On Optimal Transport, variational Mean Field Games and beyond

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Overview

1. Optimal Transport

   The three formulations of quadratic Optimal Transport

   On geodesics

2. The Schrödinger problem

   The three formulations of Schrödinger

3. Variational Mean Field Games

   Eulerian and Lagrangian formulation for MFG with quadratic Hamiltonian

4. Towards a numerical method:

Optimal Transport

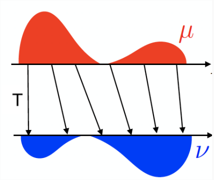
Let $\mu, \nu \in \mathcal{P}(\Omega)$, Ω compact subset ofR^n, the Optimal Transport (OT) problem is defined as follows

$$(\mathcal{MK}) \quad E_c(\mu,\nu) = \inf\{\mathcal{E}_c(\gamma) \mid \gamma \in \Pi(\mu,\nu)\} \tag{1}$$

where $\Pi(\mu,\nu) := \{\gamma \in \mathcal{P}(\Omega^2) | \quad \pi_{1,\sharp}\gamma = \mu,\ \pi_{2,\sharp}\gamma = \nu\}$ and

$$\mathcal{E}_c(\gamma) := \int c(x,y)d\gamma(x,y).$$

Solution à la Monge: the transport plan γ is deterministic (or à la Monge) if $\gamma = (Id, T)_\sharp \mu$ where $T_\sharp \mu = \nu$.



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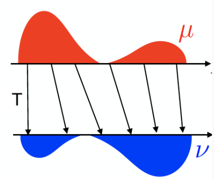

The $(\mathcal{MK})$ problem admits a dual formulation:

$$\sup \{\mathcal{J}(\phi,\psi) \mid (\phi,\psi) \in \mathcal{K}\}. \tag{2}$$

where

$$\mathcal{J}(\phi,\psi) := \int_\Omega \phi d\mu(x) + \int_\Omega \psi d\nu(y)$$

and $\mathcal{K}$ is the set of bounded and continuous functions ϕ, ψ such that $\phi(x) + \psi(y) \leq c(x,y)$.

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$$\inf \int_0^1 \int_\Omega \frac{1}{2}|v_t|^2 \rho_t dxdt \quad s.t.\ \partial_t \rho_t + \mathrm{div}(\rho_t v_t) = 0$$

$$\rho(0,\cdot) = \mu,\ \rho(1,\cdot) = \nu$$

And its "dual"

$$\sup \left\{ \int_\Omega \varphi(1,x) d\nu - \int \varphi(0,x) d\mu \;|\partial_t \varphi + \frac{1}{2}|\nabla \varphi|^2 \leq 0 \right\}$$

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The dynamic (Lagrangian) ($C = H^1([0,1];\Omega)$ and $e_t : [0,1]\to\Omega$)

$$\inf\left\{\int_C\int_0^1 \frac{1}{2}|\dot\omega|^2 dtdQ(\omega) \mid Q\in\mathcal{P}(C),\ (e_0)_\sharp Q=\mu,\ (e_1)_\sharp Q=\nu\right\}$$

On geodesics

- Quadratic optimal transport is indeed a distance between probability measures, aka the $\mathcal{W}_2^2$ Wasserstein distance, and $(\mathcal{P}(\Omega), \mathcal{W}_2)$ is a metric space;
  - Gives a way to **compare** and **interpolate** between probability measures.
  - Consider the optimal solutions for the three formulations $\gamma^\star, Q^\star, \rho_t^\star$ then

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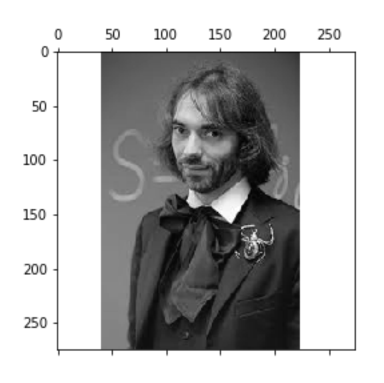

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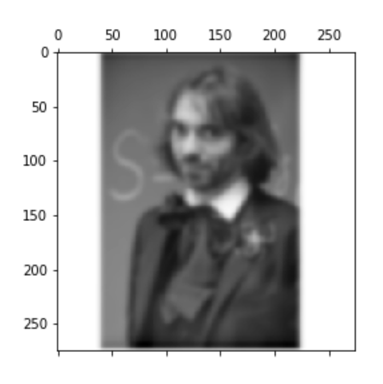

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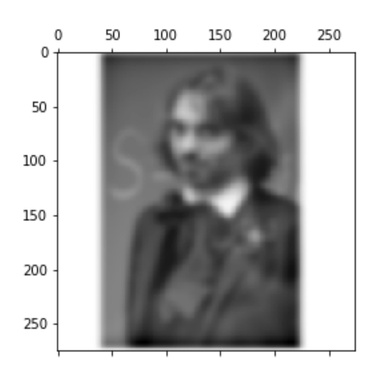

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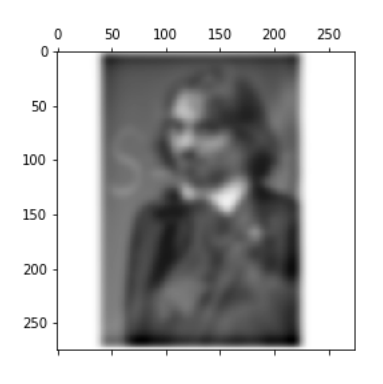

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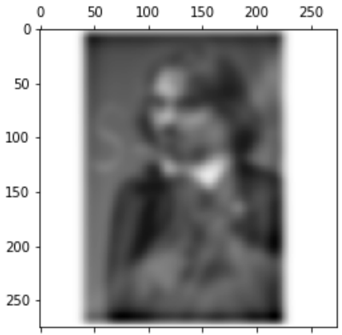

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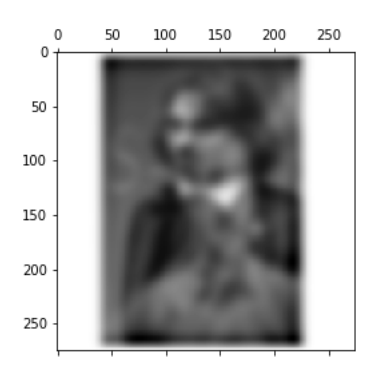

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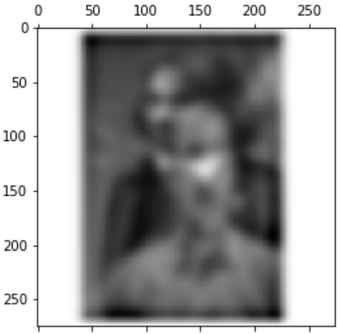

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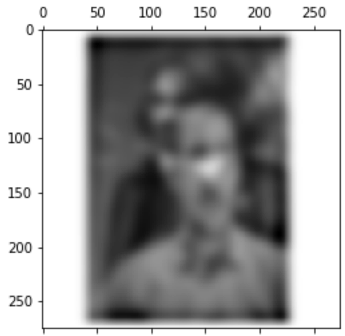

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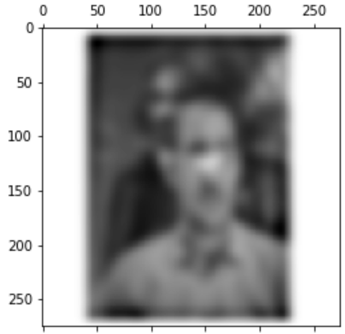

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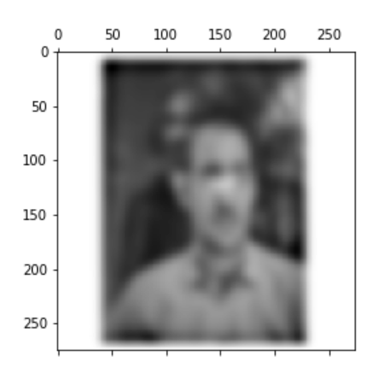

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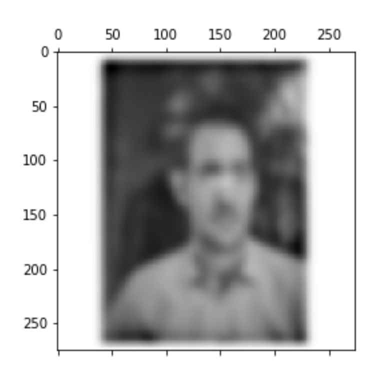

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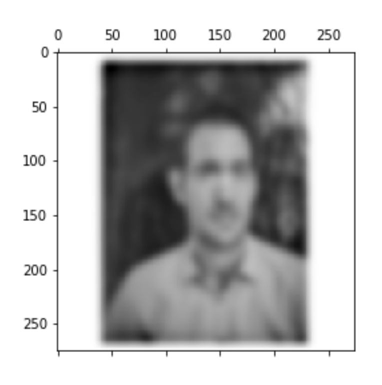

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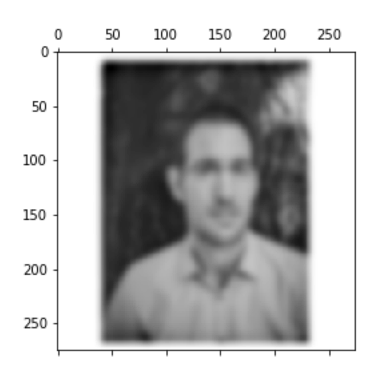

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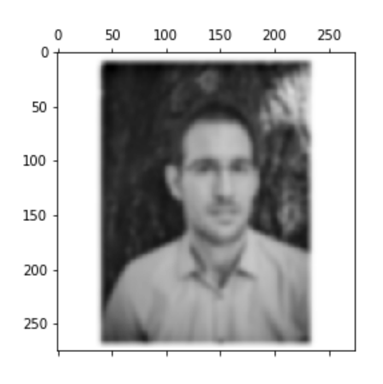

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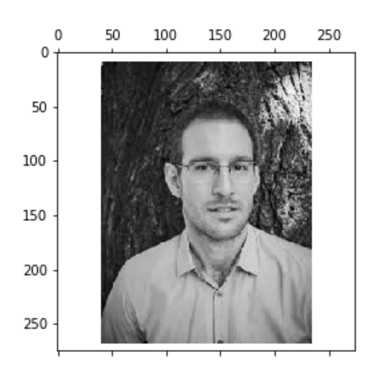

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$\Rightarrow$ **The Schrödinger problem**

Definition (Relative entropy)
Let ρ and π probability measures on Ω then the relative entropy is defined as

$$\mathcal{H}(\rho|\pi) = \begin{cases} \int_{\Omega\times\Omega} \left(\log\left(\frac{d\rho(x,y)}{d\pi(x,y)}\right) - 1\right) d\rho(x,y), & \text{if } \rho \ll \pi \\ +\infty, & \text{otherwise,} \end{cases}$$

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Remark: static formulation can be defined for a general cost function $c(x,y)$.

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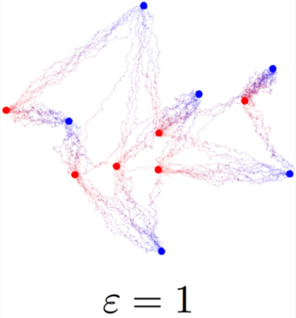
- It is easy to solve numerically: Sinkhorn algorithm (we will see it in 15 minutes...if I do not talk too much).

Convergence to optimal transport solution for quadratic cost as $\varepsilon \to 0$



The “bridge” between quadratic Monge-Kantorovich and Schrödinger

- Schrödinger pb is a **strictly convex pb** and it admits a unique almost explicit solution $\gamma^\star = a(x)b(y)e^{-c/\varepsilon}$.
- As for OT the solutions of the three formulation are related. In particular

$$Q^\star = R^{x,y,\varepsilon} \otimes \gamma^\star,$$

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- As $\varepsilon \to 0$ Schrödinger $\Gamma-$converges to optimal transport **(Léonard 2012; Guillaume Carlier, Duval, Peyré, and Bernhard Schmitzer 2017)**. This implies that minimizers converge too!
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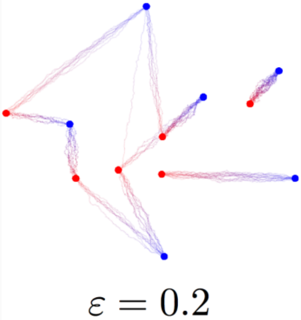

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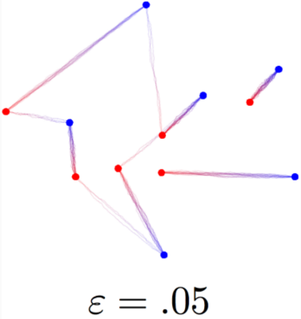

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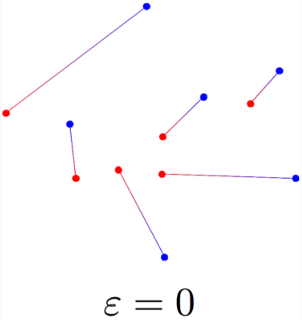

Variational Mean Field Games

Lagrangian formulation for 1st order MFG

Consider a first order MFG system then we have the following "equivalence" (see **(Lasry and Lions 2007)**)

A MFG system

$$\begin{cases} -\partial_t \varphi + \frac{1}{2}|\nabla \varphi|^2 = g(x, \rho), \ \varphi(1, x) = \Psi(x) \\ \partial_t \rho - \mathrm{div}(\rho \nabla \varphi) = 0, \ \rho(0, \cdot) = \rho_0. \end{cases}$$

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where G is the anti-derivative of g w.r.t its second variable and $F(\rho_1) = \int_\Omega \Psi d\rho_1$ is a final cost.

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where $\mathcal{G}(\rho) = \int G(x,\rho)dx$ if $\rho \ll \mathcal{L}$ and $+\infty$ otherwise.

A Lagrangian formulation via Entropy minimization

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Towards a numerical method:

The discretised (in time) problems

We can solve the Lagrangian problems by firstly discretising them in time as follows

Regularized 1st order MFG

$$\inf \int_{\Omega^{N+1}} K_N dQ_N(x_0, \cdots, x_N) + \varepsilon \mathcal{H}(Q_N|\mathcal{L}) + \sum_{i=1}^{N-1} \int_{\Omega} G(x, \pi_{i,\sharp} Q_N) dx_i + F(\pi_{N,\sharp} Q_N)$$

$$\text{s.t. } Q_N \in \mathcal{P}(\Omega^{N+1}),\ \pi_{0,\sharp} Q_N = \rho_0,$$

where $K_N = \frac{1}{2N} \sum_{i=0}^{N-1} |x_{i+1} - x_i|^2$ and $\pi_i : \Omega^{N+1} \to \Omega$ is the canonical projection.

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2nd order MFG

$$\inf \left\{ \mathcal{H}(Q_N|R_N^\varepsilon) + \sum_{i=1}^{T-1} \int_\Omega G(x, \pi_{i,\sharp} Q_N) dx_i + F(\pi_{N,\sharp} Q_N) \mid \pi_{0,\sharp} Q_N = \rho_0 \right\},$$

where $R_N^\varepsilon \overset{def}{=} \prod_{n=0}^N \xi_{n,n+1}$ and $\xi_{ij} = \exp^{-\dfrac{|x_i - x_j|^2}{2N\varepsilon}}$.

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Remarks: (i) for small ε the regularized 1st MFG approximate the unreg pb (ii) both problems can be re-written in the same way.

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IDEA: an alternate coordinate ascent algorithm (equivalent to a generalised Sinkhorn)

The dual problem

The Lagrangian problem can be re-written as the following dual optimization problem:

$$\sup_{(\phi_0,\cdots,\phi_N)} -\tilde{F}^\star(-\phi_0) - \frac{1}{N}\sum_{k=1}^{N-1} G^\star(-\phi_k) - F^\star(-\phi_N) - \int \left(\exp(\oplus_{k=0}^{N}\phi_k) - 1\right) R_N^\epsilon,$$

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Proposition ((J.-D. Benamou, G. Carlier, S. Di Marino, and L. Nenna 2018))

Strong duality holds, namely $\sup = \inf$.
Moreover, denoting by $\phi_k^\star$ *and* $Q_N^\star$ *the optimal solutions to the dual and primal problem respectively, it follows that the unique solution to the primal has the form*

$$Q_N^\star(x_0,\cdots,x_N) := \left(\otimes_{k=0}^N e^{\phi_k^\star(x_k)}\right) R_N^\varepsilon(x_0,\cdots,x_N). \tag{3}$$

A coordinate ascent algorithm (or generalised Sinkhorn)

Generalizing a result of **(Peyré 2015; Chizat, Peyré, B. Schmitzer, and Vialard 2016)**, we get the iterative method computing a sequence of potentials (denoted with the superscripts $.^{(n)}$) :

$$\phi_0^{(n)} := \operatorname{argmax}_\phi - \tilde{F}^\star(-\phi) - \int \exp(\phi) I_k^\phi \mathrm{d}x_1 \cdots \mathrm{d}x_N,$$

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Remarks:

- For many interesting energies F and G, the relaxed maximizations can be computed point-wise in space and analytically;
- with the same method one can compute dynamic optimal transport by imposing $G = 0$ and $F = i_{\rho_1}$

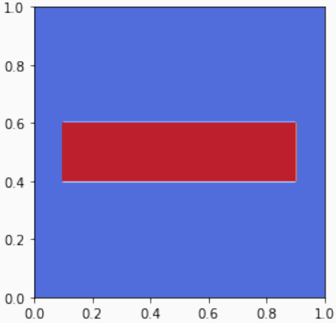
Sinkhornizing the world!!

- Wasserstein Barycenter **(Jean-David Benamou, Guillaume Carlier, Cuturi, Luca Nenna, and Peyré 2015)**;
- Matching for teams **(Luca Nenna 2016)**;
- Optimal transport with capacity constraint **(Jean-David Benamou, Guillaume Carlier, Cuturi, Luca Nenna, and Peyré 2015)**;
- Partial Optimal Transport **(Jean-David Benamou, Guillaume Carlier, Cuturi, Luca Nenna, and Peyré 2015; Chizat, Peyré, B. Schmitzer, and Vialard 2016)**;
- Multi-Marginal Optimal Transport **(Luca Nenna 2016; J.-D. Benamou, G. Carlier, and L. Nenna 2016; Jean-David Benamou, Guillaume Carlier, and Luca Nenna 2018; Jean-David Benamou, Guillaume Carlier, Cuturi, Luca Nenna, and Peyré 2015)**;
- Wasserstein Gradient Flows (JKO) **(Peyré 2015)**;
- Unbalanced Optimal Transport **(Chizat, Peyré, B. Schmitzer, and Vialard 2016)**;
- Cournot-Nash equilibria **(Blanchet, Guillaume Carlier, and Luca Nenna 2017)**
- Mean Field Games **(J.-D. Benamou, G. Carlier, S. Di Marino, and L. Nenna 2018)**;
- Grand Canonical Optimal transport **(Simone Di Marino, Lewin, and Luca Nenna 2022)**;
- and more...

Dynamic OT

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0,1]^2$ with $N \times N$ points $N = 250$;
- Given final density;



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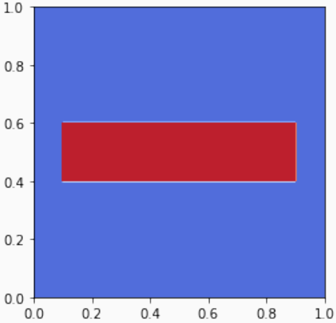

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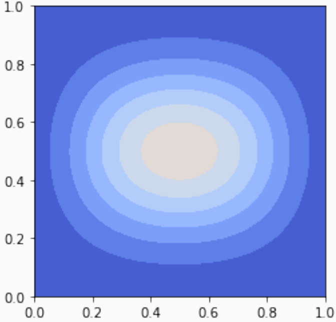

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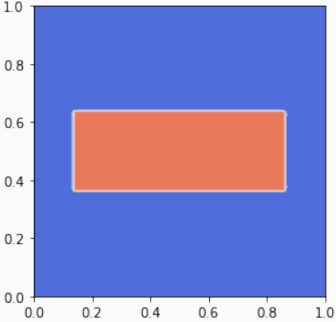

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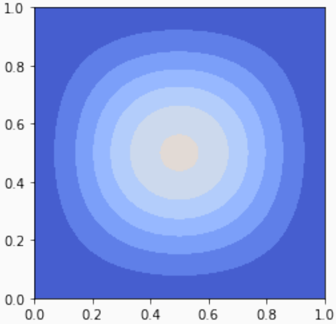

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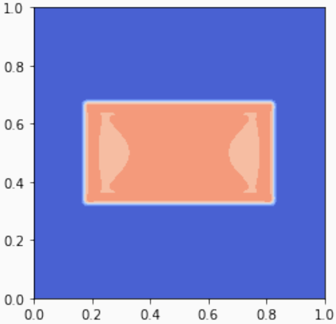

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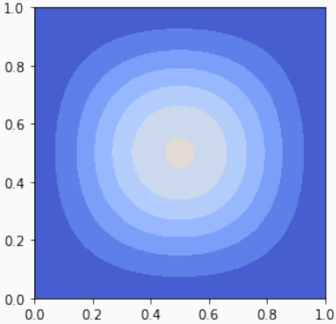

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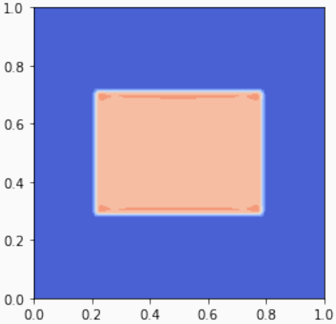

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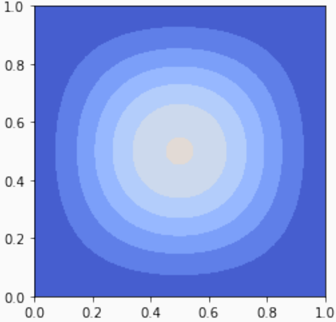

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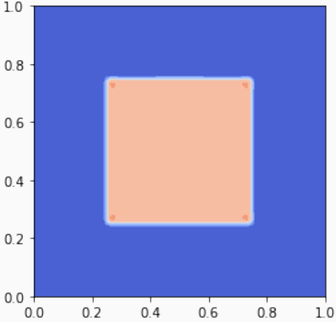

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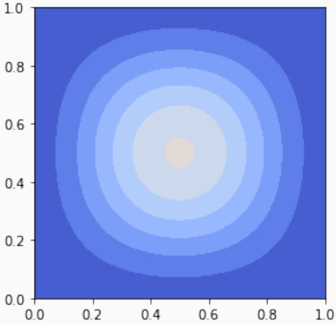

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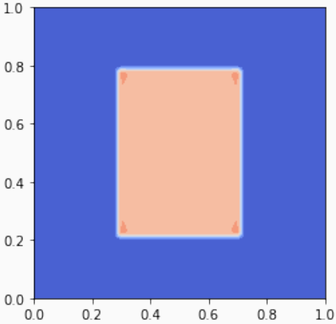

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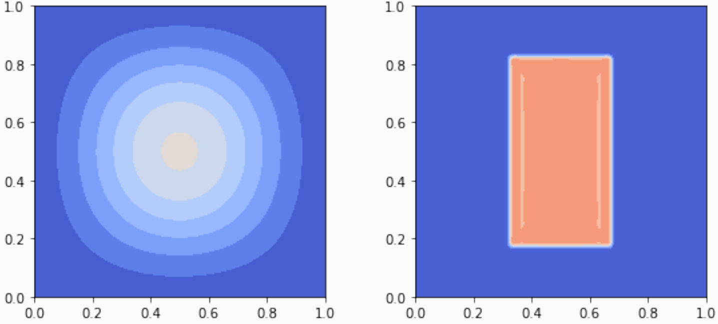
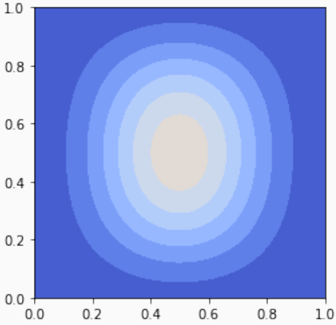


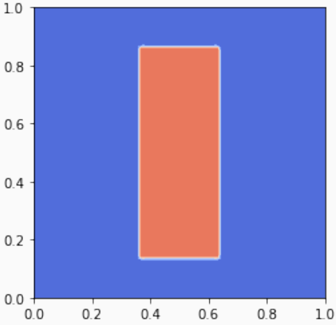
$\varepsilon = 1$ $\varepsilon = 0.0005$

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0, 1]^2$ with $N \times N$ points $N = 250$;
- Given final density;



$\varepsilon = 1$

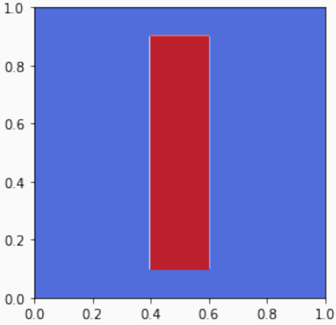


$\varepsilon = 0.0005$

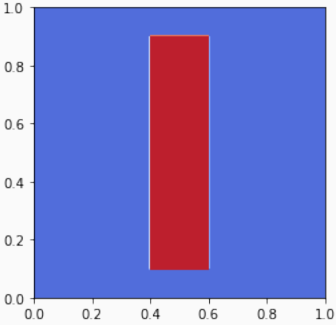
Dynamic OT

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0,1]^2$ with $N \times N$ points $N = 250$;
- Given final density;



$\varepsilon = 1$

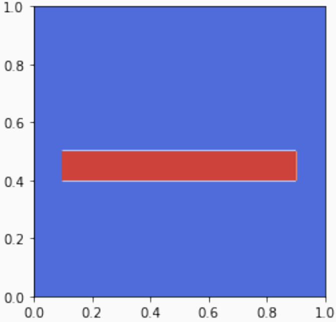


$\varepsilon = 0.0005$

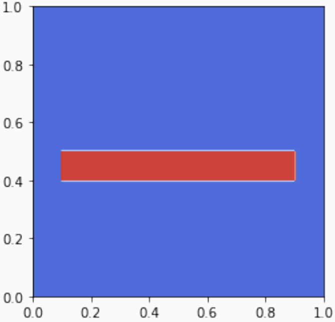
Planning MFG with obstacles on the torus, behaviour as $\varepsilon \to 0$

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0, 1]^2$ with $N \times N$ points $N = 250$;
- Given final density + obstacles;



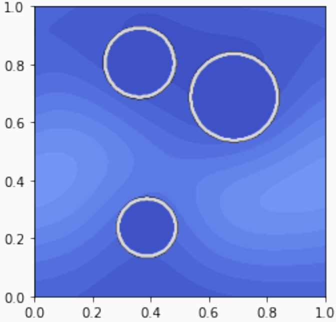
$\varepsilon = 1$



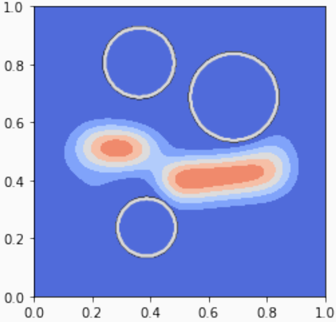
$\varepsilon = 0.01$

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0, 1]^2$ with $N \times N$ points $N = 250$;
- Given final density + obstacles;



$\varepsilon = 1$

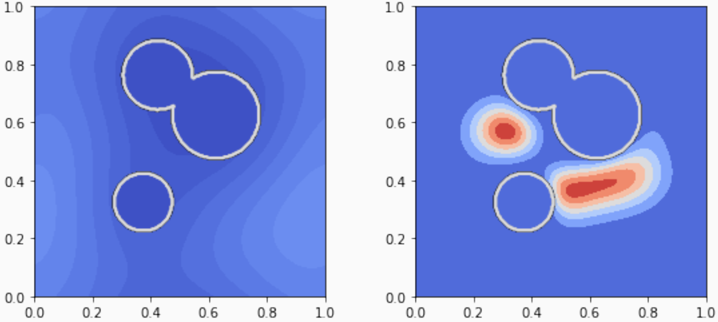


$\varepsilon = 0.01$

Planning MFG with obstacles on the torus, behaviour as $\varepsilon \to 0$

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0, 1]^2$ with $N \times N$ points $N = 250$;
- Given final density + obstacles;

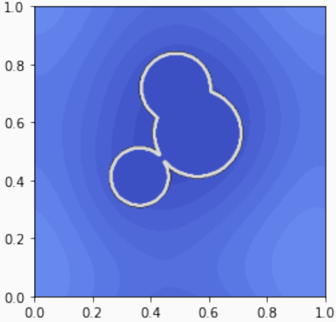


$\varepsilon = 1$ $\varepsilon = 0.01$

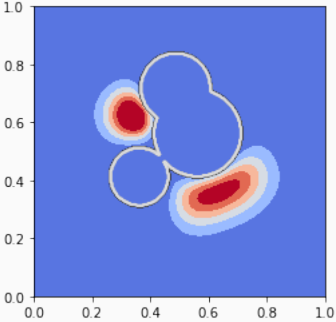
Planning MFG with obstacles on the torus, behaviour as $\varepsilon \to 0$

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0,1]^2$ with $N \times N$ points $N = 250$;
- Given final density + obstacles;



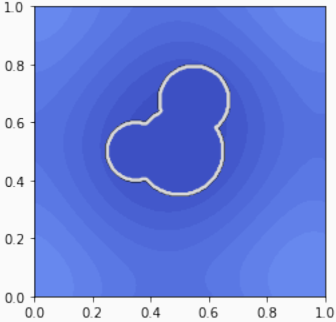
$\varepsilon = 1$



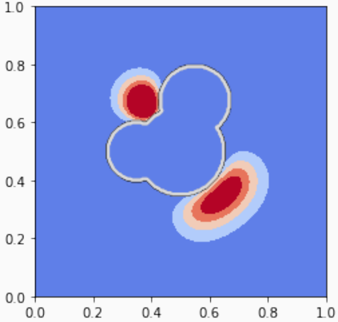
$\varepsilon = 0.01$

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0,1]^2$ with $N \times N$ points $N = 250$;
- Given final density + obstacles;



$\varepsilon = 1$

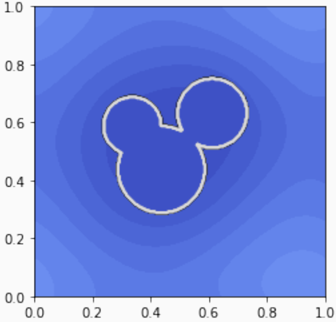


$\varepsilon = 0.01$

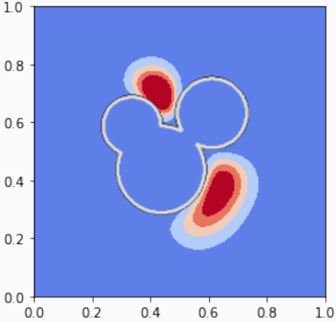
Planning MFG with obstacles on the torus, behaviour as $\varepsilon \to 0$

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0,1]^2$ with $N \times N$ points $N = 250$;
- Given final density + obstacles;



$\varepsilon = 1$

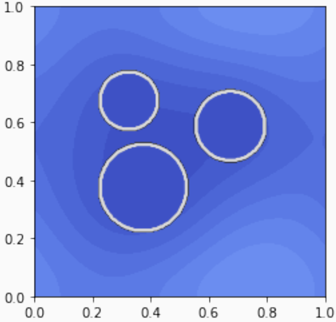


$\varepsilon = 0.01$

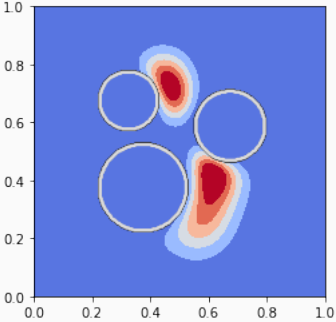
Planning MFG with obstacles on the torus, behaviour as $\varepsilon \to 0$

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0, 1]^2$ with $N \times N$ points $N = 250$;
- Given final density + obstacles;



$\varepsilon = 1$

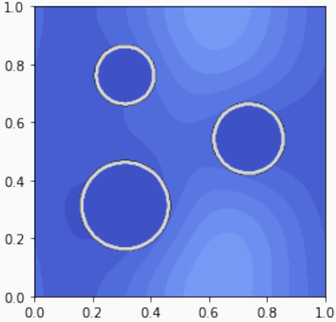


$\varepsilon = 0.01$

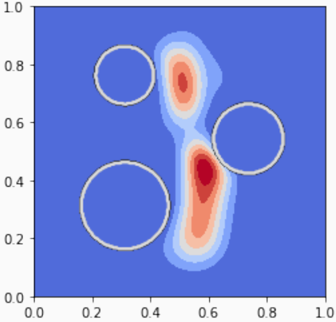
Planning MFG with obstacles on the torus, behaviour as $\varepsilon \to 0$

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0,1]^2$ with $N \times N$ points $N = 250$;
- Given final density + obstacles;



$\varepsilon = 1$

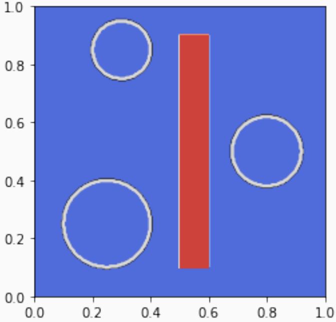


$\varepsilon = 0.01$

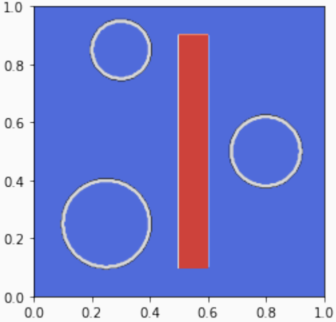
Planning MFG with obstacles on the torus, behaviour as $\varepsilon \to 0$

Data:

- $T = 32$ time steps;
- grid: uniform discretization of $[0,1]^2$ with $N \times N$ points $N = 250$;
- Given final density + obstacles;



$\varepsilon = 1$



$\varepsilon = 0.01$

Thank you!