CEMRACS 2023 mini-project proposal

2 pages maximum (references included)

Title: Learning local Dirichlet-to-Neumann maps for multi-scale urban flood modeling

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1. **Context**

This mini-project is a part of the ANR project Top-up (math.unice.fr/~brenner/top-up.html) and will be dealing with approximation of the nonlinear elliptic PDEs involving highly variable coefficients. More specifically we will be interested in replacing some of the components of the traditional scientific computing methods such as Ritz-Galerkin [6, 7] and Domain Decomposition methods [5] with models resulting from the training of the artificial neural networks.

Multi-scale PDEs are relevant for multiple scientific and industrial applications. Here our major motivation stems from the modeling of floods in urbanized areas based on diffusive shallow water model [1], where the multi-scale character of the problem results from of the numerous structural features, such as building, walls, cars, etc., having strong hydraulic impact.

The numerical challenges associated with the multi-scale PDEs arise from the contrast between the typical domain size and the scale at which the material properties are represented. As the result, when discretized with Finite Elements (FE), they lead to very large systems of algebraic equations. Efficient linear solvers for such systems can be constructed, for example, within the framework of the two-level Domain Decomposition (DD) methods, which follows the divide and conquer paradigm based on the partitioning of the spatial domain. Then, the fine-scale FEM approximation is recovered iteratively, using the solution of the local problems together with the so-called coarse correction based on a low-dimensional approximation space. For multi-scale problems the performance of two-level DD methods is strongly impacted by the quality of the coarse approximation space; moreover, if the coarse space is rich enough it can be used to define a standalone approximation method. The latter approach leads to the family of multi-scale approximation methods such as, for example, the MsFEM [4].

Both Ms and DD methods are well established for linear problems and some viable strategies for the extension to the nonlinear PDEs are known. However, certain features of those methods appear inherently linearity based. In particular, traditional MsFEM and, in some extend, DD methods relies on the reuse of the computations. For example, the stiffness matrix of the MsFEM method (as well as its basis) can be computed only once, while being reused for multiple right-hand-sides, or within an iterative multigrid-like algorithm. Similarly, the iterative substructuring DD methods could benefit from the precomputed LU decomposition of the local matrices. Unfortunately, those reuse strategy do not extend naturally to the nonlinear problems and some further model reduction must be mobilized in order to accelerate the evaluation of the local Dirichlet-to-Neumann (DtN) operators [3, 2].

2. **Description and objectives**

In this mini-project we propose to revisit the traditional MsFEM with machine learning tools. We aim to extend it to the nonlinear problems by means of learning the local nonlinear Dirichlet-to-Neumann maps. The training will be performed based on the local FE computations. The resulting learning based multi-scale method is going to be tested on some nonlinear models PDEs using heterogeneous domains based on realistic urban geometries. The set of the nonlinear model problem would typically include the nonlinear advection diffusion equation and the p-Laplace problem, for which Python FE implementation will be provided.

3. **Proposed methodology**

Following the methodology of the linear MsFEM we propose to construct a low-dimensional approximations of the nonlinear PDEs using the neural network (NN) approximation of the local DtN operators that are going to be learned during the training stage. Unlike the MsFEM we will not compute the multi-scale basis, instead, based on FEM-generated data, the NN will be trained to replicate the action of the local nonlinear DtN maps on some coarse subset of the trace space. Once the training is completed the surrogate DtN operators will be used to form the global coarse problem based on Ritz or Petrov-Galerkin formulation. The latter is nonlinear problem is going to be solved by quasi-Newton method.

4. **Software requirements**

To generate the training sets the mini-project will rely on the prototype FEM software (in Python) that implements model nonlinear PDEs as well as the infrastructure for MsFEM and DD methods. The neural networks will be implemented in TensorFlow.

5. **References**

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