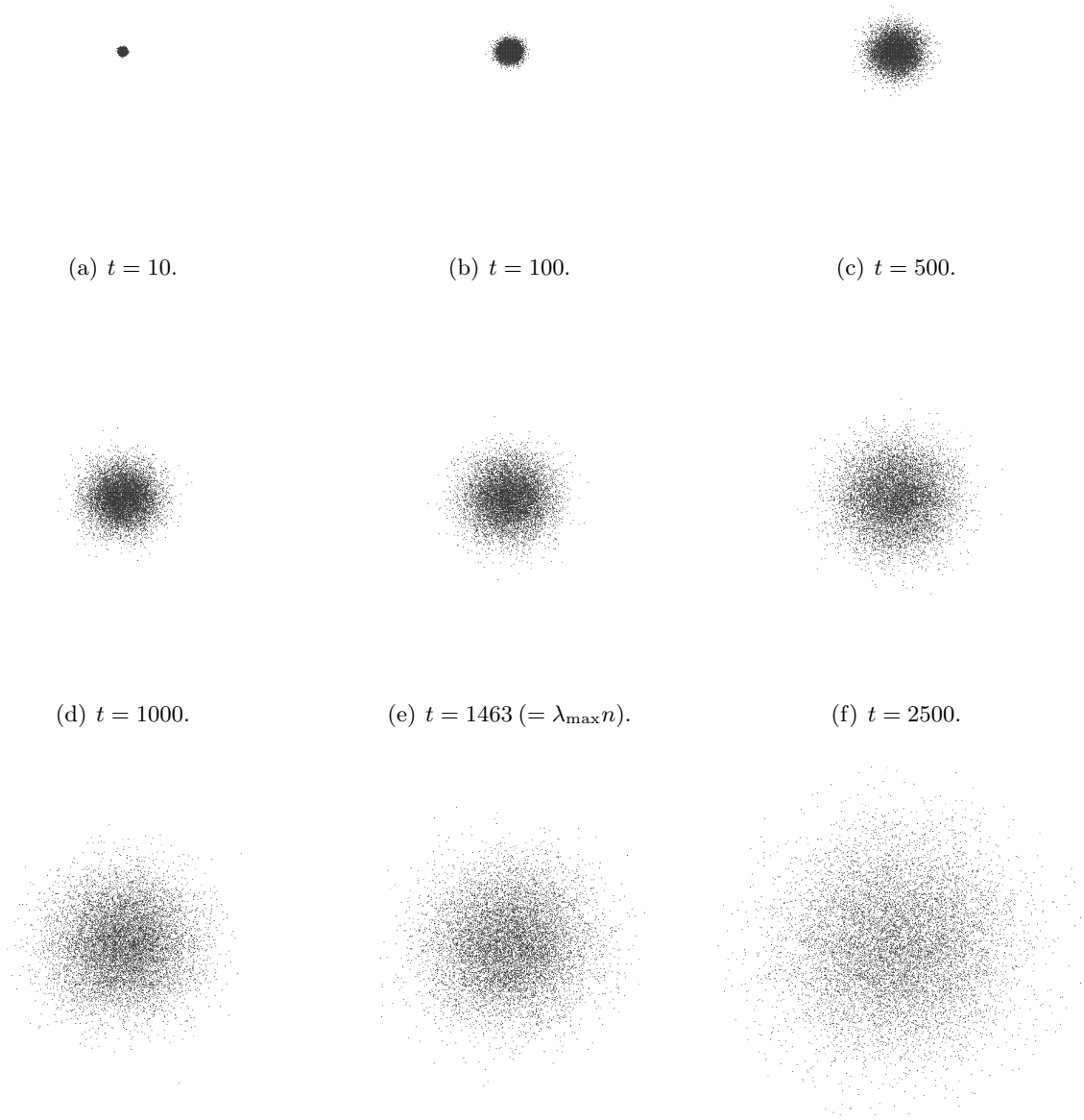
$$\xi(p) = |p - 1/2|^{-4/3+o(1)}$$

as $p \to 1/2$ (here and in the sequel we restrict ourselves to the triangular lattice). This type of estimates (and the underlying techniques) then enabled us to show that the description by [63] is indeed valid for gradient percolation, and to confirm rigorously that the 7/4 dimension shows up in this context.

The goal of the present paper is to investigate "diffusion front" models mathematically and to use our results concerning gradient percolation to show that their geometry and roughness can also be rigorously estimated.

Let us now describe our main model in more detail. We start at time $t = 0$ with a large number n of particles located at the origin, and we let them perform independent random walks. At each time t, we then look at the sites containing at least one particle. These occupied sites can be regrouped into connected components, or "clusters", by connecting two occupied sites if they are adjacent on the lattice.

As time t increases, different regimes arise, as depicted on Figure 6.1. Roughly speaking and



(a) $t = 10$. (b) $t = 100$. (c) $t = 500$.

(d) $t = 1000$. (e) $t = 1463\ (= \lambda_{\max} n)$. (f) $t = 2500$.

(g) $t = 3977\ (= \lambda_c n)$. (h) $t = 5000$. (i) $t = 10000$.

Figure 6.1: Different stages of evolution as the time t increases, for $n = 10000$ particles.

in the large n-limit, at first, a very dense cluster around the origin forms. This clusters grows as long as t remains small compared to n. When t gets comparable to n, the cluster first continues to grow up to some time $t_{\max} = \lambda_{\max} n$, then starts to decrease and it finally dislocates at some critical time $t_c = \lambda_c n$ – and never re-appears. The constants $\lambda_{\max}$ and λ_c are different and their ratio is equal to $1/e$.

To be more specific, the phase transition corresponding to λ_c can be described as follows:

- When $t = \lambda n$ for a $\lambda < \lambda_c$, then (with probability going to one as $n \to \infty$), the origin contains a macroscopic cluster. Its boundary is roughly a circle – we will provide a formula for its radius in terms of λ and n– and its roughness can be described via the dimension $7/4$. This is a *dense phase*.

- When $t = \lambda n$ for a $\lambda > \lambda_c$, then all the clusters – in particular the cluster of the origin – are very small, and the whole picture is "dominated" by a subcritical percolation picture. This is a *dilute phase*.

Actually, alongside these two principal regimes, at least two other interesting cases could be distinguished: the critical regime when t is very close to $\lambda_c n$, and an intermediate regime when t is very small compared to n ($t \ll n^\alpha$ for any $\alpha > 0$, typically $t = \log n$), for which the boundary is smoother. We will not discuss these regimes in the present paper.

We will study some variants of this model and in particular, we will consider the case where a source of particles at the origin does create new particules at the origin in a stationary way. One can for instance think of the formation of an ink stain. If new particles arrive at the origin at some fixed rate $\mu > 0$, then after a long time, one observes a macroscopic cluster near the origin, that grows as time passes. We will explain briefly how such a model can be described using the same techniques and ideas.

6.2 Setup and statement of the results

6.2.1 Description of the model

As previously mentioned, our results will be obtained by comparing our model with an (inhomogeneous) independent percolation process, for some well-chosen occupation parameters. This process is now rather well-understood, we will thus be able to get precise estimates on quantities related to fractal dimensions. Recall that site percolation on a lattice $G = (V, E)$, where V is the set of vertices (or "sites"), and E is the set of edges (or "bonds") can be described as follows. We consider some family of parameters $(p(z), z \in V)$, and we declare each site z occupied with probability $p(z)$, vacant with probability $1 - p(z)$, independently of the other sites. We are then often interested in the connectivity properties of the set of occupied sites, that we group into connected components, or "clusters".

From now on, we will focus on the triangular lattice $\mathbb{T} = (\mathbb{V}^T, \mathbb{E}^T)$, since at present this is the lattice for which the most precise properties, like the existence of critical exponents, are known – a key step being conformal invariance in the scaling limit [69], that remains open on other lattices. However, let us mention that most results remain valid on the common regular lattices like $\mathbb{Z}^2$, except those explicitly referring to the value of some critical exponents, like the fractal dimension of the boundary (see Section 8.1 in [55] for a discussion of this issue). In the standard homogeneous case where $p(z) \equiv p \in [0, 1]$, a celebrated result of Kesten asserts that there is a phase transition at the critical parameter $p_c = 1/2$: when $p \leq 1/2$ there is (a.s.) no infinite cluster, while for $p > 1/2$ there is (a.s.) a unique infinite cluster. In the following, we

call "interface" a curve on the dual hexagonal lattice bordered by occupied sites on one side, and by vacant sites on the other side (the boundary of a finite cluster for instance).

We assume that a fixed number n of particles is deposited at the origin at time $t = 0$, and we let these particles perform independent random walks on the lattice, with equal transition probabilities 1/6 on every edge. We then look at the "occupied" sites containing at least one particle at each time t. We denote by $\pi_t = (\pi_t(z), z \in \mathbb{V}^T)$ the distribution after t steps of a simple random walk starting from 0, so that the probability of occupation for a site z is

$$\tilde{p}_{n,t}(z) = 1 - (1 - \pi_t(z))^n \simeq 1 - e^{-n\pi_t(z)}, \tag{6.1}$$

and we expect

$$\pi_t(z) \simeq \frac{C_T}{t} e^{-\|z\|^2/t}$$

for some constant C_T depending only on the lattice. This type of bound is usually known as a Local Central Limit Theorem, for our purpose we will need a strong version of it that we recall in a next section. We also denote by $N_{n,t}(z)$ the number of particles at site z.

Roughly speaking, $\tilde{p}_{n,t}(z)$ is maximal for $z = 0$, and $\tilde{p}_{n,t}(0)$ decreases to 0 as the particles evolve. Hence, different regimes arise according to t: if $p_{n,t}(0) > 1/2$, there is a giant component around the origin, and its boundary is located in a region where the probability of occupation is close to $p_c = 1/2$. On the other hand, if $p_{n,t}(0) < 1/2$ the model behaves like sub-critical percolation, so that all the different clusters are very small.

As usual in the statistical physics literature, the notation $f \asymp g$ means that there exist two constants $C_1, C_2 > 0$ such that $C_1 g \leq f \leq C_2 g$, while $f \approx g$ means that $\log f / \log g \to 1$ – we will also sometimes use the notation "$\simeq$", which has no precise mathematical meaning.

6.2.2 Main ingredients

Local Central Limit Theorem

To describe the fine local structure of the boundary, we need very precise estimates for π_t. We will use the following strong form of the Local Central Limit Theorem:

$$\pi_t(z) = \frac{\sqrt{3}}{2\pi t} e^{-\|z\|^2/t} + O\left(\frac{1}{t^2}\right) \tag{6.2}$$

uniformly for $z \in \mathbb{V}^T$. This equation means that locally, the estimate given by the CLT holds uniformly: the constant $\sqrt{3}/2$ comes from the "density" of sites on the triangular lattice.

This result can be obtained by sharpening the proofs of [72], as done in the case of $\mathbb{Z}^2$ by Lawler [45]. For the sake of completeness, we provide a proof in an appendix. Note that the estimate in [72], with a $o(1/t)$, would be enough to get localization, but establishing properties like the fractal dimension requires this strong form, with a $O(1/t^2)$.

Poisson approximation

If we took instead a Poissonian number of particles $N \sim \mathcal{P}(n)$, the thinning property of the Poisson distribution would immediately imply that the random variables $(N_{n,t}(z), z \in \mathbb{V}^T)$ are independent of each other, with $N_{n,t}(z) \sim \mathcal{P}(n\pi_t(z))$. We thus simply get an independent percolation process with parameters

$$p_{n,t}(z) = 1 - e^{-n\pi_t(z)}. \tag{6.3}$$

As will soon become clear, this Poisson approximation turns out to be valid, since we only need to observe a negligible fraction of the particles to fully describe the boundary: what we observe locally looks very similar to the "Poissonian case". For that, we use the classical Chen-Stein bound (see Theorem 2.M p.34 in [6], or [21]). Assume that $W = X_1 + \ldots + X_k$ is a sum of k independent random variables, with $X_i \sim \mathcal{B}(p_i)$. Then for $\lambda = \mathbb{E}[W] = \sum_{i=1}^k p_i$, we have

$$\left\|\mathcal{L}(W) - \mathcal{P}(\lambda)\right\|_{TV} \leq \frac{1-e^{-\lambda}}{\lambda} \sum_{i=1}^k p_i^2. \tag{6.4}$$

Now, assume that we look at what happens in a (small) subset A of the sites. The distribution in A can be obtained by first drawing how much particles fall in A, and then by choosing accordingly the configuration in A. Each particle has a probability $\pi_t(A) = \sum_{z\in A} \pi_t(z)$ to be in A, so that the number of particles in A is $N_t(A) \sim \mathcal{B}(n, \pi_t(A))$, and Chen-Stein bound entails that ($\lambda = n\pi_t(A)$ in this case)

$$\left\|\mathcal{L}(N_t(A)) - \mathcal{P}(n\pi_t(A))\right\|_{TV} \leq \frac{1}{n\pi_t(A)} n\pi_t(A)^2 = \pi_t(A). \tag{6.5}$$

We can thus couple $N_t(A)$ with $N_A \sim \mathcal{P}(n\pi_t(A))$ so that they are equal with probability at least $1 - \pi_t(A)$, which tends to 1 if A is indeed negligible. Hence to summarize, we can ensure that the configuration in A coincide with the "Poissonian configuration", i.e. with the result of an independent percolation process, with probability at least $1 - \pi_t(A)$.

Radial gradient percolation

We use earlier results on near-critical percolation, especially Kesten's paper [43]. All the results that we will use here are stated and derived in [55], and we follow the notations of this paper. In particular, we use the basis $(1, e^{i\pi/3})$, so that $[a_1, a_2] \times [b_1, b_2]$ refers to the parallelogram of corners $a_i + b_j e^{i\pi/3}$ $(i, j \in \{1, 2\})$.

We recall that the correlation length is defined as

$$L_\epsilon(p) = \min\{n \text{ s.t. } \mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,n])) \leq \epsilon\} \tag{6.6}$$

when $p < 1/2$ ($\mathcal{C}_H$ denoting the existence of a horizontal occupied crossing), and the same with vacant crossings when $p > 1/2$ (so that $L_\epsilon(p) = L_\epsilon(1-p)$), for any $\epsilon \in (0, 1/2)$, and that we have

$$L_\epsilon(p) \asymp L_{\epsilon'}(p) \approx |p - 1/2|^{-4/3} \tag{6.7}$$

for any two $0 < \epsilon, \epsilon' < 1/2$. We will also use the property of exponential decay w.r.t. L: for any $\epsilon \in (0, 1/2)$ and any $k \geq 1$, there exist some universal constants $C_i = C_i(k, \epsilon)$ such that

$$\mathbb{P}_p(\mathcal{C}_H([0,n] \times [0,kn])) \leq C_1 e^{-C_2 n / L_\epsilon(p)} \tag{6.8}$$

uniformly for $p < 1/2$ and $n \geq 1$.

The aforementioned approximation allows to use results concerning the independent Bernoulli case: we will consider a "radial" gradient percolation, a percolation process where the parameter decreases with the distance to the origin. Let us first recall the description obtained in a strip $[0, \ell_N] \times [0, N]$ with a constant vertical gradient, i.e. with parameter $p(y) = 1 - y/N$: we will explain later how to adapt these results in our setting. The characteristic phenomenon is the appearance of a unique "front", an interface touching at the same time the occupied cluster at the bottom of the strip and the vacant cluster at the top, and:

(i) This front remains localized around the line $y = y_c = \lfloor N/2 \rfloor$, where $p(y) \simeq 1/2$.

(ii) Its fluctuations are of order

$$\sigma_N = \sup\left\{\sigma \text{ s.t. } L(p(y_c + \sigma)) \geq \sigma\right\} \approx N^{4/7}. \tag{6.9}$$

(iii) In the strip of width $2\sigma_N$ around y_c, its behavior is comparable to that of critical percolation (in particular its fine structure can be described via the exponents for critical percolation).

6.2.3 Statement of the results

In view of the previous sections, let us introduce

$$\bar{\pi}_t(r) = \frac{\sqrt{3}}{2\pi t} e^{-r^2/t} \tag{6.10}$$

(so that $\bar{\pi}_t(\|z\|) = \pi_t(z) + O(1/t^2)$ by Eq.(6.2)), and the associated "Poissonian" parameters

$$\bar{p}_{n,t}(r) = 1 - e^{-n\bar{\pi}_t(r)}. \tag{6.11}$$

Since $\bar{p}_{n,t}$ is a decreasing continuous function of r, and tends to 0 when $r \to \infty$, there is a unique r such that $\bar{p}_{n,t}(r) = 1/2$ *iff* $\bar{p}_{n,t}(0) \geq 1/2$, or equivalently *iff* $t \leq t_c = \lambda_c n$, with

$$\lambda_c = \frac{\sqrt{3}}{2\pi \log 2} \quad (\simeq 0.397\ldots). \tag{6.12}$$

We introduce in this case

$$r^*_{n,t} = (\bar{p}_{n,t})^{-1}(1/2). \tag{6.13}$$

In particular, $r^*_{n,t} = 0$ when $t = \lambda_c n$. We can check that

$$r^*_{n,t} = \sqrt{t \log \frac{\lambda_c}{t/n}}. \tag{6.14}$$

An easy calculation shows that $r^*_{n,t}$, as a function of t, increases on $(0, \lambda_{\max} n]$ and then decreases on $[\lambda_{\max} n, \lambda_c n]$ (until being equal to 0), with

$$\lambda_{\max} = \frac{\lambda_c}{e} = \frac{\sqrt{3}}{2e\pi \log 2} \quad (\simeq 0.146\ldots). \tag{6.15}$$

In particular, the ratio $t_{\max}/t_c$ is equal to $1/e$, and is thus independent of n. For future reference, note also that

- When $t = \lambda n$, with $\lambda < \lambda_c$, $r^*_{n,t}$ increases as $\sqrt{t} \asymp \sqrt{n}$:

$$r^*_{n,t} = \sqrt{\log \frac{\lambda_c}{\lambda}} \sqrt{t}.$$

- When $t = n^\alpha$, with $0 < \alpha < 1$,

$$r^*_{n,t} = \sqrt{t} \sqrt{\left(\frac{1}{\alpha} - 1\right) \log t + \log \lambda_c} = \sqrt{\frac{1}{\alpha} - 1} \sqrt{t \log t} + O\left(\frac{\sqrt{t}}{\sqrt{\log t}}\right).$$

We also introduce

$$\sigma_{n,t}^{\pm} = \sup\left\{\sigma \text{ s.t. } L(p(\lfloor r_{n,t}^{*}\rfloor \pm \sigma)) \geq \sigma\right\} \tag{6.16}$$

that will measure the fluctuations of the interface. We will see that $\sigma_{n,t}^{\pm} \approx t^{2/7}$ under suitable hypotheses on n and t. We are now in a position to state our main result. We denote by S_r (resp. $S_{r,r'}$) the circle of radius r (resp. the annulus of radii $r < r'$) centered at the origin.

Theorem 87. *Consider some sequence $t_n \to \infty$. Then*

(i) If $t_n \geq \lambda n$ for some $\lambda > \lambda_c$, then there exists a constant $c = c(\lambda)$ such that

$$\mathbb{P}_{n,t}(\textit{every cluster is of diameter} \leq c\log n) \to 1$$

when $n \to \infty$.

(ii) If $n^{\alpha} \leq t_n \leq \lambda' n$ for some $\alpha > 2/3$ and $\lambda' < \lambda_c$, then for any fixed small $\epsilon > 0$, the following properties hold with probability tending to 1 as $n \to \infty$:

(a) There is a unique interface γ_n surrounding $\partial S_{t^{\epsilon}}$.

(b) This interface remains localized in the annulus of width $4t^{2/7+\epsilon}$ around $\partial S_{r_{n,t}^{*}}$ (recall that $r_{n,t}^{*} \approx \sqrt{t}$), and its fluctuations (compared to this circle) are larger than $t^{2/7-\epsilon}$.

Remark 88. *The hypothesis $\alpha > 2/3$ is due to technical limitations only i.e. to the $O(1/t^2)$ error bound in our version of the CLT that we use for Eq.(6.10). This is the only instance where it is used.*

6.3 Proofs

Let us start with item (i), which is the simpler one. In this case, the hypothesis $t_n \geq \lambda n$ implies that for any site z,

$$\bar{p}_{n,t}(\|z\|) \leq \bar{p}_{n,t}(0) \leq 1 - e^{-\frac{\lambda_c}{\lambda}\log 2} = \rho < 1/2. \tag{6.1}$$

Since

$$\pi_t(z) = \bar{\pi}_t(\|z\|) + O(1/t^2) = \bar{\pi}_t(\|z\|) + O(1/n^2)$$

uniformly on z, we also have

$$p_{n,t}(z) = 1 - e^{-n\pi_t(z)} = 1 - e^{-n\bar{\pi}_t(\|z\|)+O(1/n)} \leq \rho' < 1/2 \tag{6.2}$$

for some $\rho' > \rho$.

This enables us to dominate our model by a subcritical percolation process. Take some $\delta > 0$, and consider the model with $N \sim \mathcal{P}((1+\delta)n)$ particles: $N \geq n$ with probability tending to 1, and in this case the resulting configuration dominates the original one. Moreover, if we choose δ such that

$$\lambda_c(1+\delta)n \leq \frac{\lambda + \lambda_c}{2}n,$$

the hypothesis $t_n \geq \tilde{\lambda} n$ holds for some $\lambda_c < \tilde{\lambda} < \lambda$, so that the bound (6.2) is satisfied by the associated parameters for some $\rho' < \rho'' < 1/2$.

Let us assume furthermore that $t \leq n^3$, and subdivide S_t into $\sim t^2$ horizontal and vertical parallelograms of size $\frac{c}{2}\log n \times c\log n$. Any cluster of diameter larger than $c\log n$ would have

to cross one of these parallelograms "in the easy direction", which – using the exponential decay property – has a probability at most

$$C_0 t^2 e^{-C(\rho'')c\log n} \leq C_0 n^{6-C(\rho'')c}, \tag{6.3}$$

that tends to 0 as $n \to \infty$ for c large enough.

In fact, the case $t \geq n^3$ can be handled more directly. We dominate our model by using $N \sim \mathcal{P}(2n)$: $N \geq n$ with high probability, and we have $\bar{p}_{2n,t}(\|z\|) \leq \bar{p}_{2n,t}(0) = O(1/n^2)$, so that

$$p_{2n,t}(z) = \bar{p}_{2n,t}(\|z\|) + O(1/n^2) = O(1/n^2).$$

Hence, the probability to observe two neighboring sites is at most

$$\sum_{z\in\mathbb{V}^T}\sum_{z'\sim z} p_{2n,t}(z)p_{2n,t}(z') \leq 6 \max_{z'\in\mathbb{V}^T} p_{2n,t}(z') \times \left(\sum_{z\in\mathbb{V}^T} p_{2n,t}(z)\right) = O\left(\frac{1}{n}\right),$$

since $\sum_{z\in\mathbb{V}^T} p_{2n,t}(z) \leq 2n$.

Let us turn now to item (ii). First of all, the hypothesis on t_n ensures that

$$C_{\lambda'}\sqrt{t} \leq r^*_{n,t} \leq C_\alpha\sqrt{t\log t} \tag{6.4}$$

Let us study the behavior of $\bar{p}_{n,t}$ around $r^*_{n,t}$: we have

$$\frac{\partial}{\partial r}\bar{p}_{n,t}(r) = (1-\bar{p}_{n,t}(r)) \times \big(\log(1-\bar{p}_{n,t}(r))\big) \times \left(\frac{2r}{t}\right). \tag{6.5}$$

The hypothesis $t_n \leq \lambda' n$ also implies that

$$\bar{p}_{n,t}(0) \geq 1/2 + 2\delta$$

for some $\delta > 0$ (like in Eq.(6.1)). We can thus define

$$r^-_{n,t} = (\bar{p}_{n,t})^{-1}(1/2+\delta) \quad \text{and} \quad r^+_{n,t} = (\bar{p}_{n,t})^{-1}(1/2-\delta)$$

as we defined $r^*_{n,t}$. A formula similar to Eq.(6.14) apply in this case, in particular we have

$$C^\pm_{\lambda'}\sqrt{t} \leq r^\pm_{n,t} \leq C^\pm_\alpha\sqrt{t\log t}. \tag{6.6}$$

Hence, we get from Eq.(6.5) that for $r \in [r^-_{n,t}, r^+_{n,t}]$,

$$-\frac{C_1\sqrt{\log t}}{\sqrt{t}} \leq \frac{\partial}{\partial r}\bar{p}_{n,t}(r) \leq -\frac{C_2}{\sqrt{t}}. \tag{6.7}$$

We deduce

$$\frac{C_2}{\sqrt{t}}\big|r - r^*_{n,t}\big| \leq \big|\bar{p}_{n,t}(r) - 1/2\big| \leq \frac{C_1\sqrt{\log t}}{\sqrt{t}}\big|r - r^*_{n,t}\big|. \tag{6.8}$$

Now, for any z in the annulus $S_{r^-_{n,t}, r^+_{n,t}}$,

$$p_{n,t}(z) - \bar{p}_{n,t}(z) = e^{-n\bar{\pi}_t(\|z\|)}\big(1 - e^{n(\bar{\pi}_t(\|z\|)-\pi_t(z))}\big), \tag{6.9}$$

and using $\bar{\pi}_t(\|z\|) - \pi_t(z) = O(1/t^2)$, we get

$$p_{n,t}(z) - \bar{p}_{n,t}(z) = O(n/t^2) = O(1/t^{\alpha'}) \tag{6.10}$$

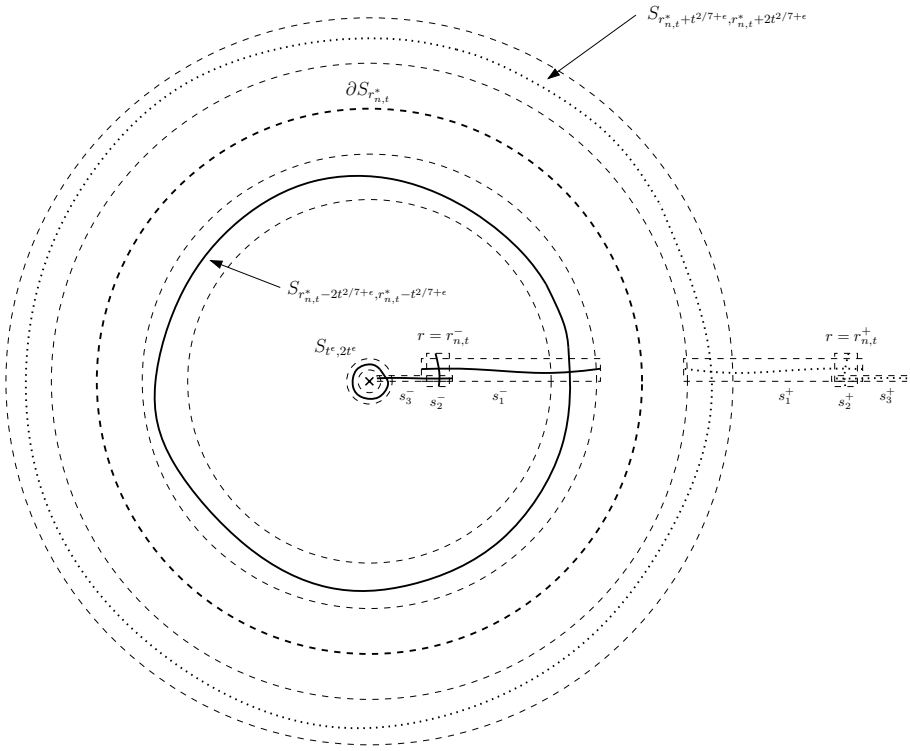


Figure 6.2: Construction showing localization of the macroscopic interface.

with $\alpha' = 2 - 1/\alpha > 1/2$ (thanks to $\alpha > 2/3$). Hence, Eq.(6.8) is also valid with $p_{n,t}(z)$ (take different constants if necessary).

We now consider the configuration depicted on Figure 6.2. We take as a subset A of sites the annulus $S_{r^*_{n,t}-2t^{2/7+\epsilon},r^*_{n,t}+2t^{2/7+\epsilon}}$, together with the smaller annulus $S_{t^\epsilon,2t^\epsilon}$ and some strips connecting these annuli, $s_1^- = [r^-_{n,t}/2, r^*_{n,t} - 0.99 \times t^{2/7+\epsilon}] \times [0, t^{2/7+\epsilon}]$, $s_3^- = [0, r^-_{n,t}] \times [0, t^\epsilon]$ and some vertical strip s_2^- in the middle. We also add external strips, $s_1^+ = [r^*_{n,t}+0.99\times t^{2/7+\epsilon}, 2r^+_{n,t}]\times [0, t^{2/7+\epsilon}]$ and $s_3^+ = [r^+_{n,t}, t] \times [0, t^\epsilon]$, connected by a vertical strip s_2^+.

We claim that for this choice of A, we have

$$\pi_t(A) = O(t^{-3/14+2\epsilon}). \tag{6.11}$$

Indeed, this is a direct consequence of $\pi_t(z) = O(1/t)$ and $r^*_{n,t} = O(\sqrt{t \log t})$, except for the most external strip $s_3^+ = [r^+_{n,t}, t] \times [0, t^\epsilon]$: for this strip, we have (recall that $r^+_{n,t} \geq C^+_{\lambda'}\sqrt{t}$)

$$\begin{aligned}\pi_t(s_3^+) &= O\left(t^\epsilon \sum_{u=C^+_{\lambda'}\sqrt{t}}^{\infty} \frac{1}{u} e^{-r^2/u}\right) \\ &= O(t^{-1/2+\epsilon}).\end{aligned}$$

The correlation length in both annuli $S_{r^*_{n,t}-2t^{2/7+\epsilon},r^*_{n,t}-t^{2/7+\epsilon}}$ and $S_{r^*_{n,t}+t^{2/7+\epsilon},r^*_{n,t}+2t^{2/7+\epsilon}}$ is at most

$$L\left(\frac{1}{2} \pm \frac{C_2}{\sqrt{t}} t^{2/7+\epsilon}\right) = L\left(\frac{1}{2} \pm C_2 t^{-3/14+\epsilon}\right) \approx t^{2/7-4\epsilon/3}, \tag{6.12}$$

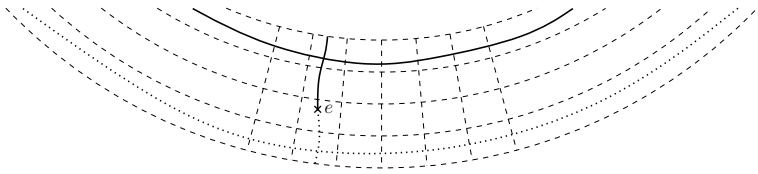


Figure 6.3: By considering sub-sections of the main annulus, we can show the existence of an edge in $S_{r^*_{n,t}-t^{2/7+\epsilon},r^*_{n,t}+t^{2/7+\epsilon}}$ possessing two arms as depicted, which implies uniqueness of the interface.

the property of exponential decay w.r.t. L (cf Eq.(6.8)) thus implies that there are circuits in these annuli, an occupied circuit in the internal one and a vacant circuit in the external one: indeed, these circuits can be constructed with roughly $r^*_{n,t}/t^{2/7} \approx t^{3/14}$ overlapping parallelograms of size $2t^{2/7} \times t^{2/7}$ (this size is at the same time $\gg t^{2/7-4\epsilon/3}$, so that crossing probabilities tend to 1 sub-exponentially fast, and $\ll t^{2/7+\epsilon}$). For the same reason, there are also occupied crossings in s_1^- and s_2^-, and vacant crossings in s_1^+ and s_2^+. Since the parameter $p_{n,t}$ is at least $1/2+\delta$ in $S_{r^-_{n,t}}$, there is also an occupied crossing in s_3^-, and an occupied circuit in $S_{t^\epsilon,2t^\epsilon}$. Since $p_{n,t}$ is at most $1/2-\delta$ outside of $S_{r^+_{n,t}}$, there is also a vacant crossing in s_3^+. As a consequence, the potential interfaces have to be located in the annulus $S_{r^*_{n,t}-2t^{2/7+\epsilon},r^*_{n,t}+2t^{2/7+\epsilon}}$.

By considering $\sim r^*_{n,t}/t^{2/7+\epsilon} \approx t^{3/14-\epsilon}$ sub-sections of length $t^{2/7+\epsilon}$ in the main annulus as on Figure 6.3, and using the same construction as for uniqueness in [54] (Proposition 7), we can then show that with probability tending to 1, there is an edge e (on the dual lattice) in $S_{r^*_{n,t}-t^{2/7+\epsilon},r^*_{n,t}+t^{2/7+\epsilon}}$ with two arms, one occupied going to $\partial S_{r^*_{n,t}-2t^{2/7+\epsilon}}$ and one vacant to $\partial S_{r^*_{n,t}+2t^{2/7+\epsilon}}$. In this case, the interface is unique, and can be characterized as the set of edges possessing two such arms.

Finally, the previous sub-sections of the main annulus also allow to see that the fluctuations of the interface are larger than $t^{2/7-\epsilon}$, by using "blocking vertical crossings" as for Theorem 6 in [54].

Remark 89. *Characterizing the interface as the set of two-arm edges allows to describe its fine local structure, by comparing it to critical percolation interfaces like in [54]. For instance:*

- *Its length – number of edges – behaves like $L_n \approx t^{2/7}\sqrt{t}(t^{2/7})^{-1/4} = t^{5/7}$ ($\alpha_2 = 1/4$ being the so-called "two-arm exponent" measuring the probability to observe two arms for critical percolation).*
- *More generally, its "discrete Minkowski dimension" is $7/4$ like for critical percolation (the number of balls of size $\approx \eta t^{2/7}$ needed to cover it increases as $\eta^{-7/4}$).*
- *A decorrelation property between the different sub-sections also holds, so that quantities like L_n are concentrated around their expectation (more precisely, $L_n/\mathbb{E}[L_n] \to 1$ in L^2 as $n \to \infty$).*

Remark 90. *Here item (ii) contains simultaneously the two cases $t \sim \lambda n$ (for $\lambda < \lambda_c$) and $t \sim n^\alpha$. However, note that these two regimes can be distinguished by the size of the "transition window", where the parameter decreases from $1/2+\delta$ to $1/2-\delta$, as shown on Figure 6.4:*

- *When $t \sim \lambda n$, the transition takes place gradually w.r.t. $\sqrt{t}$. The parameter at the origin is bounded away from 1.*

- *When $t \sim n^\alpha$, the transition is concentrated in a window of size $\sim \sqrt{t}/\sqrt{\log t}$ around $r^* \asymp \sqrt{t \log t}$. The parameter at the origin tends to 1.*

Here we used that around $r^*_{n,t}$, the local behavior of $p_{n,t}$ is similar, with an extra $\sqrt{\log t}$ factor that does not affect the exponents.

6.4 Model with a source

We briefly discuss in this section a dynamical model for the formation of an ink stain, with a source of particles at the origin. We start with no particles, and at each time $t \geq 0$, new particles are added at the origin with rate μ: the number n_t of new particles at time t has distribution $\mathcal{P}(\mu)$ for some fixed $\mu > 0$. Once arrived, the particles perform independent simple random walks. As shown on Figure 6.5, we observe in this setting a macroscopic cluster around the origin, whose size increases as the number of particles gets larger and larger.

The number of particles $N_t(z)$ on site z at time t is given by a sum of independent Poissonian random variables:

$$N_t(z) \sim \mathcal{P}(\mu\pi_t(z)) + \ldots + \mathcal{P}(\mu\pi_0(z)) \sim \mathcal{P}(\mu\rho_t(z)), \tag{6.1}$$

with

$$\rho_t(z) = \sum_{u=0}^{t} \pi_u(z). \tag{6.2}$$

We thus introduce like for the previous model

$$q_{\mu,t}(z) = 1 - e^{-\mu\rho_t(z)}. \tag{6.3}$$

For any value of μ, this parameter tends to 1 near the origin, and this dynamical model turns out to behave like the previous model. We will show that the constructions of the previous section can be adapted in this new setting.

Lemma 91. *For each $\epsilon > 0$, there exist constants $C_i = C_i(\epsilon)$ such that*

$$\rho_t(z) \geq C_1 \log t - C_2 \tag{6.4}$$

uniformly on sites z such that $\|z\| \leq t^{1/2-\epsilon}$.

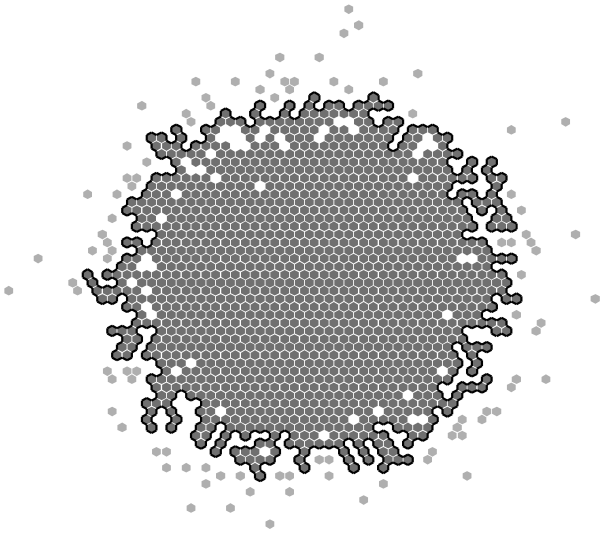
Hence in particular, $q_{\mu,t}(z) \to 1$ uniformly on $S_{t^{1/2-\epsilon}}$.

Proof. We write

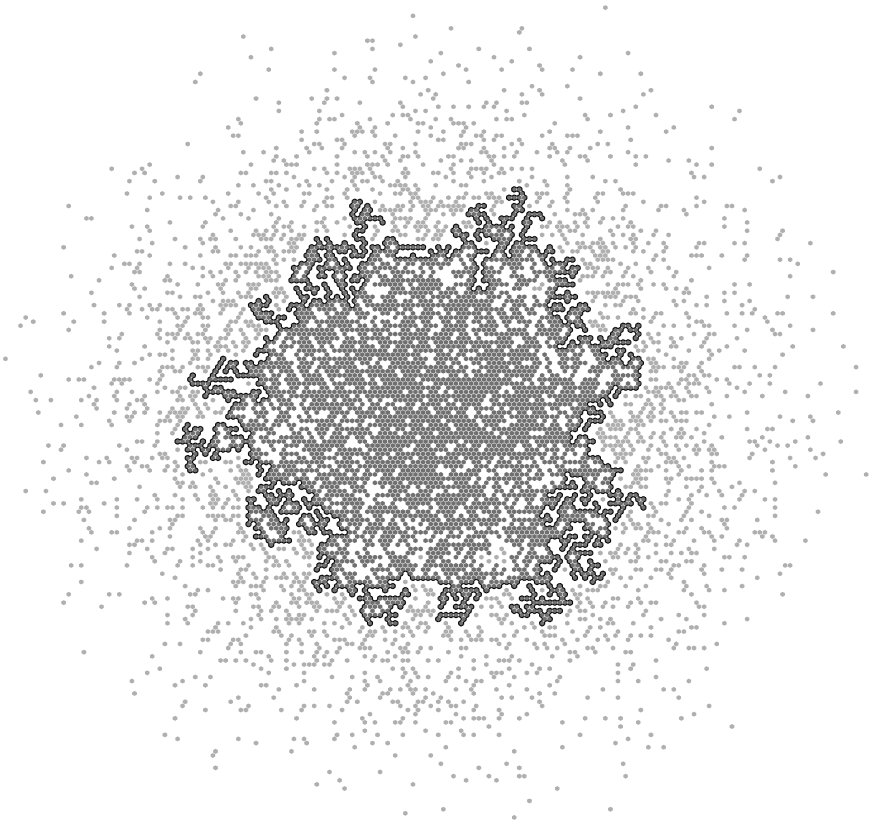
$$\rho_t(z) \geq \sum_{u=t^{1-2\epsilon}}^{t} \pi_u(z) \geq \sum_{u=t^{1-2\epsilon}}^{t} \left(\frac{\sqrt{3}}{2\pi u} e^{-\|z\|^2/u} + O\left(\frac{1}{u^2}\right) \right).$$

Since $\|z\|^2/u \leq t^{1-2\epsilon}/u \leq 1$, and $\sum_{u=t^{1-2\epsilon}}^{t} 1/u^2 \leq \sum_{u=1}^{\infty} 1/u^2 < \infty$, we get

$$\rho_t(z) \geq \frac{\sqrt{3}e^{-1}}{2\pi} \left(\sum_{u=t^{1-2\epsilon}}^{t} \frac{1}{u} \right) - C, \tag{6.5}$$

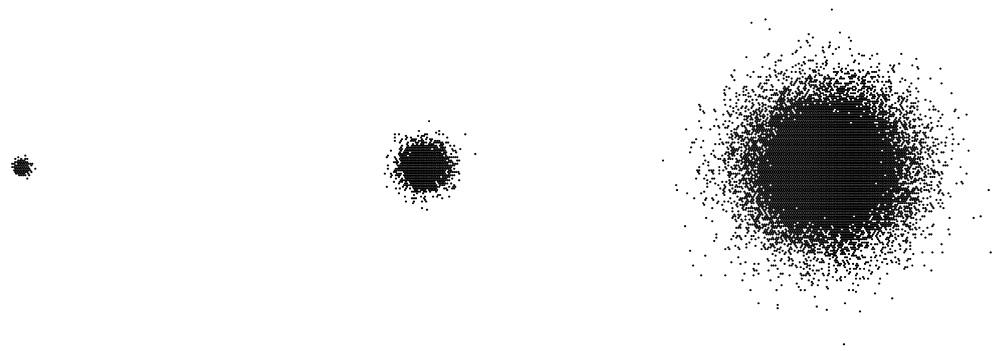


(a) $t \ll n$: localized transition.



(b) $t \sim n$: gradual transition.

Figure 6.4: Cluster of the origin in dense phase.



(a) $t = 10$. (b) $t = 100$. (c) $t = 1000$.

Figure 6.5: A "stain" growing as the time t increases, with new particles arriving at rate $\mu = 50$.

whence the result, since

$$\begin{aligned}\sum_{u=t^{1-2\epsilon}}^{t} \frac{1}{u} &= \big(\log t + \gamma + o(1)\big) - \big((1-2\epsilon)\log t + \gamma + o(1)\big) \\ &= 2\epsilon \log t + o(1).\end{aligned}$$

□

Let us now consider for $r > 0$

$$\bar{\rho}_t(r) = \frac{\sqrt{3}}{2\pi} \int_{r^2/t}^{+\infty} \frac{e^{-u}}{u} du, \tag{6.6}$$

and also

$$\bar{q}_{\mu,t}(r) = 1 - e^{-\mu \bar{\rho}_t(r)}. \tag{6.7}$$

Lemma 92. *We have*

$$\rho_t(z) = \bar{\rho}_t(\|z\|) + O\left(\frac{1}{t^{3/4}}\right) \tag{6.8}$$

uniformly on sites z such that $\|z\| \geq t^{7/16}$.

For this lemma, we will use the following a-priori upper bound on $\pi_t(z)$, providing exponential decay for the sites z at distance $\gg \sqrt{t}$: for some universal constant $C > 0$, we have

$$\pi_t(z) \leq C e^{-\|z\|^2/2t} \tag{6.9}$$

for all sites z and for any time $t \geq 0$. This is for instance a direct consequence of Hoeffding's inequality.

Proof. We write

$$\begin{aligned}\rho_t(z) &= \sum_{u=0}^{t^{3/4}} \pi_u(z) + \sum_{u=t^{3/4}+1}^{t} \pi_u(z) \\ &= \sum_{u=0}^{t^{3/4}} \pi_u(z) + \sum_{u=t^{3/4}+1}^{t} \left(\frac{\sqrt{3}}{2\pi u} e^{-\|z\|^2/u} + O\left(\frac{1}{u^2}\right)\right).\end{aligned}$$

For each $0 \le u \le t^{3/4}$, we have by Eq.(6.9)

$$\pi_u(z) \le Ce^{-\|z\|^2/2u} \le Ce^{-t^{1/8}/2} \tag{6.10}$$

since $\|z\|^2 \ge t^{7/8}$, so that the corresponding sum is a $O(t^{3/4}e^{-t^{1/8}/2}) = o(1/t^{3/4})$. We also have

$$\sum_{u=t^{3/4}+1}^{t} O\left(\frac{1}{u^2}\right) = O\left(\frac{1}{t^{3/4}}\right).$$

Let us now estimate the main sum. We can write

$$\sum_{u=t^{3/4}+1}^{t} \frac{1}{u} e^{-\|z\|^2/u} = \sum_{u=t^{3/4}+1}^{t} \frac{1}{t} f_\lambda\left(\frac{u}{t}\right), \tag{6.11}$$

with $f_\lambda(x) = e^{-\lambda/x}/x$ and $\lambda = \|z\|^2/t \ge t^{-1/8}$, and we have

$$\left| \sum_{u=t^{3/4}+1}^{t} \frac{1}{t} f_\lambda\left(\frac{u}{t}\right) - \int_{t^{-1/4}}^{1} f_\lambda(x)dx \right| \le \sum_{u=t^{3/4}+1}^{t} \int_{(u-1)/t}^{u/t} \left| f_\lambda\left(\frac{u}{t}\right) - f_\lambda(x) \right| dx. \tag{6.12}$$

By an easy calculation,

$$f'_\lambda(x) = \frac{1}{x^2}\left(\frac{\lambda}{x} - 1\right)e^{-\lambda/x} = \frac{1}{\lambda^2} g\left(\frac{\lambda}{x}\right),$$

with $g(y) = y^2(y-1)e^{-y}$, that is bounded on $[0, +\infty)$. Hence for some $M > 0$,

$$f'_\lambda(x) \le \frac{M}{\lambda^2} = O(t^{1/4}),$$

so that for $x \in [(u-1)/t, u/t]$,

$$\left| f_\lambda\left(\frac{u}{t}\right) - f_\lambda(x) \right| \le C' t^{1/4} \left| \frac{u}{t} - x \right| \le \frac{C'}{t^{3/4}}, \tag{6.13}$$

and finally

$$\sum_{u=t^{3/4}+1}^{t} \int_{(u-1)/t}^{u/t} \left| f_\lambda\left(\frac{u}{t}\right) - f_\lambda(x) \right| dx = O\left(\frac{1}{t^{3/4}}\right).$$

Putting everything together, we get

$$\rho_t(z) = \int_{t^{-1/4}}^{1} \frac{1}{x} e^{-\|z\|^2/tx} dx + O\left(\frac{1}{t^{3/4}}\right). \tag{6.14}$$

Making the change of variable $u = \|z\|^2/tx$, we obtain

$$\rho_t(z) = \int_{\|z\|^2/t}^{\|z\|^2/t^{3/4}} \frac{e^{-u}}{u} du + O\left(\frac{1}{t^{3/4}}\right), \tag{6.15}$$

which allows us to conclude, since

$$\int_{\|z\|^2/t^{3/4}}^{+\infty} \frac{e^{-u}}{u} du = O\left(\frac{t^{3/4}}{\|z\|^2} e^{-\|z\|^2/t^{3/4}}\right) = O(t^{-1/8}e^{-t^{1/8}}). \tag{6.16}$$

□

Since $q_{\mu,t}(r)$ tends to 0 as $r \to \infty$, and to 1 as $r \to 0^+$, we can introduce

$$r^*_{\mu,t} = (\bar{q}_{\mu,t})^{-1}(1/2), \tag{6.17}$$

and an easy calculation shows in this case that

$$r^*_{\mu,t} = \sqrt{F^{-1}\left(\frac{2\pi \log 2}{\mu\sqrt{3}}\right)}\sqrt{t}, \tag{6.18}$$

with $F(x) = \int_x^{+\infty} \frac{e^{-u}}{u} du$, that is a decreasing bijection from $(0, +\infty)$ onto itself. We also consider

$$r^-_{\mu,t} = (\bar{q}_{\mu,t})^{-1}(3/4) \quad \text{and} \quad r^+_{\mu,t} = (\bar{q}_{\mu,t})^{-1}(1/4), \tag{6.19}$$

and we have

$$r^\pm_{\mu,t} = C^\pm\sqrt{t}. \tag{6.20}$$

If we compute the derivative of $\bar{q}_{\mu,t}(r)$ w.r.t. r

$$\frac{\partial}{\partial r}\bar{q}_{\mu,t}(r) = (1 - \bar{q}_{\mu,t}(r)) \times \frac{\sqrt{3}\mu}{2\pi} \times \frac{e^{-r^2/t}}{r^2/t} \times \frac{2r}{t},$$

we get, similarly to Eq.(6.8), that

$$\frac{C_1}{\sqrt{t}}\left|r - r^*_{n,t}\right| \le \left|\bar{q}_{\mu,t}(r) - 1/2\right| \le \frac{C_2}{\sqrt{t}}\left|r - r^*_{n,t}\right| \tag{6.21}$$

for $r \in [r^-_{\mu,t}, r^+_{\mu,t}]$. We are thus in position to apply the same construction as in the previous section (Figure 6.2), and the properties of Theorem 87 (case (ii)) still hold: with probability tending to 1 as $t \to \infty$,

(a) There is a unique interface γ_t surrounding ∂S_{t^ϵ}.

(b) This interface remains localized in the annulus $S_{r^*_{\mu,t}-2t^{2/7+\epsilon}, r^*_{\mu,t}+2t^{2/7+\epsilon}}$ around $\partial S_{r^*_{\mu,t}}$, and its fluctuations are larger than $t^{2/7-\epsilon}$.

Remark 93. *We could also modify our model by adding a fixed number n_0 of particles at each time t: the same results apply in this case, by using a Poissonian approximation in the same set A of sites as in the previous section. Indeed, we can ensure that the configuration in A coincides with the "Poissonian configuration" with probability at least $1 - \rho_t(A)$, for*

$$\rho_t(A) = \sum_{z \in A} \rho_t(z) = \sum_{u=0}^{t} \pi_u(A). \tag{6.22}$$

Appendix: Local Central Limit Theorem on the triangular lattice

We embed the triangular lattice $\mathbb{T}$ on $\mathbb{Z}^2$ by adding NE-SW diagonals on every face, and we consider the simple random walk on this lattice

$$Z_n = X_1 + \ldots + X_n,$$

where the X_i are i.i.d., uniform on $\{(\pm 1, 0), (0, \pm 1), (\pm 1, \mp 1)\}$. This random walk is the image of the simple random walk on $\mathbb{T}$ under the mapping $x + e^{i\pi/3}y \mapsto x + iy$.

Since Z_n takes values in $\mathbb{Z}^2$, we know from the inversion formula for the characteristic function that

$$\pi_t(z) = \frac{1}{(2\pi)^2} \int_{[-\pi,\pi]^2} e^{-iz.\theta} \big[\phi(\theta)\big]^t d\theta,$$

with $\phi(\theta) = \mathbb{E}[e^{iX_1.\theta}]$. We have

$$\phi(\theta) = \frac{1}{3}\big[\cos\theta_1 + \cos\theta_2 + \cos(\theta_1 - \theta_2)\big] = 1 - \frac{1}{3}Q(\theta) + O(|\theta|^4)$$

as $\theta \to 0$, with $Q(\theta) = \theta_1^2 + \theta_2^2 - \theta_1\theta_2$.

Note that in $[-\pi,\pi]^2$, $|\phi(\theta)| = 1$ *iff* $\theta_1 = \theta_2 = 0$. The quadratic form Q being positive definite, we have

$$c_1|\theta|^2 \le Q(\theta) \le c_2|\theta|^2 \tag{6.23}$$

for some $c_1, c_2 > 0$. We then take $r > 0$ such that

$$|\phi(\theta)| \le 1 - \frac{c_1}{2}|\theta|^2$$

for $|\theta| \le r$, and for this r,

$$|\phi(\theta)| \le \rho = \rho(r) < 1$$

for $|\theta| \ge r$ in $[-\pi,\pi]^2$.

We can now follow the proof of [45]: we first cut $\pi_t(z)$ into two pieces $\pi_t(z) = I(t,z) + J(t,z)$, with

$$\begin{aligned} I(t,z) &= \frac{1}{(2\pi)^2} \int_{|\theta|\le r} e^{-iz.\theta} \big[\phi(\theta)\big]^t d\theta \\ &= \frac{1}{(2\pi\sqrt{t})^2} \int_{|\alpha|\le r\sqrt{t}} e^{-iz.\alpha/\sqrt{t}} \big[\phi(\alpha)\big]^t d\alpha \end{aligned}$$

and $|J(t,z)| \le \rho^t$. We then write

$$(2\pi\sqrt{t})^2 I(t,z) = I_0(t,z) + I_1(t,z) + I_2(t,z) + I_3(t,z),$$

with

$$I_0(t,z) = \int_{\mathbb{R}^2} e^{-iz.\alpha/\sqrt{t}} e^{-Q(\alpha)/3} d\alpha$$

and the three remaining terms are all $O(1/t)$ for the same reasons as in [45] (use Eq.(6.23)). By diagonalizing Q, it is easy to check that

$$I_0(t,z) = 2\sqrt{3}\pi e^{-(z_1^2+z_2^2+z_1z_2)/t},$$

whence the result, since $z_1^2 + z_2^2 + z_1z_2 = \|z_1 + e^{i\pi/3}z_2\|^2$.

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