Simulation Numérique en Aéroacoustique
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Schémas de discrétisation optimisés dans l'espace de Fourier

C. Bogey, J. Berland & C. Bailly

LMFA, UMR CNRS 5509

Ecole Centrale de Lyon

http://acoustique.ec-lyon.fr

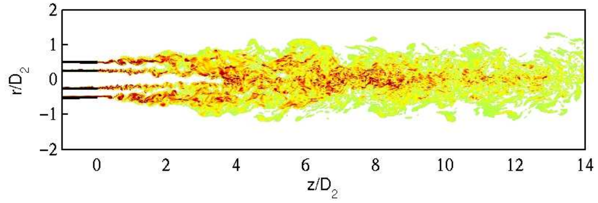
Outline - Road map

- **Background / Motivations**
- **Spatial discretization optimized in the Fourier space**
  - Finite differences for spatial derivatives
  - Selective filters for removing high-frequency waves
- **Time integration : optimized Runge-Kutta schemes**
- **Applications**
  - Acoustic test problem : diffraction by a cylinder
  - Navier-Stokes simulations (Large-Eddy Simulations)
- **Concluding remarks**

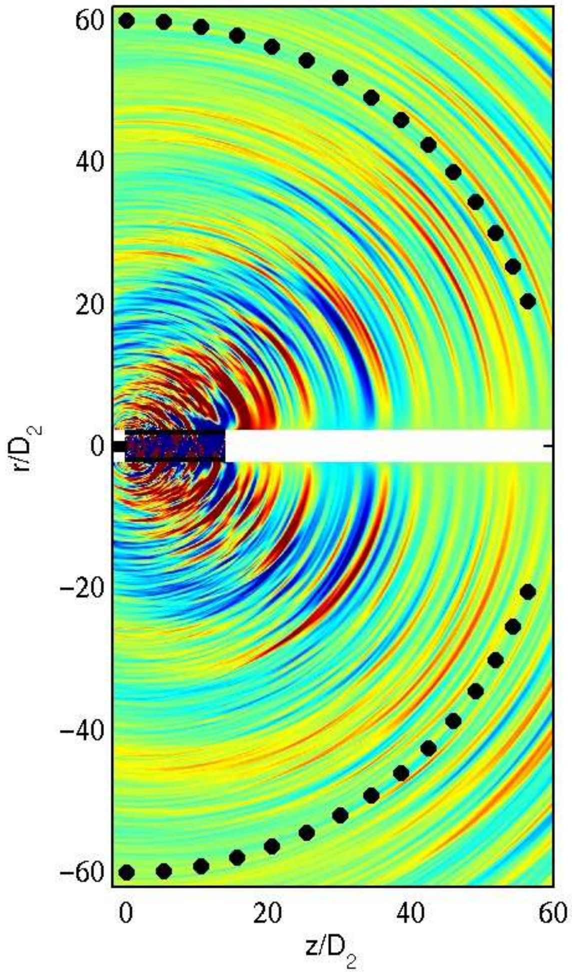
Motivations

- **Development of Computational AeroAcoustics (CAA)**
  - **direct simulation of sound generation by solving the unsteady Navier-Stokes equations for compressible flows**
  - **simulations of long-range propagation (Linearized Euler Equations)**
- **Key numerical issues in CAA**
  - **disparities in magnitudes and length scales between flow and acoustics**
  - **turbulent and sound broadband spectra**
  - **far-field propagation**

vorticity field of a coaxial jet



sound pressure field



Motivations

- **Problem model for wave propagation**
  - **1-D advection equation :**

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \qquad u(x,0) = g(x)$$

  - **exact solution** $u(x,t) = g(x - ct)$
    by Fourier-Laplace transform : dispersion relation $\omega = kc$
    elementary solution : harmonic plane wave $Ae^{i(kx-\omega t)}$
  - **numerical approximation :** $\omega_s = k_s c$

- **Development of schemes optimized in the Fourier space**
  - **space : Dispersion-Relation-Preserving schemes of Tam & Webb (JCP, 1991), spectral-like schemes of Lele (JCP, 1992)**
  - **time : Runge-Kutta algorithms of Hu et al. (JCP, 1996)**

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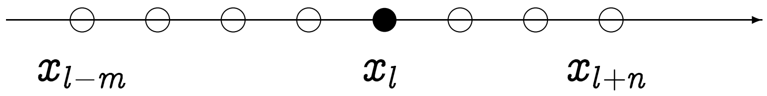
Explicit finite-difference schemes

$$\left.\frac{\partial u}{\partial x}\right|_l \simeq \frac{1}{\Delta x}\sum_{j=-m}^{n} a_j\, u_{l+j}$$



$x_l = l\Delta x$ **centered** $m = n$ **upwind** $m > n$

Particular case of the continuous relation:

$$\frac{\partial u}{\partial x} \simeq \frac{1}{\Delta x}\sum_{j=-m}^{n} a_j\, u(x + j\Delta x)$$

By Fourier transform:

$$ik\hat{u} \simeq \frac{\hat{u}}{\Delta x}\sum_{j=-m}^{n} a_j e^{ijk\Delta x} \qquad u(x) = \mathcal{F}^{-1}\left[\hat{u}(k)\right] = \int_{-\infty}^{+\infty} \hat{u}(k) e^{ikx} dk$$

Numerical dimensionless wavenumber $$k_s\Delta x = -i\sum_{j=-m}^{n} a_j e^{ijk\Delta x}$$

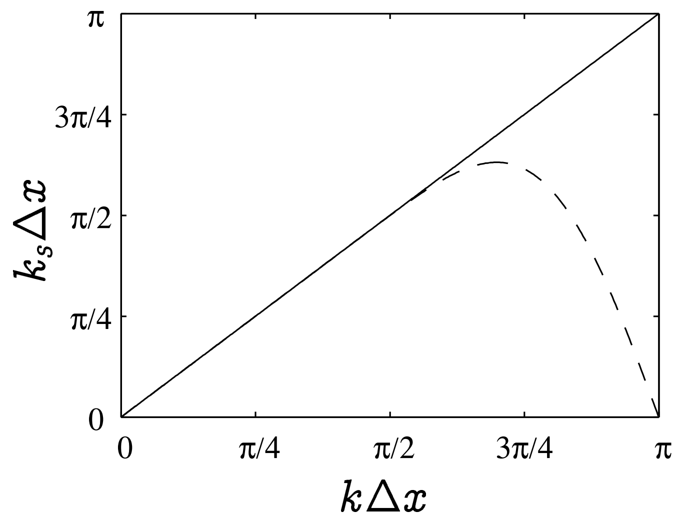
Finite differences for spatial derivatives

- **Centered schemes ($n = m$)**

  a_j is antisymmetric

  $\rightarrow$ k_s is real (no dissipation)

$$k_s\Delta x = 2\sum_{j=1}^{n} a_j \sin jk\Delta x$$



– **high-order schemes :** a_j are determined by cancelling the Taylor series
formal truncation order $\mathcal{O}(\Delta x^{2n})$

– **low-dispersion schemes :** a_j are determined by minimizing the error between the exact and numerical wavenumbers k and k_s over a large wavenumber range $k_l\Delta x \leq k\Delta x \leq k_u\Delta x$

e.g. in Bogey & Bailly (JCP, 2004), minimization of the integral error

$$\int_{\pi/16}^{\pi/2} |k_s\Delta x - k\Delta x| \, \frac{d\ln(k\Delta x)}{k\Delta x}$$

Finite differences for spatial derivatives

- **Non-centered schemes ($n \neq m$)**

  k_s has an imaginary part (providing dissipation/amplification)

  – **approximate solution for the 1-D advection equation**

$$u(x,t) = \underbrace{e^{i[k - Re(k_s)]ct}}_{\text{phase error}} \quad \cdot \quad \underbrace{e^{Im(k_s)ct}}_{\text{dissipation /amplification}} \quad \cdot \quad \underbrace{e^{ik(x-ct)}}_{\text{exact solution}}$$

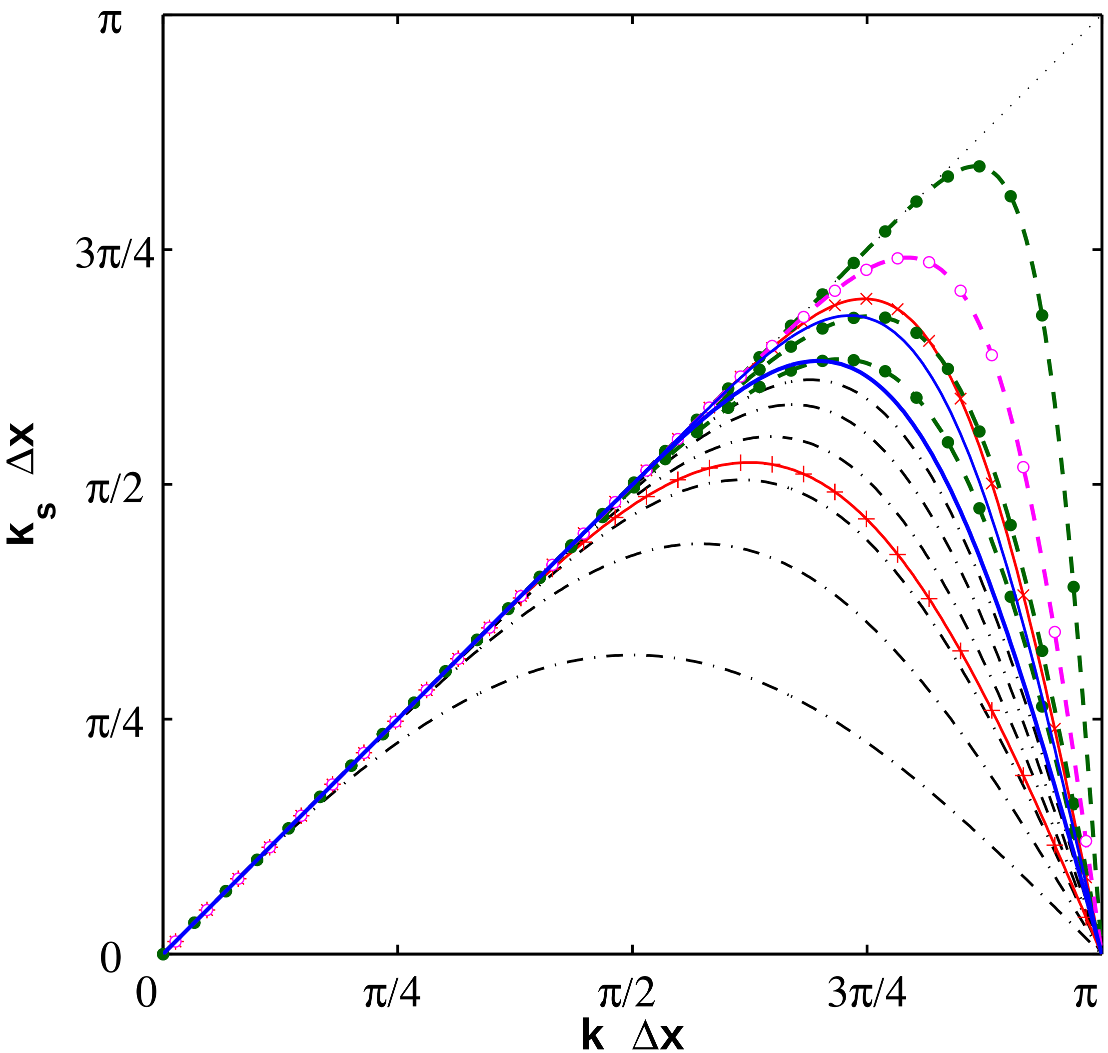
  – **minimization both of phase error and of dissipation/amplification e.g. in Berland et al. (JCP, 2006) with the integral error**

$$\int_{\pi/16}^{\pi/2} \left[(1-\alpha) \underbrace{|k\Delta x - Re(k_s \Delta x)|}_{\text{dispersion}} + \alpha \underbrace{|Im(k_s \Delta x)|}_{\text{dissipation}} \right] \frac{d(k\Delta x)}{k\Delta x}$$

  $\rightarrow$ **low-dispersion and low-dissipation schemes**

Finite differences for spatial derivatives

- **Numerical wavenumber $k_s\Delta x$ vs. exact wavenumber $k\Delta x$ for centered schemes**



–·–·– 2nd, 4th, 6th, 8th, 10th and 12th-order central differences

– + – DRP 7-pt Tam & Webb (1993)

– × – DRP 15-pt Tam (2003)

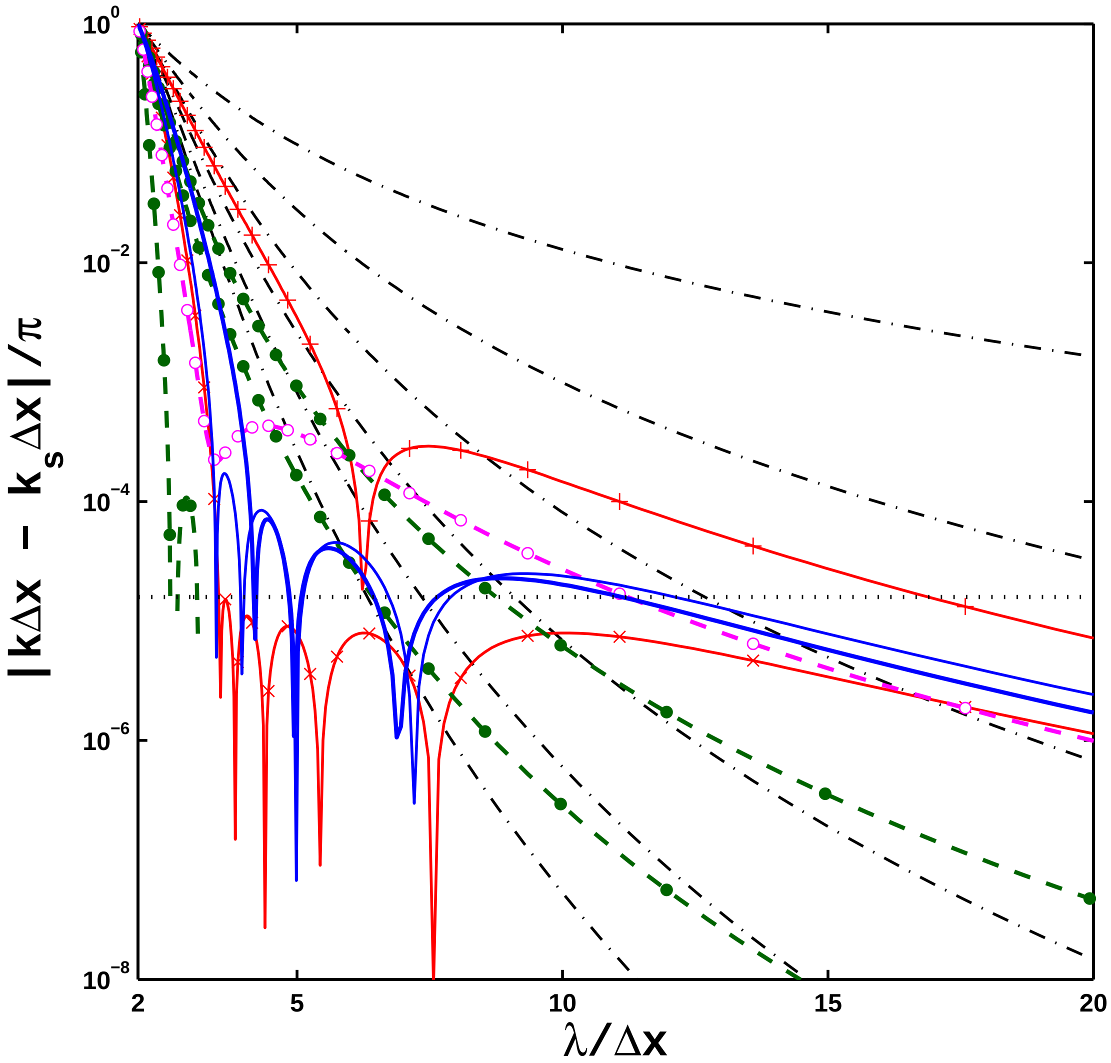
—— 11-pt and —— 13-pt schemes of Bogey & Bailly (2004)

– • – tridiag. 5-pt 6th-order and compact 8th-order schemes, pentadiag. 4th-order of Lele (1992)

– ○ – prefactored 5-pt 4th-order scheme of Ashcroft & Zhang (2003)

Finite differences for spatial derivatives

- **Phase-velocity error in terms of points per wavelength**



→ **error observed for a harmonic wave $e^{i(kx-\omega t)}$ propagating at $v_\varphi = \omega/k = ck_s/k$**

accuracy limit given by

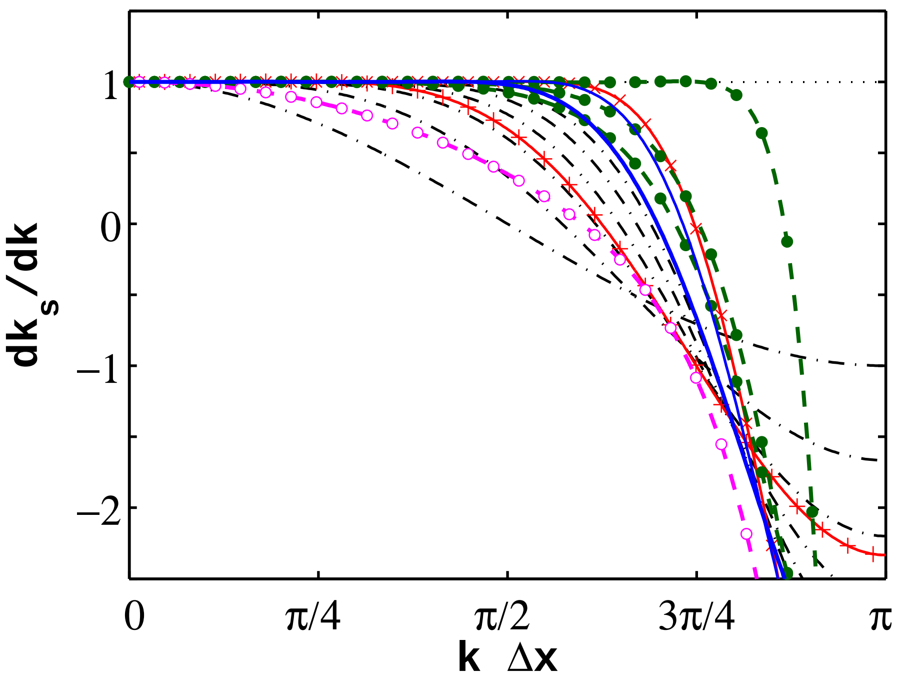
$E_{v_\varphi} = |k_s\Delta x - k\Delta x|/\pi \leq 5 \times 10^{-5}$

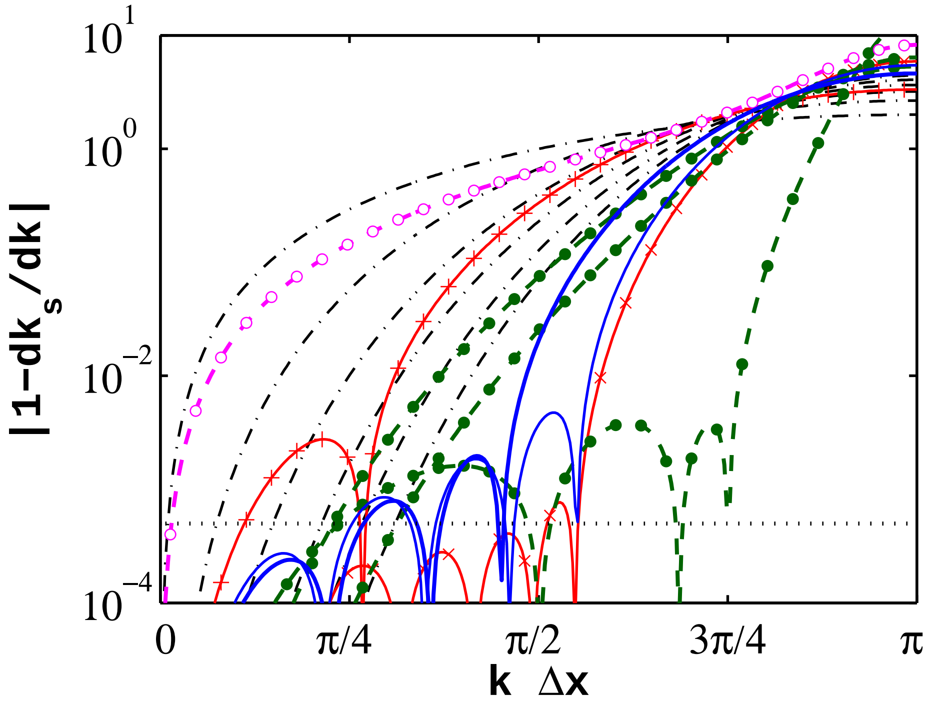
Finite differences for spatial derivatives

- **Group-velocity error as a function of the exact wavenumber**

propagation of a wave-packet at the group velocity

$$v_g = \frac{\partial \omega}{\partial k} = c\frac{\partial k_s}{\partial k} \qquad \omega = c k_s(k)$$





accuracy limit given by $E_{v_g} = |\partial k_s/\partial k - 1| \leq 5 \times 10^{-4}$

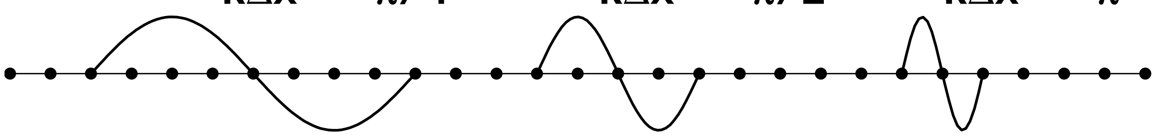
Finite differences for spatial derivatives

- **Accuracy limits**

scheme	$E_{v_\varphi} \leq 5 \times 10^{-5}$		$E_{v_g} \leq 5 \times 10^{-4}$			
	$k\Delta x\|_{\max}$	$\lambda/\Delta x\|_{\min}$	$k\Delta x\|_{\max}$	$\lambda/\Delta x\|_{\min}$	$k_{\max}\Delta x$	$k_{v_\varphi}/k_{\max}$
CFD 2nd-order	0.0986	63.7	0.0323	194.6	1.0000	0.10
CFD 4th-order	0.3439	18.3	0.2348	26.8	1.3722	0.25
CFD 6th-order	0.5857	10.7	0.4687	13.4	1.5860	0.37
CFD 8th-order	0.7882	8.0	0.6704	9.4	1.7306	0.46
CFD 10th-order	0.9550	6.6	0.8380	7.5	1.8374	0.52
CFD 12th-order	1.0929	5.7	0.9768	6.4	1.9208	0.57
DRP 7-pts 4th-order	0.4810	13.1	0.3500	18.0	1.6442	0.29
DRP 15-pts 4th-order	1.8069	3.5	1.6070	3.9	2.1914	0.82
OFD 11-pts 4th-order	1.3530	4.6	0.8458	7.4	1.9836	0.68
OFD 13-pts 4th-order	1.3486	4.7	0.7978	7.9	2.1354	0.63
CoFD 6th-order	0.8432	7.5	0.7201	8.7	1.9894	0.42
CoFD 8th-order	1.1077	5.7	0.9855	6.4	2.1334	0.52
CoFD opt. 4th-order	2.4721	7.3	0.7455	8.4	2.6348	0.33
Opt. pre. 4th-order	0.7210	8.7	0.0471	133.3	2.3294	0.31

Selective spatial filtering

- **Need for spatial filtering**
  - **grid-to-grid oscillations are not resolved by F-D schemes according to the Nyquist-Shannon theorem**
  - **the highest wave-numbers, poorly resolved by F-D, must be removed without affecting the long (physical) waves accurately discretized**



- **Explicit discrete filtering** ($\circ$ $\bullet$ $\circ$ $\circ$ $\circ$)

$$u^f(x_i) = u(x_i) - \sum_{j=-m}^{n} d_j u(x_i + j\Delta x)$$

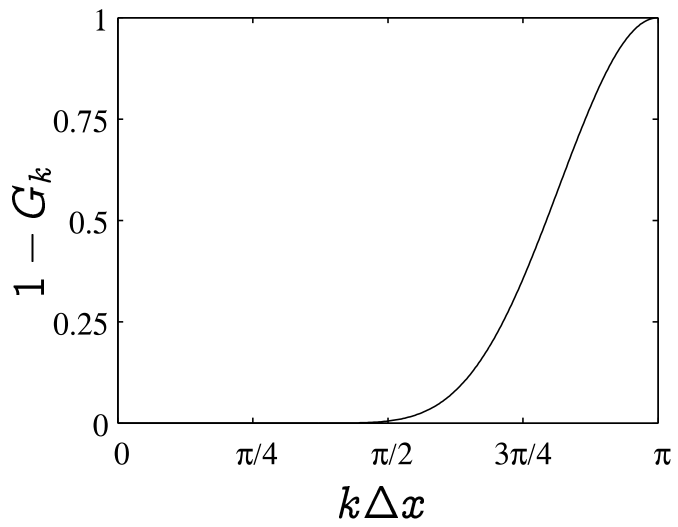
transfer function in the Fourier space $\quad G_k(k\Delta x) = 1 - \sum_{j=-m}^{n} d_j e^{ijk\Delta x}$

Selective spatial filtering

- **Requirements on the transfer function**
  - **stability :** $|G_k(k\Delta x)| < 1$
  - **removal of grid-to-grid oscillations :** $G_k(\pi) = 0$
  - **normalization :** $G_k(0) = 1$

- **Centered filters**

  $\rightarrow$ G_k **is real (no dispersion)** **(see fig.)**



- **Non-centered filters**

  $\rightarrow$ G_k **has an imaginary part (phase error)**

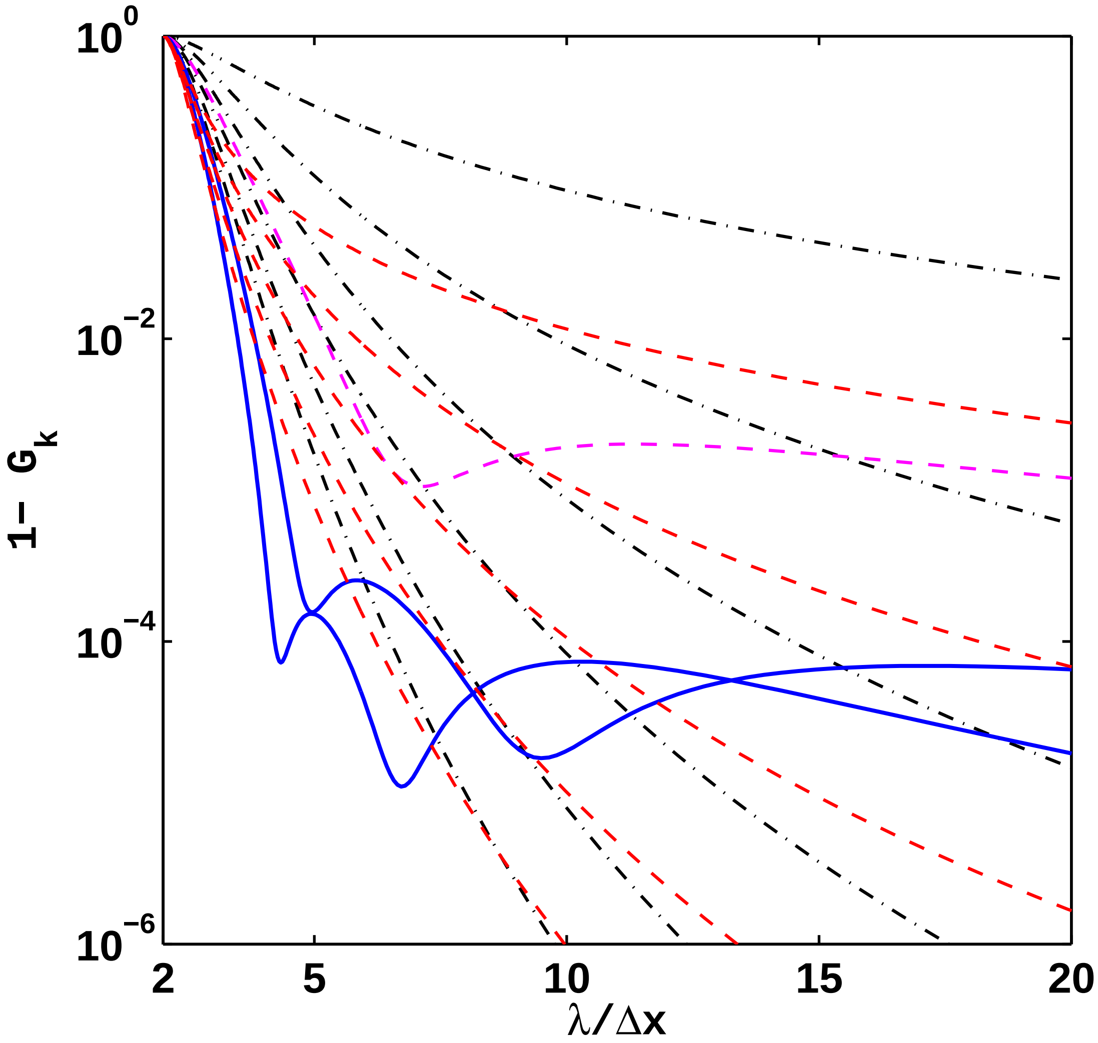
- **Optimized filtering**

  by choosing d_j minimizing an integral error

e.g., $$\int_{\pi/16}^{\pi/2} \Big[(1-\alpha) \underbrace{|1 - G_k(k\Delta x)|}_{\text{dissipation}} + \alpha \underbrace{|\phi_G(k\Delta x)|}_{\text{phase error}} \Big] \frac{d(k\Delta x)}{k\Delta x}$$

Selective spatial filtering

- **Transfer function $1 - G_k$ of centered filters**



–·–·– 2nd, 4th, 6th, 8th, 10th and 12th-order standard explicit filters

—— optimized 7-pt 2nd-order filter of Tam & Webb (1993)

—— optimized 11-pt 2nd-order and 13-pt 4th-order filters of Bogey & Bailly (2004)

– – – implicit tridiag. 2nd, 4th, 6th, 8th and 10th implicit filters ($\alpha_f = 0.4$), see Lele (1992), Gaitonde & Visbal (2000)

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Time integration

Explicit Runge-Kutta schemes

differential equation $$\frac{\partial u^n}{\partial t} = F(u^n, t) \qquad u^n(x) = u(x, n\Delta t)$$

General form of a low-storage Runge-Kutta scheme at p-stages :

$$u^{n+1} = u^n + \Delta t \sum_{i=1}^{p} b_i K^i \quad \text{with} \quad K^i = F\left(u^n + \sum_{j=1}^{i-1} a_{ij} K^j, t^n + c_i \Delta t\right)$$

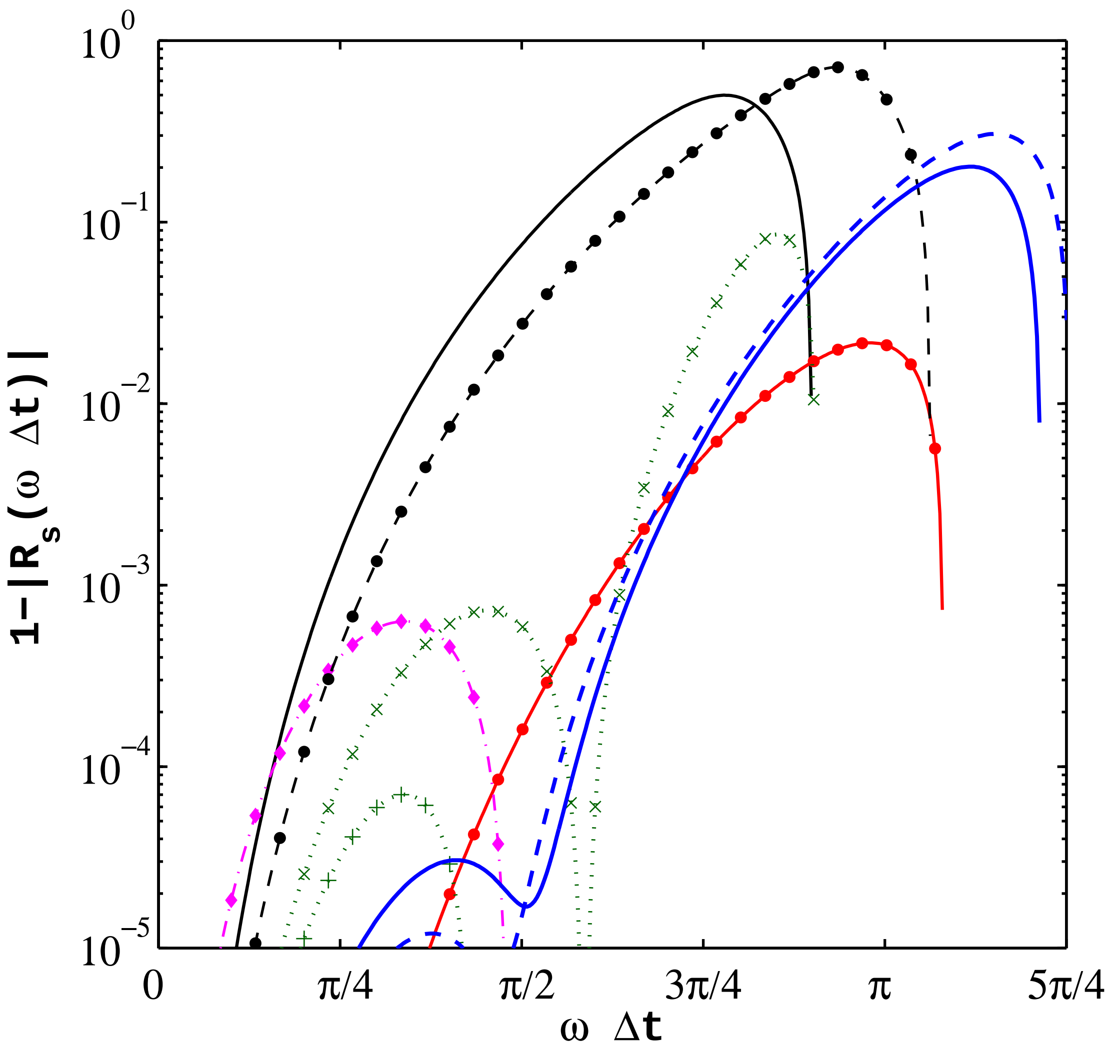
By Fourier analysis, numerical amplification factor R_s

$$R_s = \frac{\hat{u}^{n+1}}{\hat{u}^n} = 1 + \sum_{j=1}^{p} \gamma_j (-i\omega\Delta t)^j \qquad \text{exact factor} : R_e = e^{-i\omega\Delta t}$$

Optimized Runge-Kutta schemes

the coefficients γ_j are determined by minimizing the errors over a large range of pulsations $\omega\Delta t$

- **Damping factor (dissipation)**



—— 4th-order

– ● – 8th-order of Dormand & Prince (1980)

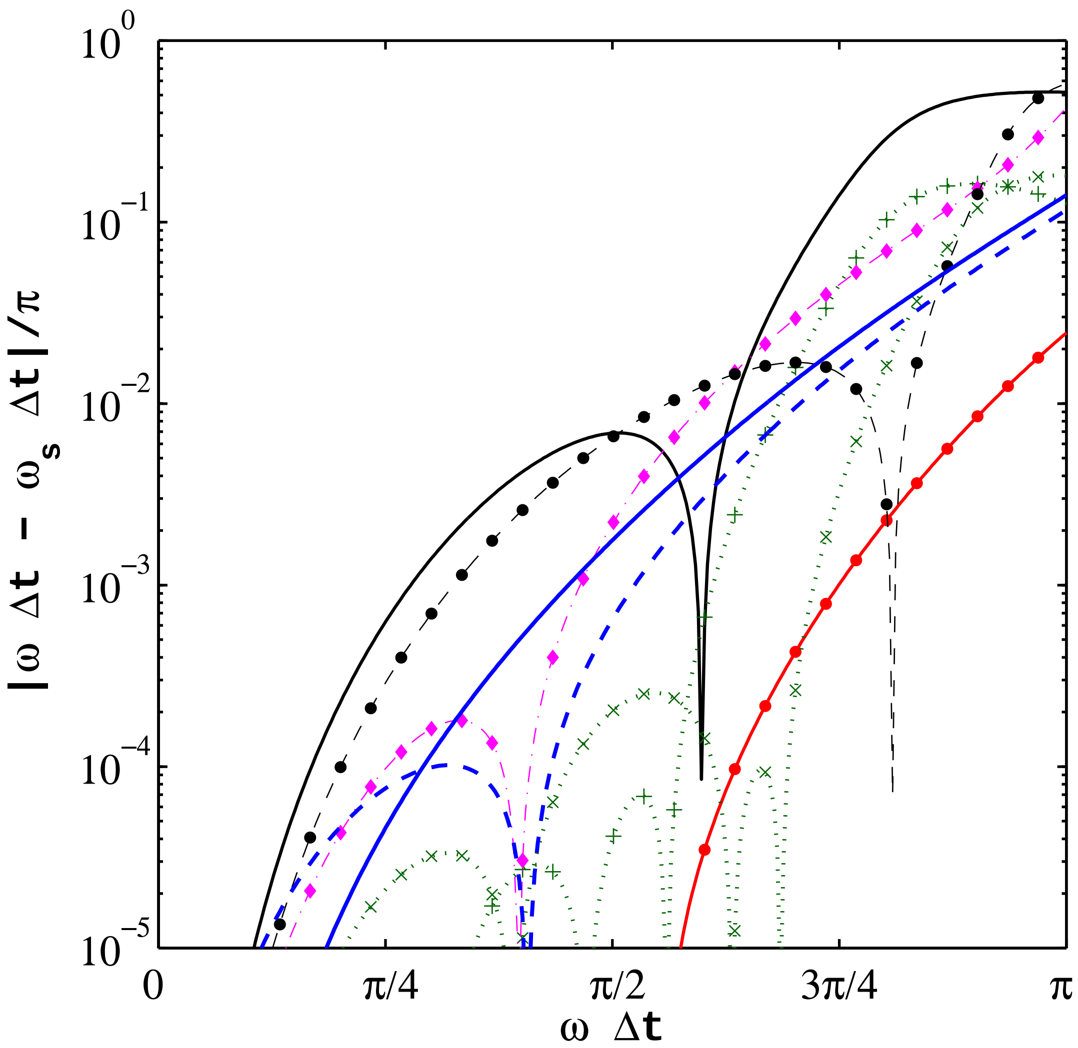
optimized - - - 2nd-order 2N ($p = 6$) of Bogey & Bailly (2004) and —— 4th-order 2N ($p = 6$) of Berland *et al.* (2006)

– ● – 4th-order 2N ($p = 5$) of Carpenter & Kennedy (1994)

$\cdots + \cdots$ LDDRK46 and $\cdots \times \cdots$ LDDRK56 of Hu *et al.* (1996)

–♦– opt. 4th-order 2N ($p = 5$) of Stanescu & Habashi

- **Phase error (dispersion)**



$$\frac{R_s}{R_e} = |R_s|\, e^{i(\omega\Delta t - \omega_s \Delta t)}$$

- **Accuracy limits**

dissipation $E_d = 1 - |R_s(\omega\Delta t)|$ & dispersion $E_\varphi = |\omega\Delta t - \omega_s\Delta t|/\pi$

CFL number β given for the opt. 11-pt FD scheme

scheme	formal	$E_d \leq 5 \times 10^{-4}$		$E_\varphi \leq 5 \times 10^{-4}$		stability	
	order	$\omega\Delta t\|_{\max}$	β	$\omega\Delta t\|_{\max}$	β	$\omega\Delta t\|_{\max}$	β
Standard RK4	4th	0.65	0.33	0.75	0.38	2.83	1.42
Standard RK8 Dormand *et al.*	8th	1.79	0.90	2.23	1.12	3.39	1.71
Stanescu *et al.*	4th	0.87	0.44	1.39	0.70	1.51	0.76
Carpenter & Kennedy	4th	0.80	0.40	0.88	0.45	3.34	1.68
Opt. LDDRK46 Hu *et al.*	4th	1.58	0.80	1.87	0.94	1.35	0.68
Opt. LDDRK56 Hu *et al.*	4th	1.18	0.59	2.23	1.13	2.84	1.43
Opt. 2N-RK Bogey *et al.*	2nd	1.91	0.96	1.53	0.77	3.94	1.99
Opt. 2N-RK Berland *et al.*	4th	1.97	0.99	1.25	0.63	3.82	1.92

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2-D Test problem - acoustic diffraction

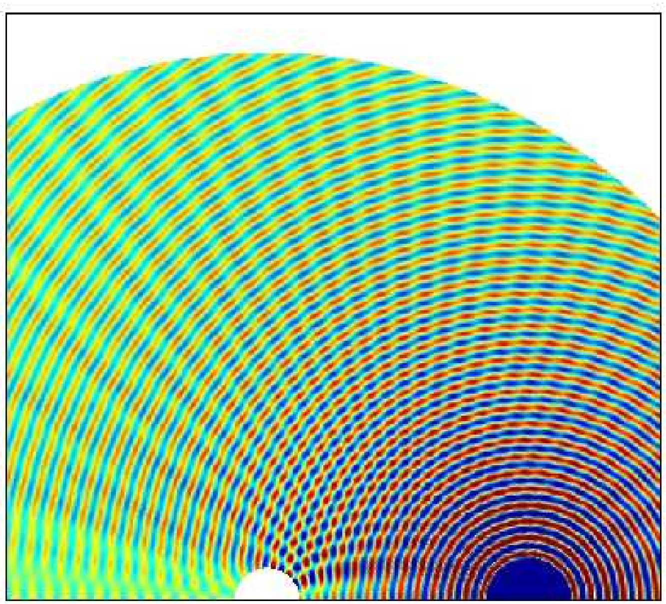
- **Acoustic diffraction by a cylinder**

  (2nd CAA Workshop, 1997)

  – **non-compact monopolar source**

  – **scattering by the cylinder**

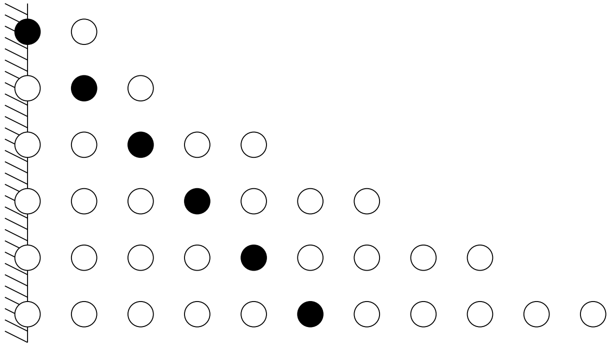
  → complex diffraction pattern sensitive to numerical accuracy



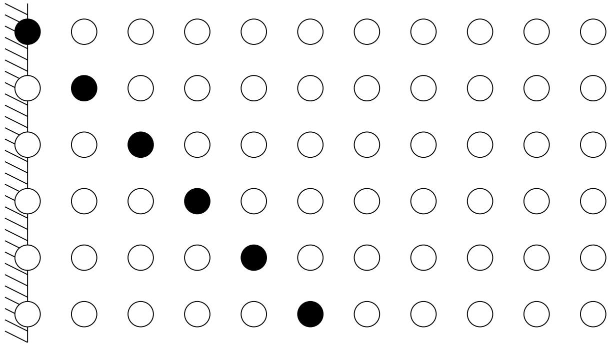
Numerical methods : optimized 11-pt F-D & filters and 6-stage Runge-Kutta

2 configurations of algorithms near the wall

centered F-D & filters



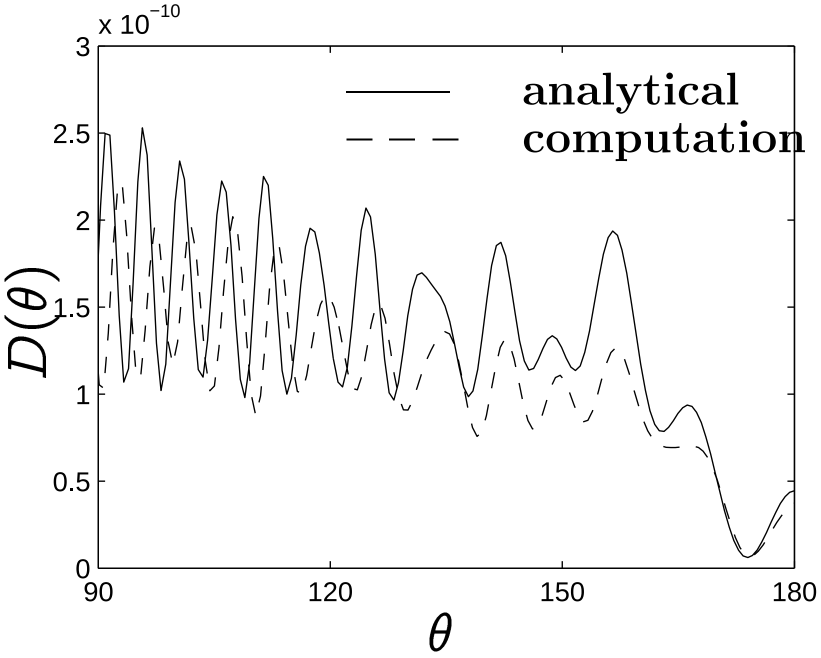
non-centered optimized 11-point F-D & filters



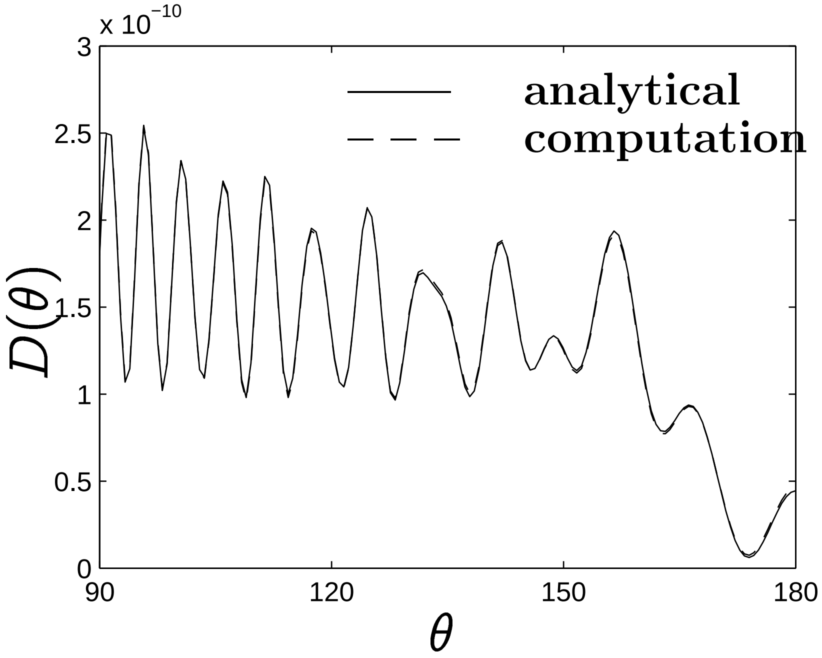
2-D Test problem - acoustic diffraction

- **Results**

Directivity $D(\theta) = rp'^2$ **at** $r = 7.5D$



centered schemes at the wall

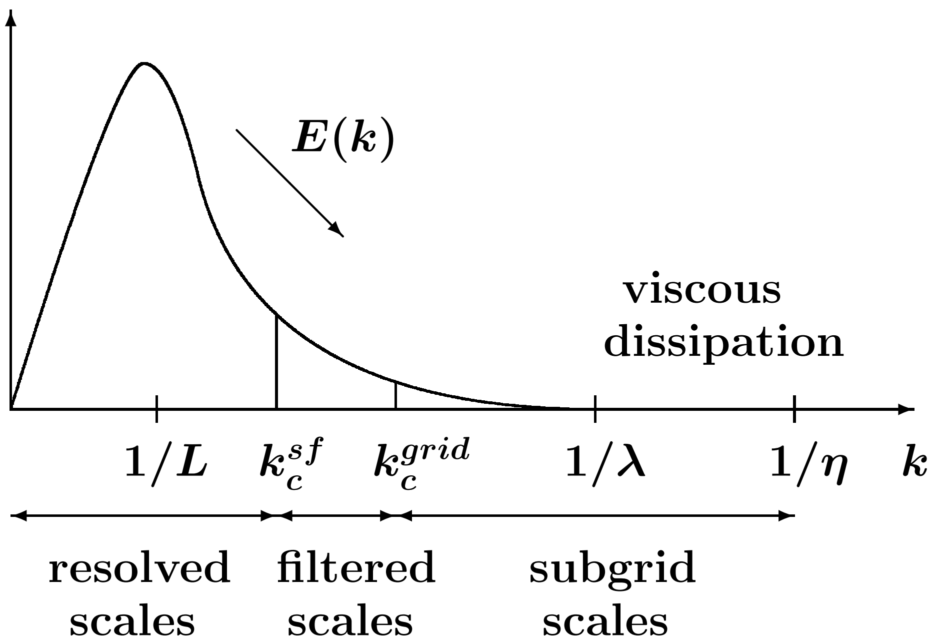


non-centered schemes at the wall

Navier-Stokes simulations (Large-Eddy Simulations)

- **Large Eddy Simulation (LES)**
  - the turbulent structures supported by the grid are computed
  - the (dissipative) effects of the subgrid scales are modelized

- **LES based on explicit filtering**



- energy draining taken into account by the filtering
- resolved scales calculated accurately by the F-D scheme, unaffected by the filtering nor by the time integration

  → flow features independent of the numerics

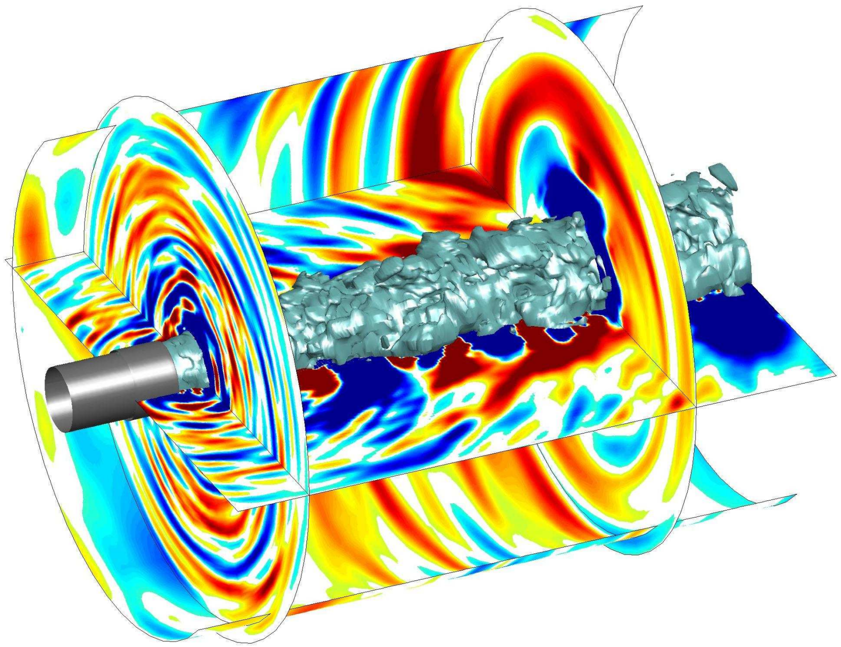
the use of optimized schemes appears appropriate for LES

Navier-Stokes simulations (Large-Eddy Simulations)

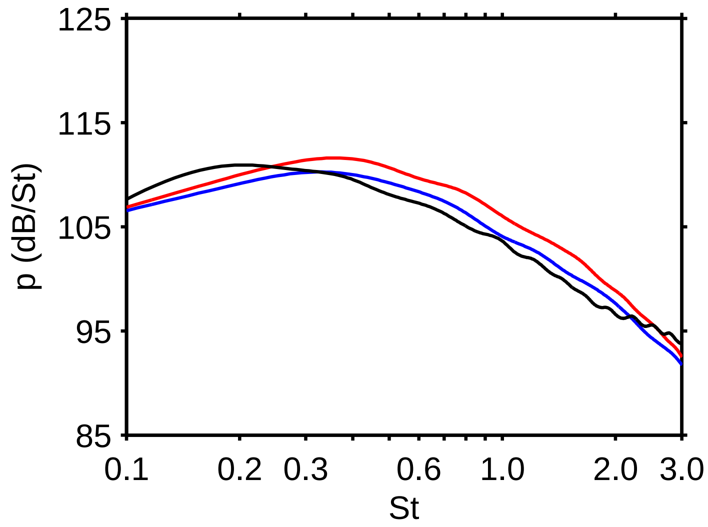
- **Investigation of noise generation**
  - subsonic and supersonic jets
  - cavity noise
  - airfoil noise

- **Noise generated by a subsonic jet** (Mach 0.9 - Reynolds 500,000)

Velocity contour and pressure field



Far-field pressure spectra at $\theta = 40^\circ$



—, — simulations

— measurements

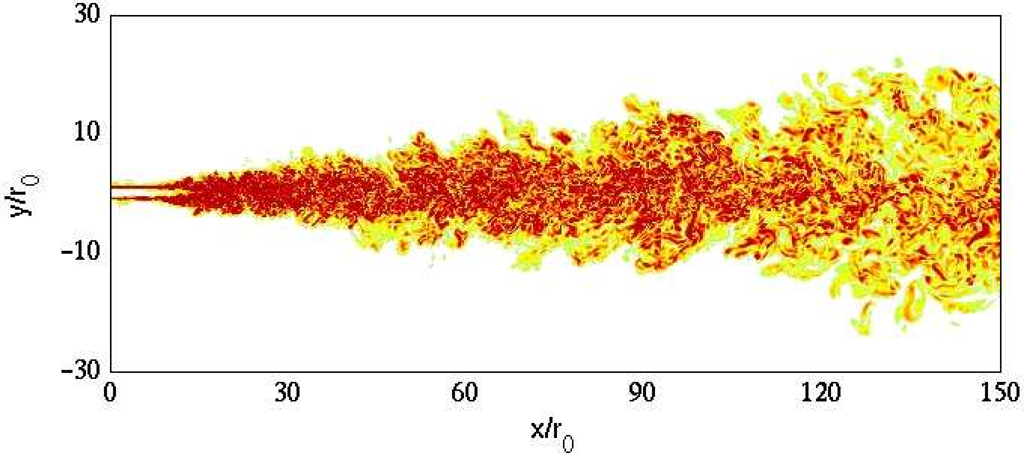
- **Investigation of turbulence**
  - simulations under controlled (physical and numerical) conditions
  - direct calculation of flow quantities including dissipation

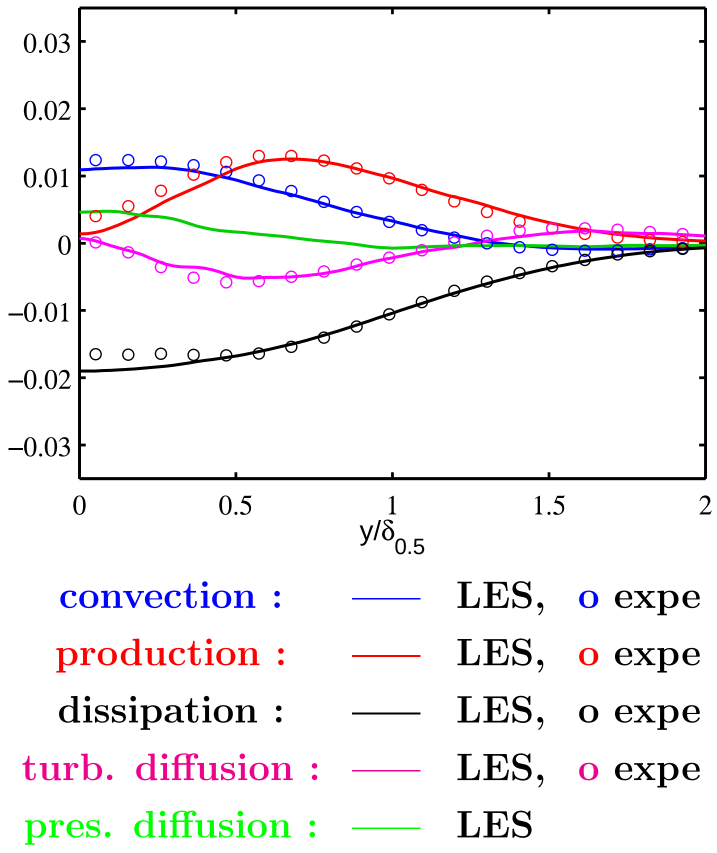
- **Energy budget in a turbulent jet**

Mach 0.9 - Reynolds 11,000

Self-similarity region for $120r_0 \leq x \leq 150r_0$

Vorticity norm





Concluding remarks

- **Optimized finite-difference methods**

  accurate / simple / efficient

  – **treatment of boundary conditions**

  – **non-uniform and curvilinear mesh**

  – **complex geometries : interpolation, overset grids, multi-domain**

  – **explicit selective filtering for Large Eddy Simulations (LES)**

- **Some difficulties**

  – **large stencils (multi-domain / parallelization)**

  – **treatment of shock-waves**