Further in the complexity of models? Application to the modelling of free-surface flows

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Joint work with
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Outline

Scientific issues

Goal: solving PDEs in a moving domain

Applications:

- Safety of coastal populations (tsunamis, floods, ...)
- Generation of energy (RME through swell, tide, ...)
- Transport of fluids (pipelines, ducts, ...)

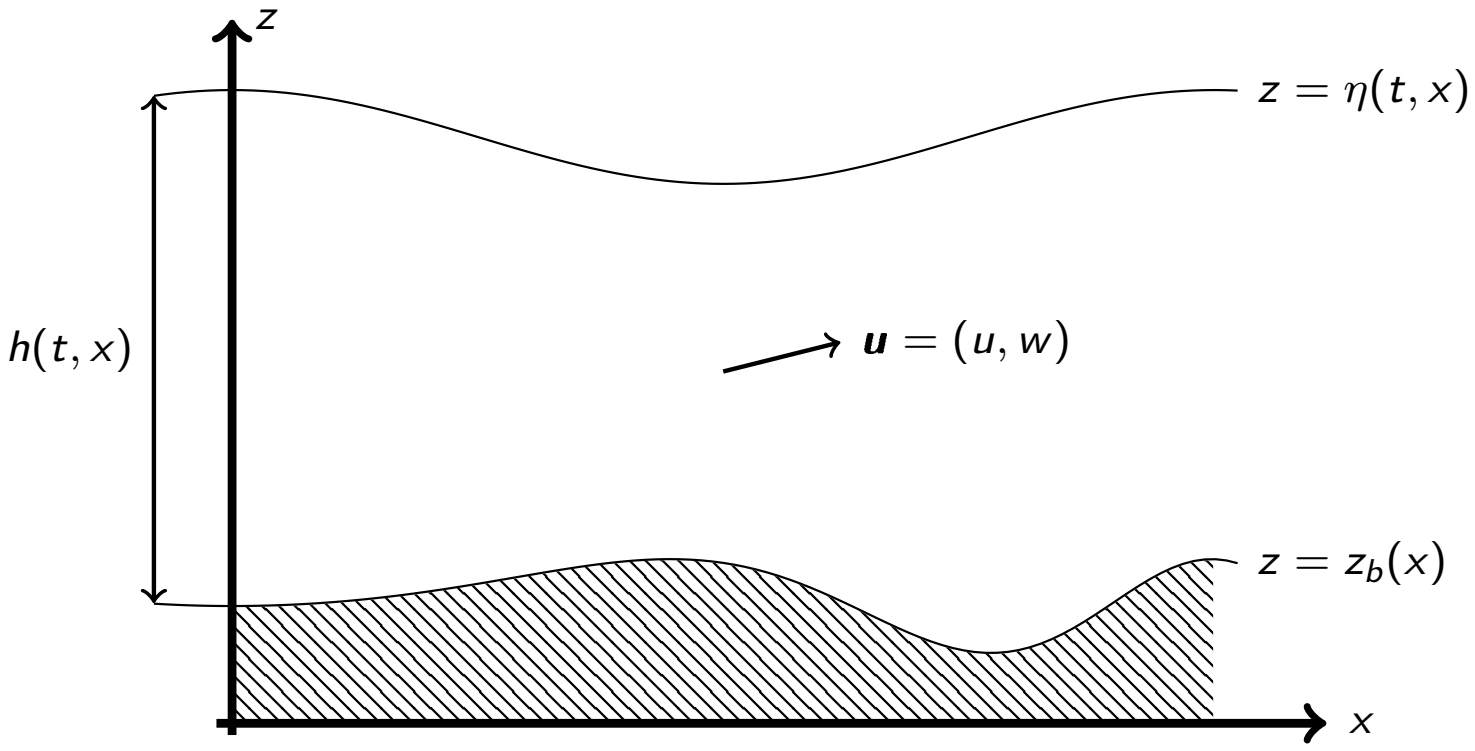
Challenges:

- The domain is an unknown in itself
- Multiphysics and multiscale

Indicators:

- Energy
- Linear dispersion relations

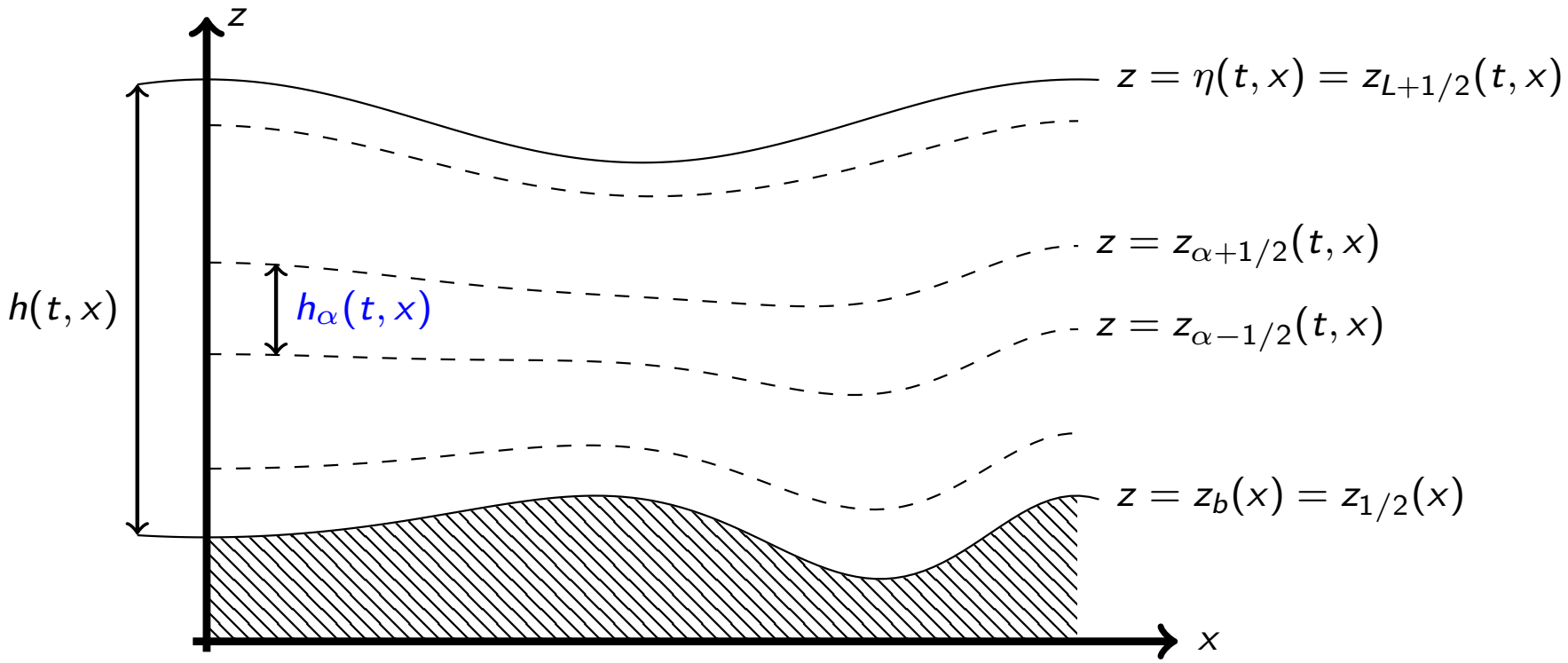
Fluid domain



Water height:

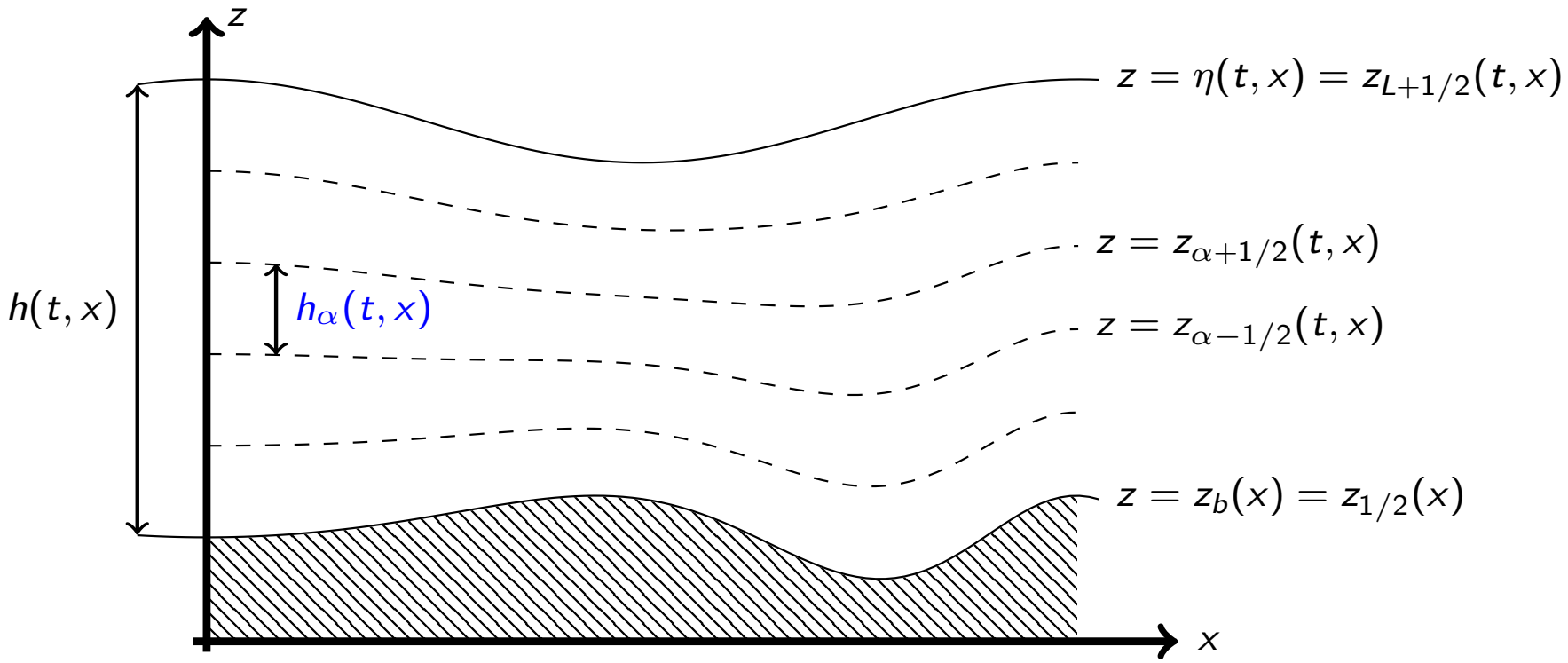
$$h(t, x) = \eta(t, x) - z_b(x)$$

Multilayer framework



Height decomposition: $h_\alpha(t,x) = \ell_\alpha h(t,x)$ with $\ell_\alpha \in (0,1)$ and $\sum_{\alpha=1}^{L} \ell_\alpha = 1$

Multilayer framework



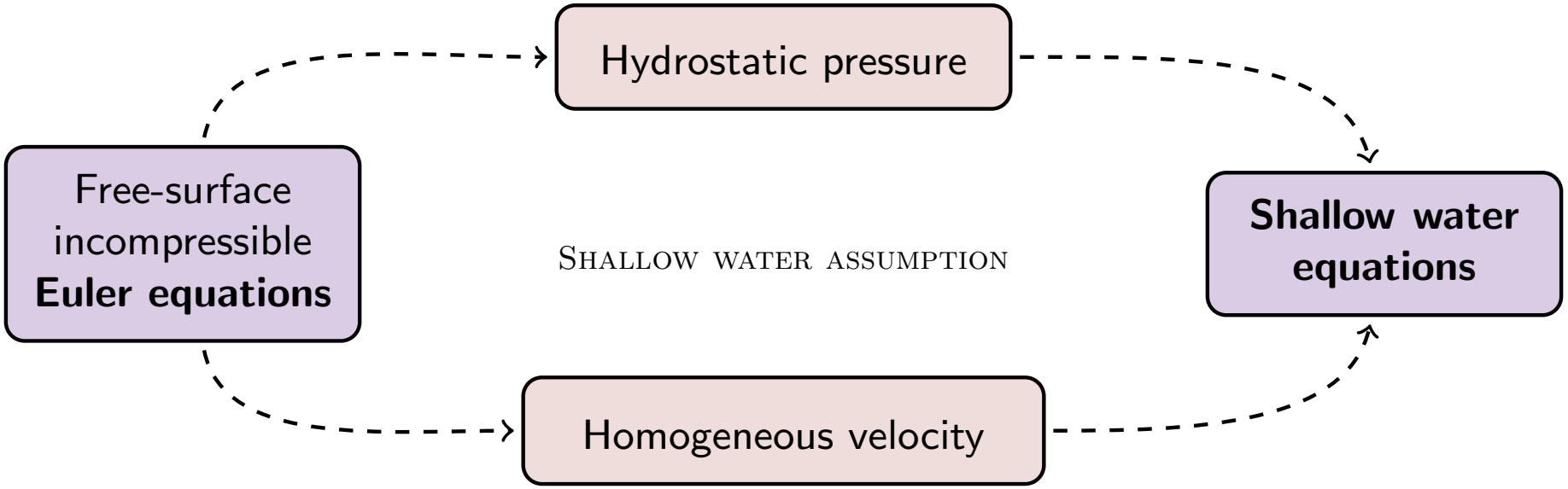
Height decomposition: $h_\alpha(t,x) = \ell_\alpha h(t,x)$ with $\ell_\alpha \in (0,1)$ and $\sum_{\alpha=1}^{L} \ell_\alpha = 1$

Homogeneous mesh: $\ell_\alpha = \dfrac{1}{L}$

Literature about free-surface flows

Free-surface
incompressible
Euler equations

Literature about free-surface flows



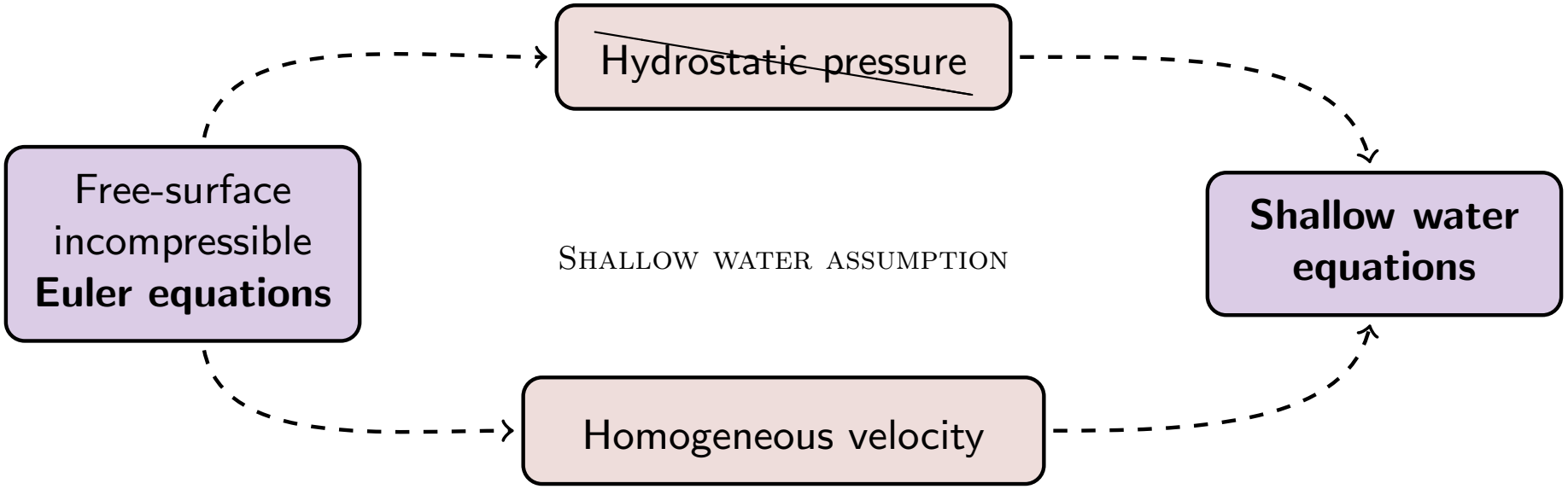
A. Barré de Saint-Venant, *Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leurs lits* (**C. R. Acad. Sci.** 73, 1871)

J.-F. Gerbeau, B. Perthame, *Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation* (**Discrete Contin. Dyn. Syst. Ser. B** 1(1), 2001)

S. Ferrari, F. Saleri, *A new two-dimensional Shallow Water model including pressure effects and slow varying bottom topography* (**Math. Model. Numer. Anal.** 38(2), 2004)

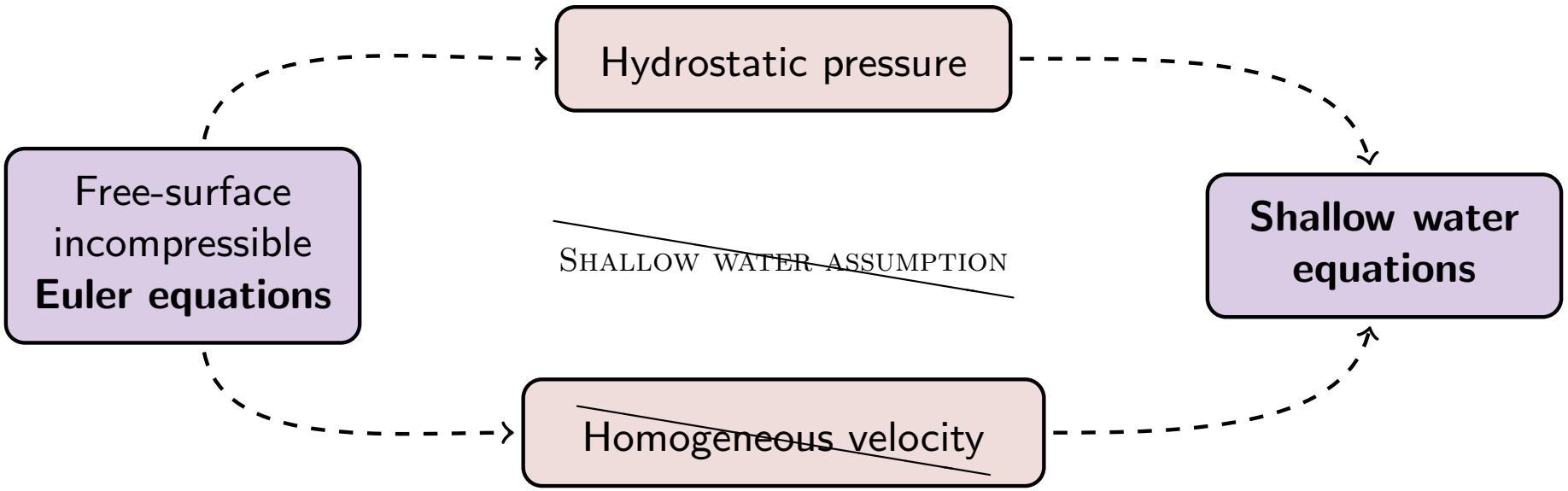
F. Marche, *Derivation of a new two-dimensional viscous shallow water model with varying topography, bottom friction and capillary effects* (**Eur. J. Mech. B Fluids** 26(1), 2007)

Literature about free-surface flows



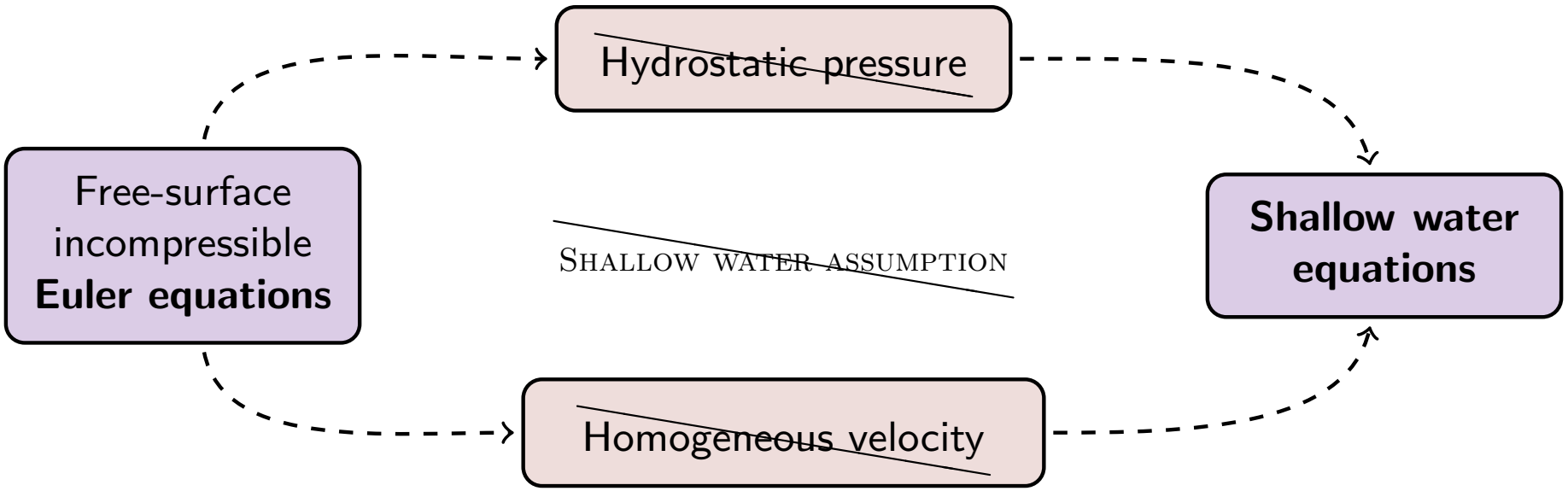
- F. Serre, *Contribution à l'étude des écoulements permanents et variables dans les canaux* (**La Houille Blanche** 6, 1953)
- A.E. Green, P.M. Naghdi, *A derivation of equations for wave propagation in water of variable depth* (**J. Fluid Mech.** 78(2), 1976)
- M.-O. Bristeau, J. Sainte-Marie, *Derivation of a non-hydrostatic shallow water model; Comparison with Saint-Venant and Boussinesq systems* (**Discrete Contin. Dyn. Syst. Ser. B** 10(4), 2008)
- D. Lannes, P. Bonneton, *Derivation of asymptotic two-dimensional time-dependent equations for surface water wave propagation* (**Phys. Fluids** 21(1), 2009)
- Peregrine '67, Madsen *et al.* '91 '96 '03 '06, Nwogu '93, Casulli *et al.* '95 '99, Yamazaki *et al.* '09, ...

Literature about free-surface flows



- E. Audusse, M.-O. Bristeau, B. Perthame, J. Sainte-Marie, *A multilayer Saint-Venant system with mass exchanges for Shallow Water flows. Derivation and numerical validation* (**Math. Model. Numer. Anal.** 45(1), 2011)
- F. Bouchut, V. Zeitlin, *A robust well-balanced scheme for multi-layer shallow water equations* (**Discrete Contin. Dyn. Syst. Ser. B** 13(4), 2010)
- E.D. Fernández-Nieto, E.H. Koné, T. Morales de Luna, R. Bürger, *A multilayer shallow water system for polydisperse sedimentation* (**J. Comput. Phys.** 238, 2013)
- Castro *et al.* '01 '04 '10, Narbona *et al.* '09 '13, ...

Literature about free-surface flows



Derivation of multilayer non-hydrostatic models

M. Zijlema, G.S. Stelling, *Further experiences with computing non-hydrostatic free-surface flows involving water waves* (**Int. J. Numer. Methods Fluids** 48(2), 2005)

Y. Bai, K.F. Cheung, *Dispersion and nonlinearity of multi-layer non-hydrostatic free-surface flow* (**J. Fluid Mech.** 726, 2013)

Euler equations

Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x (u^2 + p) + \partial_z (uw) = 0 \\ \partial_t w + \partial_x (uw) + \partial_z (p + w^2) = -g \end{cases}$$

set in the domain $\Omega(t) = \left\{ (x,z) \in \mathbb{R}^2 \;\middle|\; z_b(x) \leq z \leq \eta(t,x) \right\}$

Boundary conditions

$$\partial_t \eta(t,x) + u\big(t,x,\eta(t,x)\big)\partial_x \eta(t,x) - w\big(t,x,\eta(t,x)\big) = 0$$
$$p\big(t,x,\eta(t,x)\big) = p^{atm}(t,x)$$
$$u\big(t,x,z_b(x)\big)z_b'(x) - w\big(t,x,z_b(x)\big) = 0$$

together with well-prepared initial conditions

Pressure fields $p(t,x,z) = p^{atm}(t,x) + g\big(\eta(t,x) - z\big) + q(t,x,z)$

Euler equations

Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x (u^2 + p) + \partial_z (uw) = 0 \\ \partial_t w + \partial_x (uw) + \partial_z (p + w^2) = -g \end{cases}$$

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Boundary conditions

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$$p\big(t,x,\eta(t,x)\big) = p^{atm}(t,x)$$
$$u\big(t,x,z_b(x)\big)z_b'(x) - w\big(t,x,z_b(x)\big) = 0$$

together with well-prepared initial conditions

Pressure fields $p(t,x,z) = p^{atm}(t,x) + g\big(\eta(t,x) - z\big) + \cancel{q(t,x,z)}$

Euler equations

Model

$$\begin{cases} \partial_x u + \partial_z w = 0 \\ \partial_t u + \partial_x(u^2 + q) + \partial_z(uw) = -\partial_x(g\eta + p^{atm}) \\ \partial_t w + \partial_x(uw) + \partial_z(q + w^2) = 0 \end{cases}$$

set in the domain $\Omega(t) = \left\{(x,z) \in \mathbb{R}^2 \;\middle|\; z_b(x) \leq z \leq \eta(t,x)\right\}$

Boundary conditions

$$\partial_t \eta(t,x) + u\big(t,x,\eta(t,x)\big)\partial_x \eta(t,x) - w\big(t,x,\eta(t,x)\big) = 0$$
$$q\big(t,x,\eta(t,x)\big) = 0$$
$$u\big(t,x,z_b(x)\big)z_b'(x) - w\big(t,x,z_b(x)\big) = 0$$

together with well-prepared initial conditions

General procedure

Toy model

$$\partial_t \mathscr{R} + \partial_x(u\mathscr{R} + \mathscr{P}) + \partial_z(w\mathscr{R} + \mathscr{Q}) = \mathscr{S} \tag{1}$$

where $\mathscr{R}$, $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{S}$ take values in $\mathbb{R}^p$

Semi-discrete formulation

for $z_b(x) \leq z_-(t,x) \leq z_+(t,x) \leq \eta(t,x)$

$$\partial_t\big((z_+ - z_-)\langle\mathscr{R}\rangle\big) + \partial_x\big((z_+ - z_-)[\langle u\mathscr{R}\rangle + \langle\mathscr{P}\rangle]\big) + \mathscr{F}_+ - \mathscr{F}_- = (z_+ - z_-)\langle\mathscr{S}\rangle$$

where

$$\langle\mathscr{S}\rangle(t,x) = \frac{1}{z_+(t,x) - z_-(t,x)} \int_{z_-(t,x)}^{z_+(t,x)} \mathscr{S}(t,x,z)\,\mathrm{d}z$$

$$\mathscr{F}_\pm = \Upsilon_\pm \mathscr{R}\big(t, z_\pm(t,x)_\mp\big) - \mathscr{P}\big(t, z_\pm(t,x)_\mp\big)\partial_x z_\pm + \mathscr{Q}\big(t, z_\pm(t,x)_\mp\big)$$

$$\Upsilon_\pm = w\big(t, z_\pm(t,x)_\mp\big) - \partial_t z_\pm - u\big(t, z_\pm(t,x)_\mp\big)\partial_x z_\pm$$

Applications

Application of the previous procedure depending on the assumptions on the vertical profile of the unknowns.

Shallow water models

- $z_-(t,x) = z_b(x)$, $z_+(t,x) = \eta(t,x)$
- $\mathscr{R} \in \{1, u, w, zw, \ldots\}$, $\mathscr{P}, \mathscr{Q} \in \{0, q, zq, \ldots\}$
- Resulting models: Depth-averaged Euler, Serre – Green-Naghdi, ...

General models

- $z_-(t,x) = z_{\alpha-1/2}(t,x)$, $z_+(t,x) = z_{\alpha+1/2}(t,x)$
- $\mathscr{R} \in \{1, u, w, (z - z_\alpha)w, \ldots\}$, $\mathscr{P}, \mathscr{Q} \in \{0, q, (z - z_\alpha)q, \ldots\}$
- Resulting models: $\text{LDNH}_k(L)$, $\text{LIN-NH}_k(L)$

Mathematical structure

Resulting models

$$\begin{cases} \partial_t h + \partial_x(h\overline{u}) = 0, \\ \partial_t(h\boldsymbol{X}) + \partial_x(hu\boldsymbol{X}) + \nabla_* \boldsymbol{Q} + \boldsymbol{F} = \boldsymbol{S}, \\ \nabla_* \cdot \boldsymbol{X} = 0. \end{cases}$$

$$\boldsymbol{X} = \begin{pmatrix} u \\ w \\ \sigma \end{pmatrix}, \begin{pmatrix} u_\alpha \\ w_\alpha \\ \sigma_\alpha \end{pmatrix}, \begin{pmatrix} u_\alpha \\ \Lambda_\alpha \\ w_\alpha \\ \Phi_\alpha \\ \Psi_\alpha \end{pmatrix}, \ldots \text{ and } \boldsymbol{Q} = \begin{pmatrix} q \\ q_b \end{pmatrix}, \begin{pmatrix} q_\alpha \\ q_{\alpha-1/2} \end{pmatrix}, \begin{pmatrix} q_\alpha \\ a_{\alpha-1/2} \\ \pi_\alpha \end{pmatrix}, \ldots$$

Mathematical structure

Resulting models

$$\begin{cases} \partial_t h + \partial_x(h\overline{u}) = 0, \\ \partial_t(h\boldsymbol{X}) + \partial_x(hu\boldsymbol{X}) + \nabla_* \boldsymbol{Q} + \boldsymbol{F} = \boldsymbol{S}, \\ \nabla_* \cdot \boldsymbol{X} = 0. \end{cases}$$

$$\nabla_* \boldsymbol{Q} = \begin{pmatrix} \partial_x(hq) + \frac{\alpha^2}{2} q \partial_x z_b \\ -\alpha q \end{pmatrix}, \begin{pmatrix} \partial_x(hq) + q_b \partial_x z_b \\ -q_b \\ -2\sqrt{3}\left(q - \frac{q_b}{2}\right) \end{pmatrix}, \ldots$$

and $\nabla_* \cdot \boldsymbol{X} = \alpha w - \frac{\alpha^2}{2} u \partial_x z_b + h \partial_x u$, $\begin{pmatrix} 2\sqrt{3}\sigma + h\partial_x u \\ w - u\partial_x z_b - \sqrt{3}\sigma \end{pmatrix}, \ldots$

Mathematical structure

Resulting models

$$\begin{cases} \partial_t h + \partial_x(h\overline{u}) = 0, \\ \partial_t(h\boldsymbol{X}) + \partial_x(hu\boldsymbol{X}) + \nabla_* \boldsymbol{Q} + \boldsymbol{F} = \boldsymbol{S}, \\ \nabla_* \cdot \boldsymbol{X} = 0. \end{cases}$$

Key point

$$\boldsymbol{X} \cdot \nabla_* \boldsymbol{Q} = \partial_x \#_* - \boldsymbol{Q} \cdot (\nabla_* \cdot \boldsymbol{X})$$

Energy

$$\partial_t \left(h \frac{|\boldsymbol{X}|^2}{2} + \#_S \right) + \partial_x \left(hu \frac{|\boldsymbol{X}|^2}{2} + \#_* + \#'_S + \#_F \right) \leq 0$$

Mathematical structure

Resulting models

$$\begin{cases} \partial_t h + \partial_x(h\overline{u}) = 0, \\ \partial_t(h\boldsymbol{X}) + \partial_x(hu\boldsymbol{X}) + \nabla_* \boldsymbol{Q} + \boldsymbol{F} = \boldsymbol{S}, \\ \nabla_* \cdot \boldsymbol{X} = 0. \end{cases}$$

Key point

$$\boldsymbol{X} \cdot \nabla_* \boldsymbol{Q} = \partial_x \#_* - \boldsymbol{Q} \cdot (\nabla_* \cdot \boldsymbol{X})$$

Projection method

$$-\nabla_* \cdot \left(\frac{1}{h} \nabla_* \boldsymbol{Q} \right) = \dots$$

Well-posedness thanks to the *Lax-Milgram* theorem

Serre – Green-Naghdi equations

$$
\begin{cases}
\partial_t h + \partial_x (hu) = 0, \\
\partial_t (hu) + \partial_x \left(hu^2 + g\dfrac{h^2}{2} \right) + gh\partial_x z_b + \partial_x (hq) + q_b \partial_x z_b = 0, \\
\partial_t (hw) + \partial_x (Huw) - q_b = 0, \\
\partial_t (h\sigma) + \partial_x (h\sigma u) - 2\sqrt{3}\left[q - \dfrac{q_b}{2} \right] = 0, \\
w - u\partial_x z_b + \dfrac{h}{2}\partial_x u = 0, \\
\sigma + \dfrac{h\partial_x u}{2\sqrt{3}} = 0.
\end{cases}
$$

Serre – Green-Naghdi equations

The system also reads

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \big(\mathcal{I}_d + \mathcal{T}[h, z_b]\big)(\partial_t u + u\partial_x u) + g\partial_x(h + z_b) + \mathcal{Q}[h, z_b]u = -\partial_x p^{atm}, \end{cases}$$

where

$$\begin{aligned} \mathcal{T}[h, z_b]v &\stackrel{\text{def}}{=} \mathcal{R}_1[h, z_b](\partial_x v) + \mathcal{R}_2[h, z_b](v\partial_x z_b), \\ \mathcal{Q}[h, z_b]v &\stackrel{\text{def}}{=} -2\mathcal{R}_1[h, z_b]\left((\partial_x v)^2\right) + \mathcal{R}_2[h, z_b](v^2\partial^2_{xx} z_b), \\ \mathcal{R}_1[h, z_b]w &\stackrel{\text{def}}{=} -\frac{1}{3h}\partial_x(h^3 w) - \frac{h}{2}w\partial_x z_b, \\ \mathcal{R}_2[h, z_b]w &\stackrel{\text{def}}{=} \frac{1}{2h}\partial_x(h^2 w) + w\partial_x z_b. \end{aligned}$$

A splitting approach for the fully nonlinear and weakly dispersive Green-Naghdi model, P. Bonneton, F. Chazel, D. Lannes, F. Marche and M. Tissier. **J. Comput. Phys.**, 230(4), 2011.

Serre – Green-Naghdi equations

The system also reads

$$\begin{cases} \partial_t h + \partial_x (hu) = 0, \\ \partial_t (hu) + \partial_x \left(hu^2 + g\dfrac{h^2}{2} + \dfrac{h^2 \ddot{h}}{3} + \dfrac{h^2 \dot{u}}{2} \partial_x z_b + \dfrac{h^2 u^2}{2} \partial_{xx}^2 z_b \right) \\ \qquad\qquad + h \left(g + \dfrac{\ddot{h}}{2} + \dot{u} \partial_x z_b + u^2 \partial_{xx}^2 z_b \right) \partial_x z_b = -h \partial_x p^{atm}, \end{cases}$$

where $\dot{\xi} \stackrel{\text{def}}{=} \partial_t \xi + u \partial_x \xi$.

A rapid numerical method for solving Serre–Green-Naghdi equations describing long free surface gravity waves, N. Favrie and S. Gavrilyuk. **Nonlinearity**, 30(7), 2017.

Serre – Green-Naghdi equations

$$
\begin{cases}
\partial_t h + \partial_x (hu) = 0, \\
\partial_t (hu) + \partial_x \left(hu^2 + g\dfrac{h^2}{2} \right) + gh\partial_x z_b + \partial_x (hq) + q_b \partial_x z_b = 0, \\
12\dfrac{q}{h} - h\partial_x \left(\dfrac{\partial_x (hq)}{h} \right) - 6\dfrac{q_b}{h} - h\partial_x \left(\dfrac{q_b}{h} \partial_x z_b \right) = \mathfrak{f}(h,u), \\
\left(4 + (\partial_x z_b)^2\right) \dfrac{q_b}{h} - 6\dfrac{q}{h} + \partial_x z_b \dfrac{\partial_x (hq)}{h} = \mathfrak{f}_b(h,u).
\end{cases}
$$

Numerical strategy

Natural splitting strategy:

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\dfrac{h^2}{2}\right) + gh\partial_x z_b + \partial_x(hq) + q_b\partial_x z_b = 0, \\ \partial_t(hw) + \partial_x(huw) - q_b = 0, \\ \partial_t(h\sigma) + \partial_x(h\sigma u) - 2\sqrt{3}\left[q - \dfrac{q_b}{2}\right] = 0, \\ w - u\partial_x z_b + \dfrac{h}{2}\partial_x u = 0, \\ \sigma + \dfrac{h\partial_x u}{2\sqrt{3}} = 0. \end{cases}$$

Numerical strategy

Natural splitting strategy: hyperbolic solver

$$\begin{cases} \partial_t h + \partial_x(hu) = 0, \\ \partial_t(hu) + \partial_x\left(hu^2 + g\dfrac{h^2}{2}\right) + gh\partial_x z_b + \partial_x(hq) + q_b\partial_x z_b = 0, \\ \partial_t(hw) + \partial_x(huw) - q_b = 0, \\ \partial_t(h\sigma) + \partial_x(h\sigma u) - 2\sqrt{3}\left[q - \dfrac{q_b}{2}\right] = 0, \\ w - u\partial_x z_b + \dfrac{h}{2}\partial_x u = 0, \\ \sigma + \dfrac{h\partial_x u}{2\sqrt{3}} = 0. \end{cases}$$

Numerical strategy

Natural splitting strategy: non-hydrostatic correction

$$\begin{cases} \partial_t h + \partial_x (hu) = 0, \\ \partial_t (hu) + \partial_x \left(hu^2 + g\dfrac{h^2}{2} \right) + gh\partial_x z_b + \partial_x (hq) + q_b \partial_x z_b = 0, \\ \partial_t (hw) + \partial_x (huw) - q_b = 0, \\ \partial_t (h\sigma) + \partial_x (h\sigma u) - 2\sqrt{3}\left[q - \dfrac{q_b}{2} \right] = 0, \\ w - u\partial_x z_b + \dfrac{h}{2}\partial_x u = 0, \\ \sigma + \dfrac{h\partial_x u}{2\sqrt{3}} = 0. \end{cases}$$

Discretisation a uniform Cartesian grid

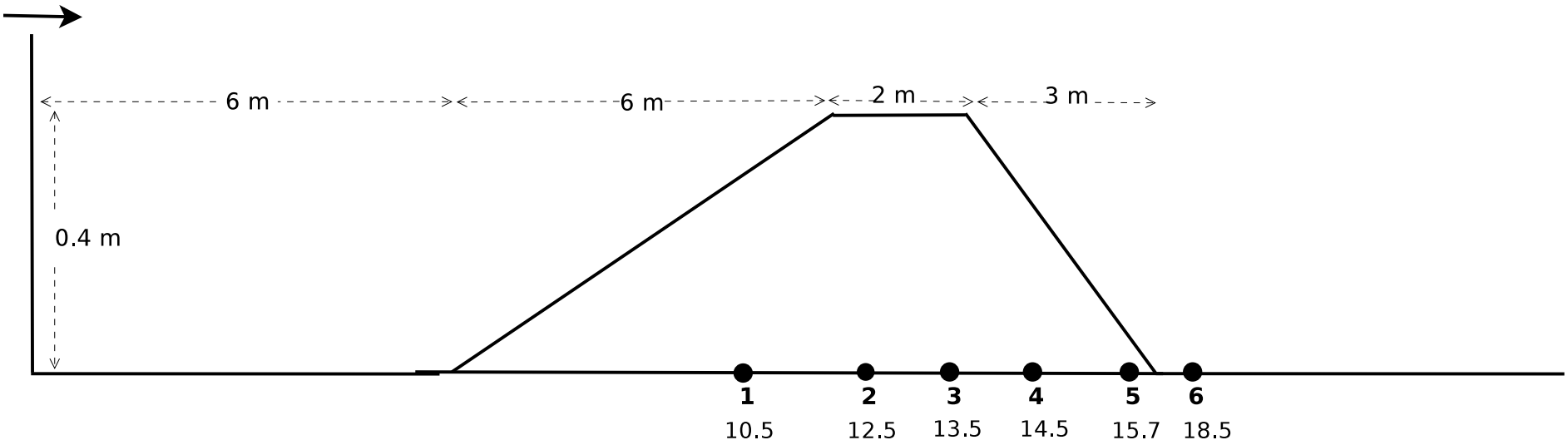
Mixed problem

$$\left(\begin{array}{c|c} \overline{\mathcal{H}}/\Delta t & \mathcal{B} \\ \hline \mathcal{B}^T & \overline{\overline{\mathbf{0}}} \end{array}\right) \left(\begin{array}{c} \boldsymbol{X} \\ \hline \boldsymbol{Q} \end{array}\right) = \left(\begin{array}{c} \overline{\mathcal{H}}\boldsymbol{X}^*/\Delta t - \widehat{\mathbf{0}} \\ \hline \widetilde{\mathbf{0}} \end{array}\right).$$

Pressure problem

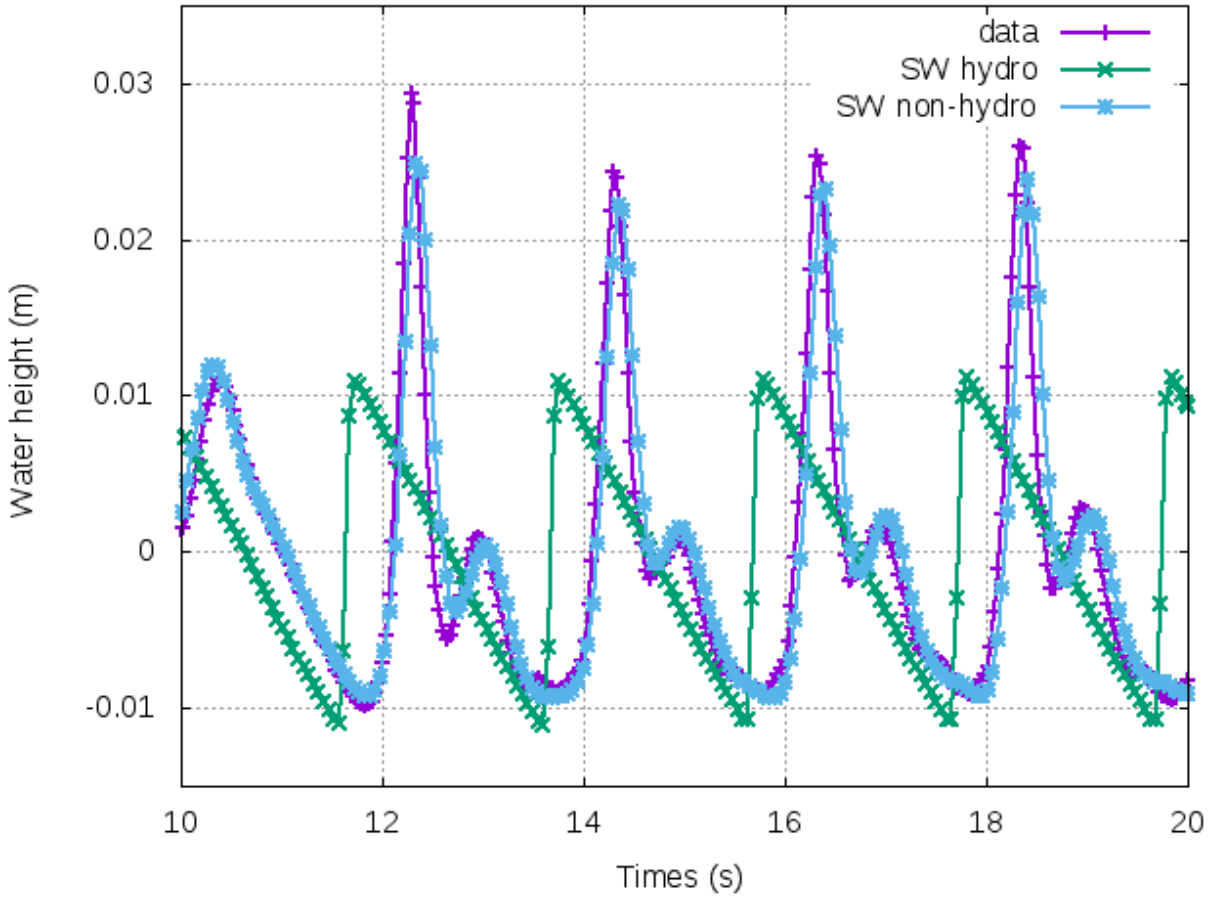
$$\underbrace{\mathcal{B}^T\overline{\mathcal{H}}^{-1}\mathcal{B}}_{:=C}\boldsymbol{Q} = \frac{\mathcal{B}^T\boldsymbol{X}^* - \widetilde{\mathbf{0}}}{\Delta t} - \mathcal{B}^T\overline{\mathcal{H}}^{-1}\widehat{\mathbf{0}},$$

Emphasis of non-hydrostatic effects

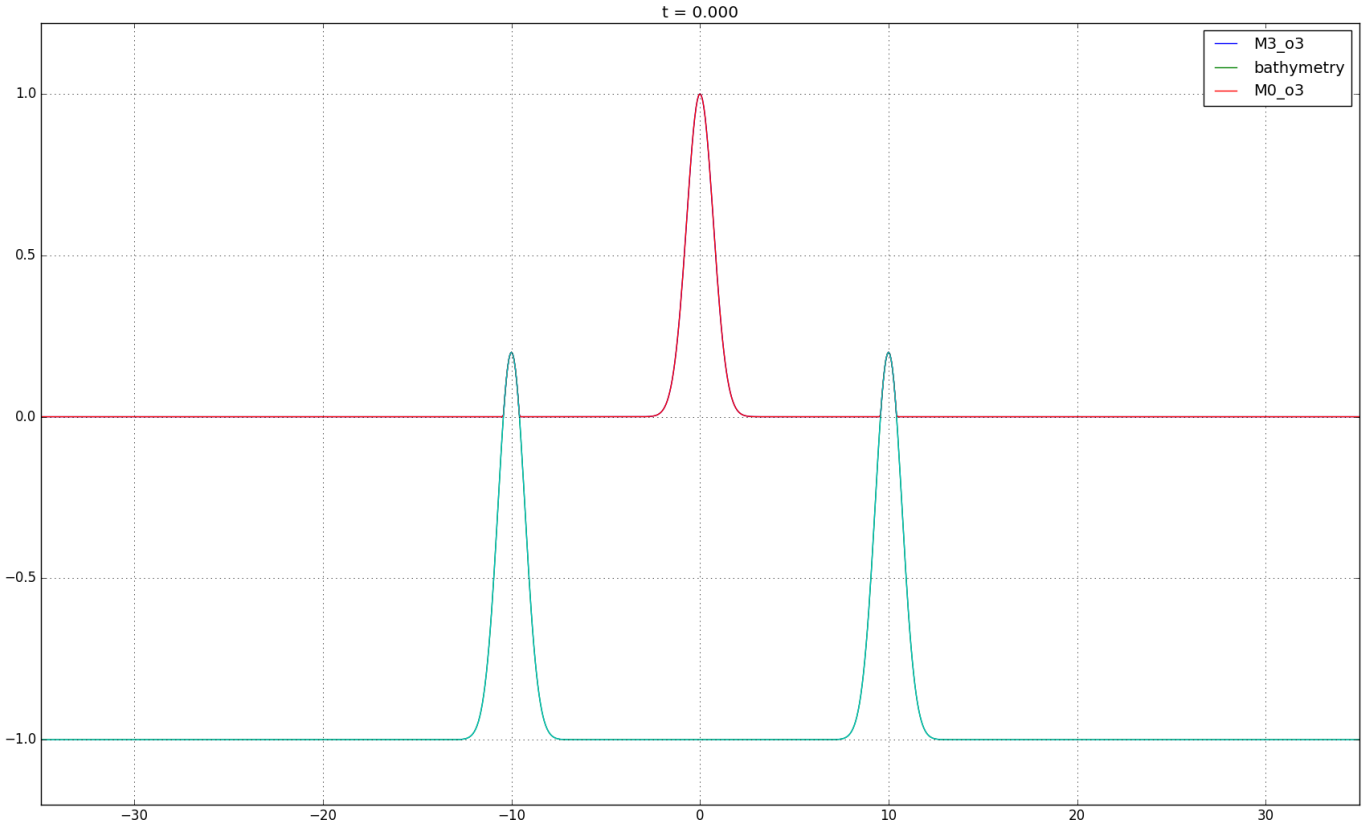


M.-W. Dingemans, *Wave propagation over uneven bottoms* (**Adv. Ser. Ocean Eng.**, 1997)

Emphasis of non-hydrostatic effects



Comparisons



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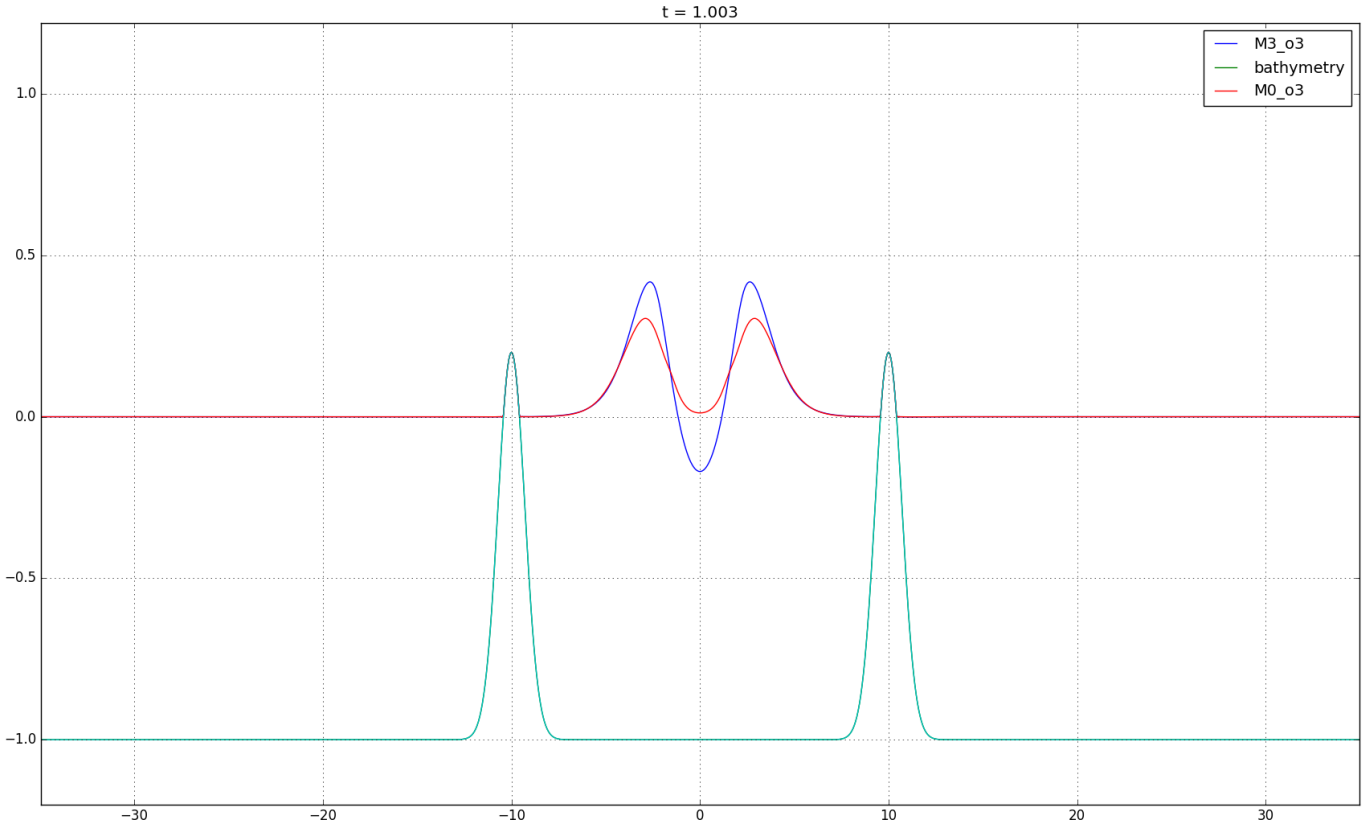

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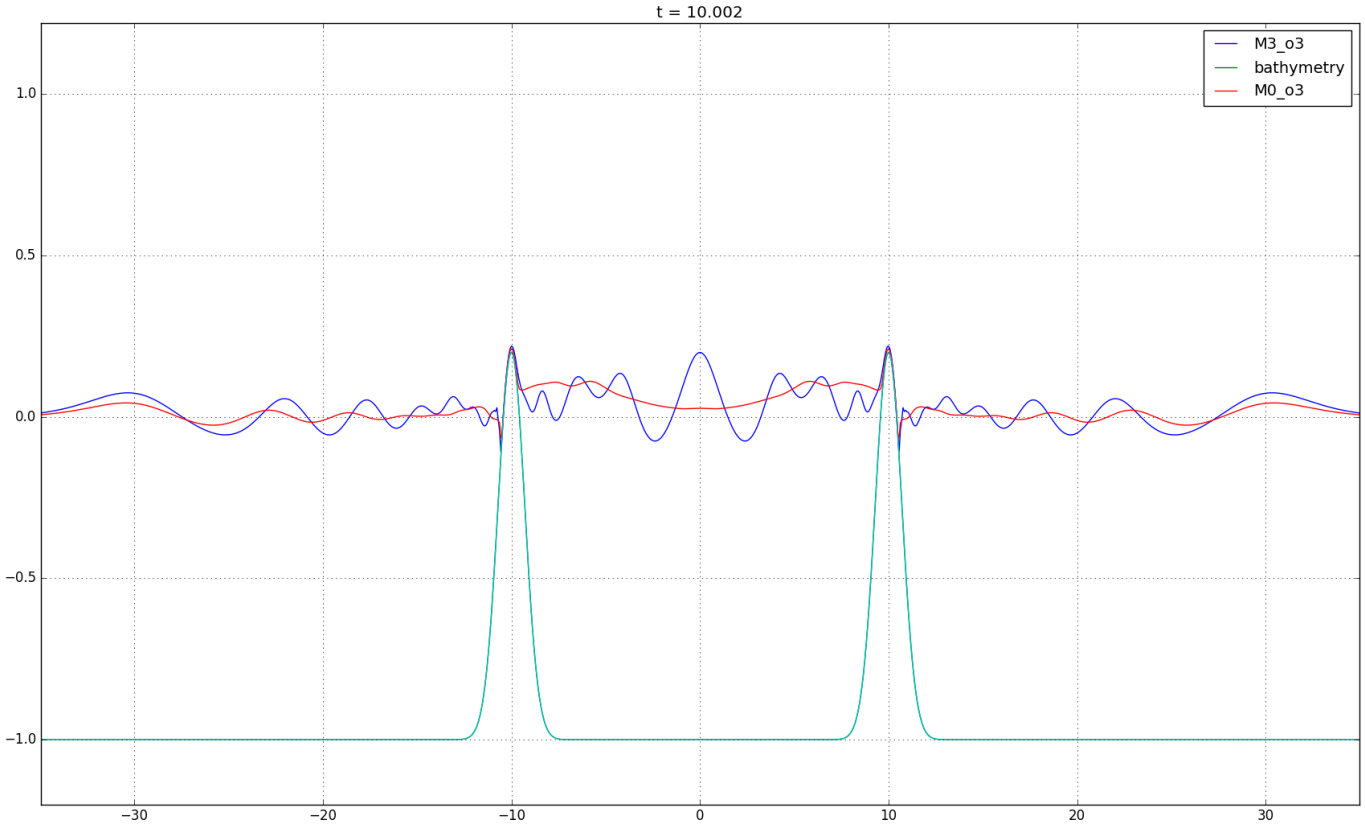


Fully discrete scheme

Mixed problem

$$\left(\begin{array}{c|c} \overline{\mathcal{H}}/(L\Delta t) & \overline{\mathcal{B}}+\overline{\mathcal{R}} \\ \hline \left(\overline{\mathcal{B}}+\overline{\mathcal{R}}\right)^T & \overline{\overline{\mathbf{0}}} \end{array}\right)\left(\begin{array}{c} \overline{\boldsymbol{X}} \\ \hline \overline{\boldsymbol{Q}} \end{array}\right)=\left(\begin{array}{c} \overline{\mathcal{H}}\overline{\boldsymbol{X}}^*/(L\Delta t)-\widehat{\mathbf{0}} \\ \hline \widetilde{\mathbf{0}} \end{array}\right).$$

Pressure problem

$$\underbrace{\left(\overline{\mathcal{B}}+\overline{\mathcal{R}}\right)^T\overline{\mathcal{H}}^{-1}\left(\overline{\mathcal{B}}+\overline{\mathcal{R}}\right)}_{:=\overline{C}}\overline{\boldsymbol{Q}}=\frac{1}{L\Delta t}\left[\left(\overline{\mathcal{B}}+\overline{\mathcal{R}}\right)^T\overline{\boldsymbol{X}}^*-\widetilde{\mathbf{0}}\right]-\left(\overline{\mathcal{B}}+\overline{\mathcal{R}}\right)^T\overline{\mathcal{H}}^{-1}\widehat{\mathbf{0}}.$$

$\overline{C}$ is blockwise tridiagonal, symmetric positive-definite.

Iterative resolution

x-direction For each layer α:

$$C_{11}\mathbf{q}_{\alpha,\circ}^{p+1} = \frac{1}{L\Delta t}\left(B_{11}^T\mathbf{U}_\alpha^* - 2\sqrt{3}\mathbf{\Sigma}_\alpha^*\right) - C_{12}^{\alpha-1/2}\mathbf{q}_{\alpha-1/2,\circ}^{p};$$

z-direction For each node x_i:

$$\mathcal{S}^{(i)}\mathbf{q}_{\circ-1/2,i}^{p+1} = D_i\mathbf{q}_{\circ,i}^{p+1}$$

where $\mathcal{S}^{(i)}$ is tridiagonal, symmetric positive-definite.

Summary

	LDNH$_0$	LDNH$_2$	LIN-NH$_0$	LIN-NH$_1$	LIN-NH$_2$
u_α	$\mathbb{P}_0$	$\mathbb{P}_0$	$\mathbb{P}_1$	$\mathbb{P}_1$	$\mathbb{P}_1$
w_α	$\mathbb{P}_0$	$\mathbb{P}_1$	$\mathbb{P}_0$	$\mathbb{P}_1$	$\mathbb{P}_2$
q_α	$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_1$	$\mathbb{P}_2$	$\mathbb{P}_3$

Dispersion relations

Let us linearise around the so-called lake-at-rest steady state $(H_0, 0, 0, 0)$.

Proposition

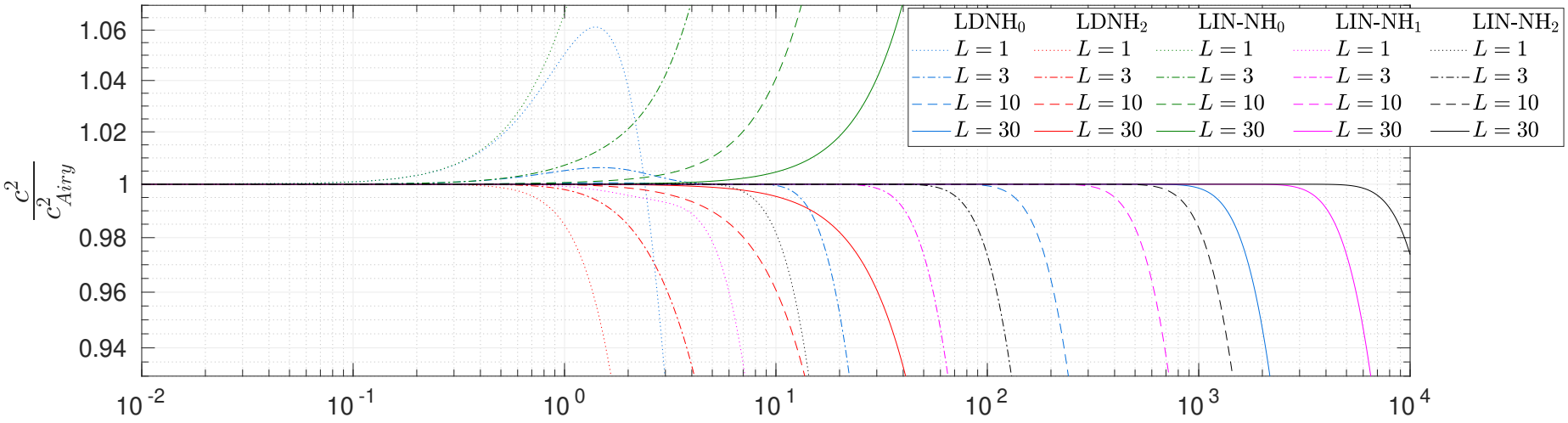
There exists a plane wave solution $\left(\hat{H}, \hat{u}_\alpha, \hat{w}_\alpha, \hat{q}_\alpha\right) e^{i(kx-\omega t)}$ *to the linearised LDNH system provided the following dispersion relation holds*

$$c_L^2 = \frac{\omega^2}{k^2 g H_0} = \frac{\mathcal{P}_L(kH_0)}{\mathcal{Q}_L(kH_0)}$$

where $\mathcal{P}_L$ *and* $\mathcal{Q}_L$ *are resp. polynomials of degree* $2(L-1)$ *and* $2L$.

Moreover, we can show that c_L^2 *goes to* $\frac{\tanh(kH_0)}{kH_0}$ *when* $L \to +\infty$.

Phase velocity



Conclusion

- Derivation of multilayer non-hydrostatic models as semi-discretisations of the Euler equations
- Analysis of physical properties (energy, dispersive effects)
- Numerical strategies and assessments

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C. Escalante-Sánchez, E.D. Fernández-Nieto, T. Morales de Luna, Y. Penel, and J. Sainte-Marie, *Numerical simulations of a dispersive model approximating free-surface Euler equations* (**J. Sci. Comput.**, 89(55), 2021).

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Thank you for your attention