Numerical methods and uncertainty quantification for kinetic equations

Lecture 5: Monte Carlo methods and Uncertainty quantification

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Outline

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Introduction

We shall present here the motivation and a general description of a method dealing with a class of problems in mathematical physics. The method is, essentially, a statistical approach to the study of differential equations, or more generally, of integro-differential equations that occur in various branches of the natural sciences.

(N.Metropolis, S.Ulam, "The Monte Carlo method", J. Am. Stat. Ass., 1949.)

Monte Carlo methods

- Nowadays Monte Carlo methods find application in a wide field of areas, including many subfields of physics, like statistical physics or high energy physics, and ranging to areas like biology, chemistry, finance, computer graphics and video games.
- Classical mathematical applications of Monte Carlo involves the computation of multidimensional integrals, the solution of partial differential equations, Markov chains and optimization problems.
- Monte Carlo methods are often used when other methods fail, since they are much less sensitive to the course of dimensionality, which plagues deterministic methods in problems with a large number of variables.
- Despite the widespread use of the methods, and numerous descriptions of them in articles and monographs[1], it is virtually impossible to find a unique notion of Monte Carlo method in the literature and the term Monte Carlo is often used to denote any numerical technique based on some kind of stochastic simulation.

[1] J.M.Hammersley, D.C.Handscomb, Monte Carlo Methods, 1964. N.Madras, Lectures on Monte Carlo methods, 2002.

Monte Carlo integration

Consider the simple integral

$$I[f] = \int_{[0,1]^d} f(x)dx, \quad d \geq 1,$$

if x is a random vector uniformly distributed in $[0,1]^d$ we have $I[f] = E[f(x)]$, where $E[\cdot]$ denotes the *expectation*. If $\{x_n\}$ is a sequence of pseudo-random vectors uniform in $[0,1]^d$ then

$$I_N[f] = \frac{1}{N}\sum_{n=1}^{N} f(x_n), \quad E[I_N[f]] = I[f].$$

For the law of large numbers it converges in probability[2]

$$\lim_{N\to\infty} I_N[f] = I[f],$$

and

$$I[f] - I_N[f] \approx \sigma_f N^{-1/2} w, \quad E[(I[f] - I_N[f])^2] = \sigma_f N^{-1/2},$$

where σ_f^2 is the variance of f and w is a normal random variable. Note that there is no dependence on the dimension.
Remark: The convergence rate for a deterministic grid based quadrature is $O(N^{-k/d})$ for an order k method. Thus Monte Carlo is "better" if $k/d \leq 1/2$.

[2] W.Feller '71, R.E.Caflisch '98

Reconstruction

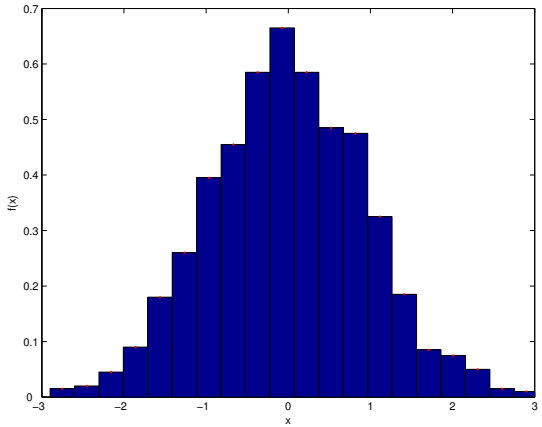
Given a set of N samples $\xi_1, \xi_2, \ldots, \xi_N$ the probability density is defined by

$$f(x) = \frac{1}{N} \sum_{k=1}^{N} \delta(x - \xi_k).$$

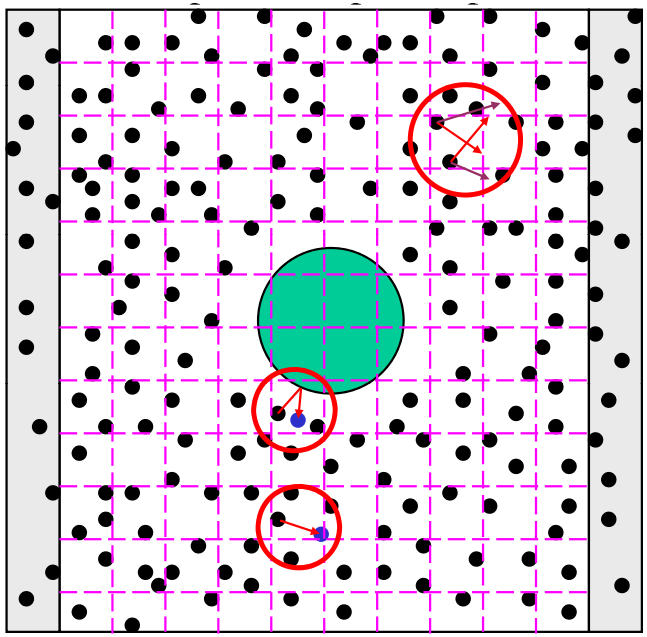
The simplest method, which produces a piecewise constant reconstruction, is based on evaluating the histogram of the samples at the cell centers of a grid

$$f(x_{j+1/2}) = \frac{1}{N} \sum_{k=1}^{N} \Psi(\xi_k - x_{j+1/2}), \quad j = \ldots, -2, -1, 0, 1, 2, \ldots$$

where $\Psi(x) = 1/\Delta x$ if $|x| \leq \Delta x/2$ and $\Psi(x) = 0$ elsewhere.



DSMC basics



- Initialize system with particles (x_i, v_i), $i = 1, \ldots, N$ (*sampling*).
- Loop over time steps of size Δt.
- Create particles at open boundaries.
- Move all the particles $x_i = x_i + v_i \Delta t$ (*transport step*).
- Process any interactions of particle and boundaries (*Maxwell's b.c.*).
- Sort particles into cells.
- Select and execute random collisions (*collision step*).
- Compute average statistical values.

BGK equations

We discuss a Monte Carlo method for the case of kinetic equations of relaxation type

$$\begin{cases} \dfrac{\partial f}{\partial t} + v \cdot \nabla_x f &= \dfrac{1}{\varepsilon}(M[f] - f), \quad v, x \in \mathbb{R}^d, \quad t > 0 \\ f(x, v, 0) &= f_0(x, v), \end{cases}$$

where $f = f(x, v, t) \geq 0$ is the number density of particles having velocity v in position x at time t and $M[f]$ is the equilibrium Maxwellian function, namely a multivariate Gaussian with the same first three moments of f

$$\int_{\mathbb{R}^d} f(x, v, t)\phi(v)\, dv = \int_{\mathbb{R}^d} M[f](x, v, t)\phi(v)\, dv, \quad \phi(v) = 1, v, |v|^2.$$

Here $\varepsilon > 0$ is the relaxation time. In particular as $\varepsilon \to 0$ we have $f = M[f]$ and thus the solution depends only on the evolution of the first three moments of f which satisfy the compressible Euler equations.

The solution is approximated by means of an operator splitting based on alternating the solution of the convection

$$\begin{cases} \dfrac{\partial f^*}{\partial t} + v\nabla_x f^* &= 0 \\ f^*(x,v,0) &= f_0(x,v) \end{cases}$$

and homogeneous relaxation steps

$$\begin{cases} \dfrac{\partial f}{\partial t} &= \dfrac{1}{\varepsilon}(M[f]-f) \\ f(x,v,0) &= f^*(x,v,t) \end{cases}$$

Note that both steps can be again solved exactly and yield the approximation

$$\begin{aligned} f^*(x,v,t) &= f_0(x-vt), \\ f(x,v,t) &= e^{-t/\varepsilon} f^*(x,v,t) + (1-e^{-t/\varepsilon}) M[f^*](x,v,t). \end{aligned}$$

A Monte Carlo method is then derived by direct sampling from the above solutions of the two splitting steps.

Monte Carlo approximation

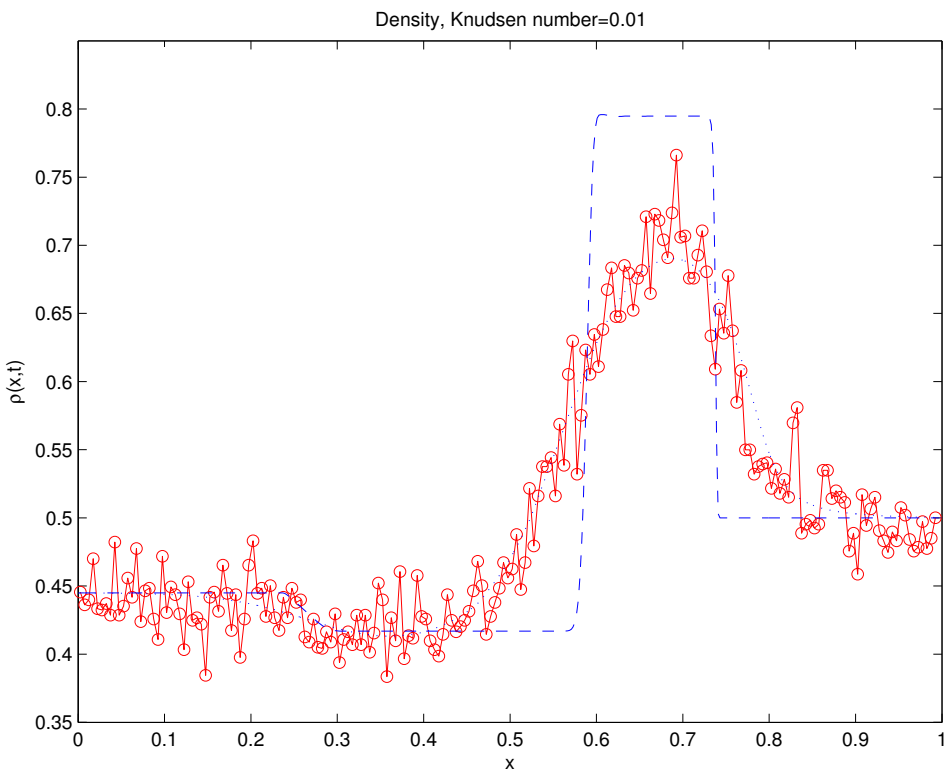
Let us assume f_0 a multivariate probability density. Given a set of samples $(\xi_1^0, v_1^0), \dots, (\xi_N^0, v_N^0)$, where $\xi_i, v_i \in \mathbb{R}^d$ we can sample directly from the exact solutions of the operator splitting steps.
A new set of samples $(\xi_1, v_1), \dots, (\xi_N, v_N)$ is obtained as follows

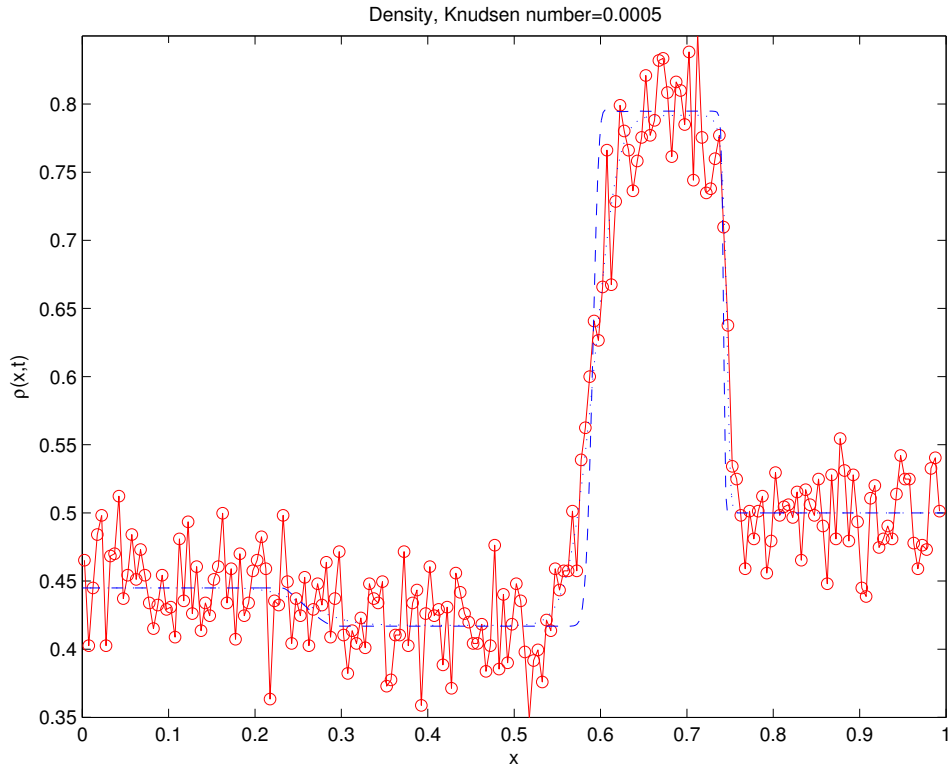
1. First compute $\xi_i = \xi_i^0 + v_i^0 t, \quad i = 1, \dots, N$
2. On a space grid of M points reconstruct the first three moments

$$\int_{\mathbb{R}^d} f^*(x_j, v, t)\phi(v)\, dv, \quad \phi(v) = 1, v, |v|^2 \quad j = 1, \dots, M.$$

3. In each space cell j given a sample (ξ_i, v_i^0)
   - with probability $1 - e^{-t/\varepsilon}$ replace v_i^0 with a velocity sample from $M[f^*]$ (for example using Box-Muller algorithm),
4. otherwise set $v_i = v_i^0$.

Lax shock tube





Solution at $t = 0.05$ for Lax shock tube, $M = 200$ and 500 particles per cell with $\Delta t = 0.1$. Left: density for $\varepsilon = 0.01$. Right: density for $\varepsilon = 0.0005$.

The kinetic model

In the Boltzmann description of RGD, the density $f = f(x, v, t)$ of particles follows the equation

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f), \quad x \in \Omega \subset \mathbb{R}^3, v \in \mathbb{R}^3,$$

The parameter $\varepsilon > 0$ is called *Knudsen number* and it is proportional to the mean free path between collisions. The bilinear *collisional operator* $Q(f, f)$ is given by

$$Q(f, f)(v) = \int_{\mathbb{R}^3} \int_{S^2} B(|v - v_*|, \omega)(f(v')f(v'_*) - f(v)f(v_*))dv_* d\omega,$$

where ω is a vector of the unitary sphere $S^2 \subset \mathbb{R}^3$ and for simplicity the dependence of f on x and t has been omitted.
The *collisional velocities* (v', v'_*) are given by the relations

$$v' = \frac{1}{2}(v + v_* + |q|\omega), \quad v'_* = \frac{1}{2}(v + v_* + |q|\omega),$$

where $q = v - v_*$ is the relative velocity.

DSMC for the collision step

- We will describe the *classical DSMC methods* due to Nanbu in the case of spatially homogeneous Boltzmann equations[3].
- We assume that the kinetic equations can be written in the form

$$\frac{\partial f}{\partial t} = \frac{1}{\varepsilon}[P(f,f) - \mu f],$$

where $\mu > 0$ is a constant and $P(f,f)$ is a non negative bilinear operator s.t.

$$\frac{1}{\mu}\int_{\mathbb{R}} P(f,f)(v)\phi(v)\,dv = \int_{\mathbb{R}} f(v)\phi(v)\,dv, \quad \phi(v) = 1, v, v^2.$$

- For the BGK equation $P(f,f) = \mu M(\rho,u,T)(v)$, for the Boltzmann equation in the Maxwellian case

$$P(f,f) = Q^+(f,f)(v) = \int_{\mathbb{R}^3}\int_{S^2} b_0(\cos z) f(v') f(v'_*)\,d\omega\,dv_*,$$

and $\mu = 4\pi\rho$.

- The case of general VHS kernels is different and it will not discussed.

[3] G.Bird '63, K.Nanbu '83

Nanbu's method (DSMC no time counter)

- We assume that f is a probability density, i.e. $\rho = \int_{-\infty}^{+\infty} f(v,t)\, dv = 1$.
- Consider a time interval $[0, t_{\max}]$, and discretize it in n_{TOT} intervals of size Δt.
- Let $f^n(v)$ be an approximation of $f(v, n\Delta t)$. The forward Euler scheme writes

$$f^{n+1} = \left(1 - \frac{\mu \Delta t}{\epsilon}\right) f^n + \frac{\mu \Delta t}{\epsilon} \frac{P(f^n, f^n)}{\mu}.$$

- Clearly if f^n is a probability density both $P(f^n, f^n)/\mu$ and f^{n+1} are probability densities. Thus the equation has the following probabilistic interpretation.
- *Physical level*: a particle with velocity v_i will not collide with probability $(1 - \mu\Delta t/\epsilon)$, and it will collide with probability $\mu\Delta t/\epsilon$, according to the collision law described by $P(f^n, f^n)(v)$.
- *Monte Carlo level*: to sample v_i from f^{n+1} with probability $(1 - \mu\Delta t/\epsilon)$ we sample from f^n, and with probability $\mu\Delta t/\epsilon$ we sample from $P(f^n, f^n)(v)/\mu$.

Note that $\Delta t \leq \epsilon/\mu$ to have the probabilistic interpretation. For the BGK model the algorithm is straightforward since sampling from $P(f, f)/\mu$ is simply sampling from a Maxwellian.

Maxwellian case

We consider the case where the collision kernel does not depend on the relative velocity.

Algorithm[Nanbu for Maxwell molecules]:

1. *compute the initial velocity of the particles,* $\{v_i^0, i = 1, \dots, N\}$, *by sampling them from the initial density* $f_0(v)$
2. for $n = 1$ to n_{TOT}
   for $i = 1$ to N
   with probability $1 - \mu\Delta t/\epsilon$
   - *set* $v_i^{n+1} = v_i^n$

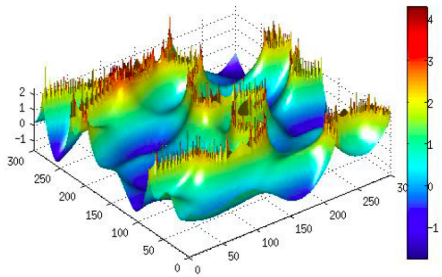
   with probability $\mu\Delta t/\epsilon$
   - *select a random particle* j
   - *compute* v_i' *by performing the collision between particle* i *and particle* j
   - *assign* $v_i^{n+1} = v_i'$

   end for
   end for

Nanbu's algorithm is not conservative, i.e. momentum and energy are conserved only in the mean, but not at each collision. A conservative algorithm is obtained selecting independent particle pairs, instead of single particles.

Outline

Uncertainty in kinetic equations



Kinetic equations play a major rule in modeling large interacting particles systems. *Uncertainty* may be due to various reasons, like lack of knowledge on the microscopic interactions or incomplete informations at the boundaries.

- Recently, kinetic equations have found new applications in socio-economy and life-sciences[4]. In all these emerging fields the derivation from first principles is not possible and the models are based on *empirical observations*.
- The development of numerical methods presents several difficulties due to the high dimensionality and the intrinsic structural properties of the solution[5]. These aspects are even more challenging in presence of uncertainties which contribute to *the curse of dimensionality*.

[4] J.A. Carrillo, M. Fornasier, G. Toscani, F. Vecil,'10; S.-Y. Ha, E. Tadmor '08; P. Degond, S. Motsch '07; L.P., G. Toscani '13

[5] G.D., L.Pareschi, M.Zanella '17; J.Hu, S.Jin, R. Shu '16,'17,'19; L.Liu, S.Jin '17; Y.Zhu, S.Jin '17; S.Jin, H.Lu '17; J.A. Carrillo, L.Pareschi, M. Zanella '18; J. Hu, E. Daus, J. Shi, L. Liu '21; L.Pareschi, W. Yubo '21 ...

Kinetic equations with uncertain parameters

We consider kinetic equations with uncertainties of the general form

$$\partial_t f + v \cdot \nabla_x f = \frac{1}{\varepsilon} Q(f, f),$$

where $f = f(z, x, v, t)$, $t \geq 0$, $x \in \mathcal{D} \subset \mathbb{R}^{d_x}$, $v \in \mathbb{R}^{d_v}$, $\varepsilon > 0$ is the *Knudsen number* and $z \in \Omega \subseteq \mathbb{R}^{d_z}$ is a *r.v.*. Examples are the *Boltzmann equation*

$$Q(f, f)(z, x, v, t) = \int_{\mathbb{S}^{d_v - 1} \times \mathbb{R}^{d_v}} B(v, v_*, \omega, z)(f(v')f(v'_*)_* - f(v)f(v_*))\, dv_*\, d\omega$$

or by *Vlasov-Fokker-Planck* type models

$$Q(f, f) = \nabla_v \cdot [\mathcal{P}[f]f + \nabla_v(Df)]$$

where $\mathcal{P}[\cdot]$ is a non–local operator of the form

$$\mathcal{P}[f](z, x, v, t) = \int_{\mathbb{R}^{d_x}} \int_{\mathbb{R}^{d_v}} P(x, x_*; v, v_*, z)(v - v_*) f(z, x_*, v_*, t) dv_* dx_*,$$

and $D(z, v) \geq 0$ describes the local relevance of the diffusion.

Notations

We denote by $p(z)$ the *probability density function (PDF)* of the random input z. The *expectation* of $f(z,x,v,t)$ with respect to the random field is then defined as

$$\mathbb{E}[f](x,v,t) = \int_\Omega f(z,x,v,t)p(z)\,dz.$$

Similarly we define the other statistical quantities of interest with respect to the random field, like *variance*

$$\mathrm{Var}(f)(x,v,t) = \int_\Omega (f(z,x,v,t) - \mathbb{E}[f](x,v,t))^2 p(z)\,dz,$$

and covariance

$$\mathrm{Cov}(f,g)(x,v,t) = \int_\Omega (f(z,x,v,t) - \mathbb{E}[f](x,v,t))(g(z,x,v,t) - \mathbb{E}[g](x,v,t))p(z)\,dz.$$

Outline

Standard Monte-Carlo (MC)

We consider deterministic interaction operator $Q(f,f)$ and random initial data $f(z,x,v,0) = f_0(z,v,v)$. We assume a *deterministic* discretization in v, x and t.

Algorithm (Standard Monte Carlo method)

1. **Sampling**: *Sample M independent identically distributed (i.i.d.) initial data f_0^k, $k = 1,\dots,M$ from the random field f_0 and approximate these over the grid characterized by Δv and Δx.*
2. **Solving**: *For each realization f_0^k the underlying kinetic equation is solved by the deterministic solver. We denote the solutions at time t^n by $f^{k,n}_{\Delta v,\Delta x}$, $k = 1,\dots,M$.*
3. **Estimating**: *Estimate the expected value of the random solution field $\mathbb{E}[f^n]$ by its arithmetic average*

$$E_M[f^n_{\Delta v,\Delta x}] = \frac{1}{M}\sum_{k=1}^{M} f^{k,n}_{\Delta v,\Delta x}.$$

Computational considerations

- The only (data) *interaction* between different samples is in step 3, when ensemble averages are computed. The MC algorithm is non-intrusive and easily parallelizable.
- The typical *error estimate* that one obtains is of the type

$$\|\mathbb{E}[f](x,v,t^n) - E_M[f^n_{\Delta v,\Delta x}]\|_{L_2(\Omega;\mathcal{B})} \leq C\left(\sigma_f M^{-1/2} + \Delta v^q + \Delta x^p\right)$$

  where $\sigma_f^2 = \|\mathrm{Var}(f)\|_{\mathcal{B}}$, $C > 0$ depends on time and on the initial data, and q, p characterize the *accuracy* of the discretizations in the phase-space (time error is neglected).
- It is possible to equilibrate the discretization and the sampling errors in the a-priori estimate taking

$$M = O(\Delta x^{-2p}), \quad \Delta v = O(\Delta x^{p/q}).$$

- In order to have *comparable errors* the number of samples should be extremely large.

Micro-macro decomposition and variance reduction

The idea is to take advantage of the large time behavior of the kinetic equation.We describe the method in the space homogeneous case $f = f(z, v, t)$. We introduce the *micro–macro decomposition* [6]

$$f(z, v, t) = f^{\infty}(z, v) + g(z, v, t),$$

where $f^{\infty}(z, v)$ is the steady state solution $Q(f^{\infty}, f^{\infty}) = 0$ and $g(z, v, t)$ is s.t.

$$m_{\phi}(g) := \int_{\mathbb{R}^{d_v}} \phi(v) g(z, v, t) dv = 0, \quad \phi(v) = 1, v, |v|^2/2.$$

We consider the micro-macro decomposition applied to the *homogeneous problem*

$$\frac{\partial f}{\partial t} = Q(f, f),$$

with initial data $f(z, x, v, 0) = f_0(z, x, v)$.

[6] T.-P. Liu, S.-H. Yu '04; M. Lemou, L. Mieussens '08

Micro-macro decomposition and variance reduction

Under suitable assumptions, $f(z,v,t)$ *exponentially decays* to the equilibrium solution $f^\infty(z,v)$, therefore $g(z,v,t)$ exponentially decays to $g^\infty(z,v) \equiv 0$. We can decompose the expected value of the solution as

$$\mathbb{E}[f](v,t) = \int_\Omega f^\infty(z,v)p(z)dz + \int_\Omega g(z,v,t)p(z)dz = \mathbb{E}[f^\infty](v) + \mathbb{E}[g](v,t).$$

Using a *Monte Carlo estimator* directly on $\mathbb{E}[f]$ based on M samples we have

$$\|\mathbb{E}[f](v,t) - E_M[f](v,t)\|_{L_2(\Omega;\mathcal{B})} \simeq \sigma_f M^{-1/2}.$$

However, since $f^\infty(z,v)$ is known, we can assume that $\mathbb{E}[f^\infty](v)$ is evaluated with a negligible error and use the Monte Carlo estimator only on $\mathbb{E}[g]$ to get

$$\begin{aligned}\|\mathbb{E}[f](v,t) - \mathbb{E}[f^\infty](v) - E_M[g](v,t)\|_{L_2(\Omega;\mathcal{B})} \\ = \|\mathbb{E}[g](v,t) - E_M[g](v,t)\|_{L_2(\Omega;\mathcal{B})} \simeq \sigma_g M^{-1/2}.\end{aligned}$$

Equilibrium as control variate

Therefore, we can use this property taking f^∞ as a *control variate* of f. Given M samples $f^k(v,t)$, $k = 1, \dots, M$ we can write [7]

$$\mathbb{E}[f](v,t) \approx E_M^\lambda[f](v,t) = \frac{1}{M}\sum_{k=1}^M f^k(v,t) - \lambda\left(\frac{1}{M}\sum_{k=1}^M f^{\infty,k}(v) - \mathbf{f}^\infty(v)\right),$$

where $\mathbf{f}^\infty(v) = \mathbb{E}[f^\infty(\cdot, v)]$ or an approximation with a negligible error. $E_M^\lambda[f]$ is clearly an *unbiased estimator* for any choice of $\lambda \in \mathbb{R}$, with $E_M^0[f] = E_M[f]$ being the *standard MC estimator* and

$$E_M^1[f](v,t) = \mathbf{f}^\infty(v) + \frac{1}{M}\sum_{k=1}^M (f^k(v,t) - f^{\infty,k}(v)) = \mathbf{f}^\infty(v) + E_M[g](v,t),$$

the *micro-macro* MC estimator [8].

[7] J.M. Hammersley, D.C. Handscomb '94

[8] G. D., L. Pareschi, M. Zanella '18

Main properties

Let us consider the random variable

$$f^{\lambda}(z,v,t) = f(v,z,t) - \lambda(f^{\infty}(z,v) - \mathbf{f}^{\infty}(v)).$$

We have $\mathbb{E}[f^{\lambda}] = \mathbb{E}[f]$, $E_M^{\lambda}[f](v,t) = E_M[f^{\lambda}](v,t)$ and its variance is

$$\mathrm{Var}(f^{\lambda}) = \mathrm{Var}(f) + \lambda^2 \mathrm{Var}(f^{\infty}) - 2\lambda \mathrm{Cov}(f, f^{\infty}).$$

Proposition 2

The quantity $\lambda^* = \dfrac{\mathrm{Cov}(f, f^{\infty})}{\mathrm{Var}(f^{\infty})}$ *minimizes* $\mathrm{Var}(f^{\lambda})$ *at* (v,t) *and gives*

$$\mathrm{Var}(f^{\lambda^*}) = (1 - \rho^2_{f,f^{\infty}})\mathrm{Var}(f),$$

where $\rho_{f,f^{\infty}} \in [-1,1]$ *is the correlation coefficient of* f *and* f^{∞}*. We have*

$$\lim_{t\to\infty} \lambda^*(v,t) = 1, \qquad \lim_{t\to\infty} \mathrm{Var}(f^{\lambda^*})(v,t) = 0 \qquad \forall\, v \in \mathbb{R}^{d_v}.$$

Time dependent control variate

To improve the MC estimate we consider as *control variate* a time dependent approximation $\tilde{f}(z,v,t)$, whose evaluation is cheaper than $f(z,v,t)$, s.t. $m_\phi(\tilde{f}) = m_\phi(f)$ for some moments and that $\tilde{f} \to f^\infty$ as $t \to \infty$. For example, using the *BGK model*

$$\frac{\partial \tilde{f}}{\partial t} = (f^\infty - \tilde{f}).$$

In terms of function decomposition this would correspond to write

$$f(z,v,t) = \tilde{f}(z,v,t) + \tilde{g}(z,v,t),$$

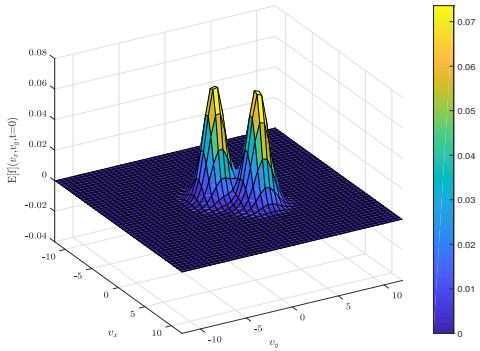
with $m_\phi(\tilde{g}) = 0$ for the same moments and $\tilde{g}(z,v,t) \to 0$ as $t \to \infty$.
The *control variate estimate* then reads

$$\mathbb{E}[f](v,t) \approx \tilde{E}_M^\lambda[f](v,t) = \frac{1}{M}\sum_{k=1}^M f^k(v,t) - \lambda\left(\frac{1}{M}\sum_{k=1}^M \tilde{f}^k(v,t) - \tilde{\mathbf{f}}(v,t)\right),$$

with $\tilde{\mathbf{f}}(v,t) = \mathbb{E}[\tilde{f}]$ or an accurate approximation, and where the optimal value for λ in terms of variance reduction is $\lambda^* = \mathrm{Cov}(f,\tilde{f})/\mathrm{Var}(\tilde{f})$.

Numerical examples: 2D homogeneous Boltzmann equation

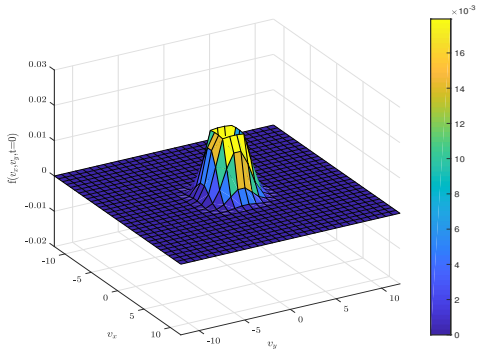
We use the fast spectral method with $N = 64$ in the velocity space[9].



Test 1: Uncertain initial data

$$f_0(z,v) = \frac{\rho_0}{2\pi}\left(\exp\left(-\frac{|v-(2+sz)|^2}{\sigma}\right)+\exp\left(-\frac{|v+(1+sz)|^2}{\sigma}\right)\right)$$

with $s = 0.2$, $\rho_0 = 0.125$, $\sigma = 0.5$ and z uniform in $[0,1]$.



Test 2: Uncertain collision kernel

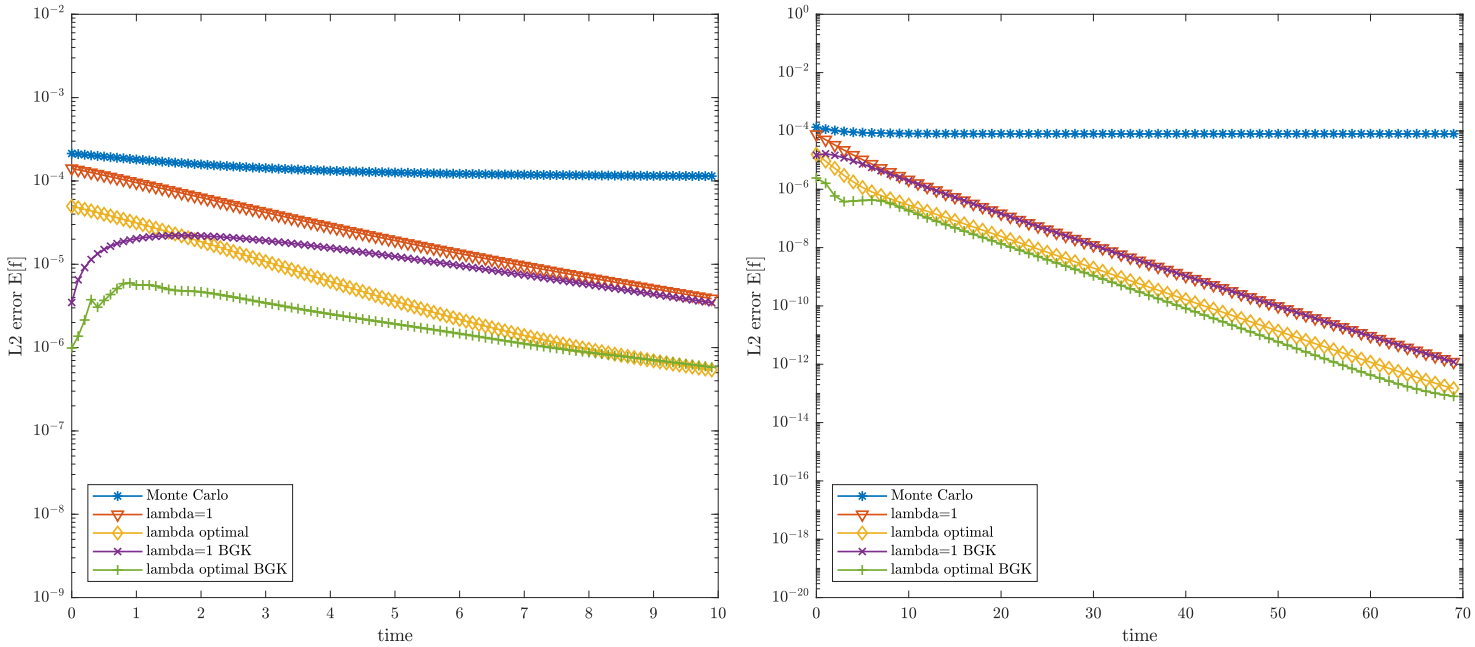
$$f_0(z,v) = \frac{1}{2\pi^2}|v|^2\exp\left(-\frac{|v|^2}{2}\right).$$

The uncertainty is in the frequency of collision $B(z) = 1 + sz$ with $s = 0.2$ and z uniform in $[0,1]$.

[9] C.Mouhot, L.Pareschi '05; F.Filbet, L.Pareschi, T. Rey '15

Test 1: error in time

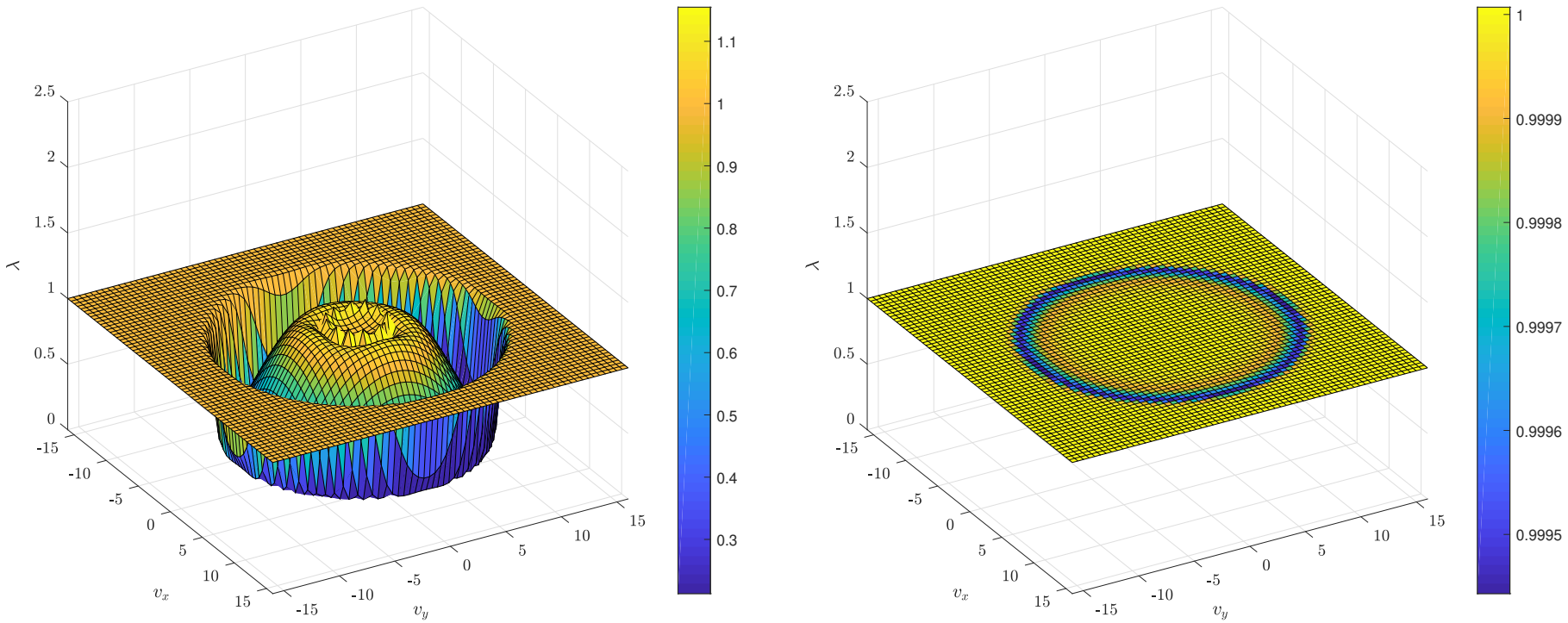
MSCV with various control variates



Evolution of the L_2 norm of the error over time with $M = 10$.

Test 1: Optimal value

MSCV with BGK



Optimal value $\lambda^*(v,t)$ at $t = 10$ and $t = 50$. MSCV with BGK as control variate.

Multiple control variates

Given $\tilde{f}_1, \dots, \tilde{f}_L$ approximations of $f(z, v, t)$ we can consider the random variable

$$f^{\lambda_1,\dots,\lambda_L}(z, v, t) = f(z, v, t) - \sum_{h=1}^{L} \lambda_h (\tilde{f}_h(z, v, t) - \tilde{\mathbf{f}}_h(v, t)),$$

where $\tilde{\mathbf{f}}_h(v, t) = \mathbb{E}[\tilde{f}_h(z, v, t)]$ and *variance* given by

$$\mathrm{Var}(f^{\lambda_1,..,\lambda_L}) = \mathrm{Var}(f) + \sum_{h=1}^{L} \lambda_h^2 \mathrm{Var}(\tilde{f}_h)$$

$$+ 2 \sum_{h=1}^{L} \lambda_h \left(\sum_{\substack{k=1 \\ k \neq h}}^{L} \lambda_k \mathrm{Cov}(\tilde{f}_h, \tilde{f}_k) - \mathrm{Cov}(f, \tilde{f}_h) \right),$$

or in *vector form*

$$\mathrm{Var}(f^{\Lambda}) = \mathrm{Var}(f) + \Lambda^T C \Lambda - 2\Lambda^T b$$

where $\Lambda = (\lambda_1, \dots, \lambda_L)^T$, $b = (\mathrm{Cov}(f, f_1), \dots, \mathrm{Cov}(f, f_L))^T$ and $C = (c_{ij})$, $c_{ij} = \mathrm{Cov}(f_i, f_j)$ is the symmetric $L \times L$ covariance matrix.

Minimizing the variance

The *optimal values* λ_h^*, $h = 1, \dots, L$ minimizing the above variance are found by equating to zero the partial derivatives with respect to λ_h

$$\frac{\partial \operatorname{Var}(f^{\lambda_1,\dots,\lambda_L})}{\partial \lambda_h} = 0, \quad h = 1, \dots, L.$$

This corresponds to solve the following *linear system*

$$\operatorname{Cov}(f, \tilde{f}_h) = \sum_{k=1}^{L} \lambda_k \operatorname{Cov}(\tilde{f}_h, \tilde{f}_k), \quad h = 1, \dots, L,$$

As an example, let us consider the case $L = 2$, where $\tilde{f}_1 = f_0$ and $\tilde{f}_2 = f^\infty$.

An example:

The optimal values λ_1^* and λ_2^* are readily found and are given by

$$\begin{aligned}\lambda_1^* &= \frac{\mathrm{Var}(f^\infty)\mathrm{Cov}(f,f_0) - \mathrm{Cov}(f_0,f^\infty)\mathrm{Cov}(f,f^\infty)}{\Delta},\\ \lambda_2^* &= \frac{\mathrm{Var}(f_0)\mathrm{Cov}(f,f^\infty) - \mathrm{Cov}(f_0,f^\infty)\mathrm{Cov}(f,f_0)}{\Delta},\end{aligned}$$

where $\Delta = \mathrm{Var}(f_0)\mathrm{Var}(f^\infty) - \mathrm{Cov}(f_0,f^\infty)^2$.
Using M samples for both control variates the optimal estimator reads

$$\tilde{E}_M^{\lambda_1^*,\lambda_2^*}(v,t) = E_M[f](v,t) - \lambda_1^*\left(E_M[f_0](v) - \mathbf{f}_0(v)\right) - \lambda_2^*\left(E_M[f^\infty](v) - \mathbf{f}^\infty(v)\right)$$

with $E_M[\cdot]$ the standard MC estimator with M samples.
By the same arguments as in Proposition 2, since $\lim_{t\to\infty} f(v,t) = f^\infty(v)$ we get

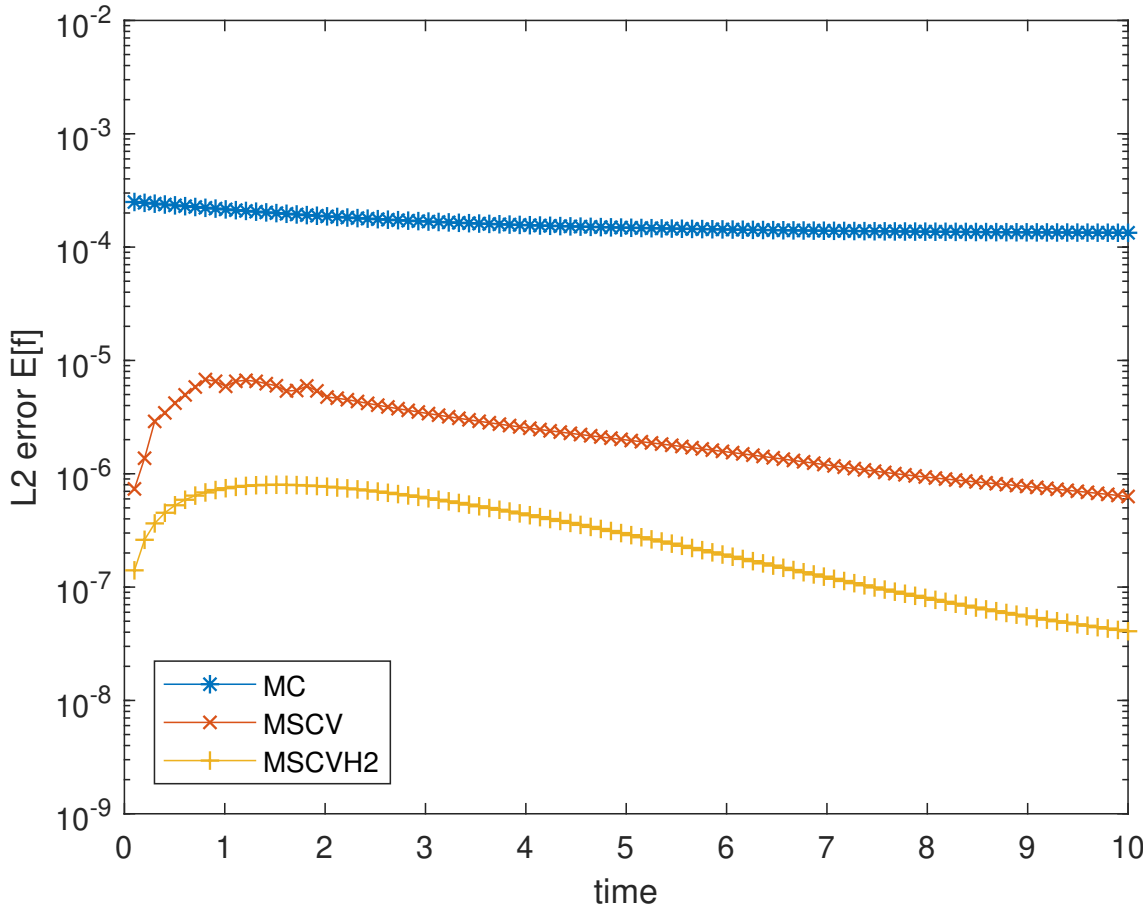
$$\lim_{t\to\infty}\lambda_1^* = 0, \qquad \lim_{t\to\infty}\lambda_2^* = 1,$$

and thus, the *variance* of the estimator vanishes asymptotically in time

$$\lim_{t\to\infty}\tilde{E}_M^{\lambda_1^*,\lambda_2^*}(v,t) = \mathbf{f}^\infty(v).$$

Test 2: error in time

Multiple MSCV with



Hierarchical methods

Now, the control variates $f_1, \dots, f_L$ represent kinetic models with *increasing level of fidelity*. We estimate $\mathbb{E}[f]$ with M_L samples using f_L as control variate

$$\mathbb{E}[f] \approx E_{M_L}[f] - \hat{\lambda}_L \left(E_{M_L}[f_L] - \mathbb{E}[f_L] \right).$$

Next, to estimate $\mathbb{E}[f_L]$ we use $M_{L-1} \gg M_L$ samples and consider f_{L-1} as control variate

$$\mathbb{E}[f_L] \approx E_{M_{L-1}}[f_L] - \hat{\lambda}_{L-1} \left(E_{M_{L-1}}[f_{L-1}] - \mathbb{E}[f_{L-1}] \right).$$

Recursively we can construct estimators for the remaining expectations of the control variates $\mathbb{E}[f_{L-2}], \mathbb{E}[f_{L-3}], \dots, \mathbb{E}[f_2]$ using $M_{L-3} \ll M_{L-4} \ll \dots \ll M_1$

$$\mathbb{E}[f_2] \approx E_{M_1}[f_2] - \hat{\lambda}_1 \left(E_{M_1}[f_1] - \mathbb{E}[f_1] \right),$$

and we stop with the final estimate

$$\mathbb{E}[f_1] \approx E_{M_0}[f_1],$$

with $M_0 \gg M_1$.

Optimality conditions

The total variance of the resulting estimator is

$$\mathrm{Var}(E_L^{\hat{\Lambda}}[f]) = \lambda_1^2 M_0^{-1}\mathrm{Var}(f_1) + \sum_{h=1}^{L} M_h^{-1}\left\{\lambda_{h+1}^2\mathrm{Var}(f_{h+1}) + \lambda_h^2\mathrm{Var}(f_h) - 2\lambda_{h+1}\lambda_h\mathrm{Cov}(f_{h+1}, f_h)\right\}.$$

By direct differentiation we get the tridiagonal system for $h = 1, \dots, L$

$$\begin{aligned} &M_{h-1}^{-1}\left\{\lambda_h\mathrm{Var}(f_h) - \lambda_{h-1}\mathrm{Cov}(f_h, f_{h-1})\right\} \\ &+M_h^{-1}\left\{\lambda_h\mathrm{Var}(f_h) - \lambda_{h+1}\mathrm{Cov}(f_{h+1}, f_h)\right\} = 0, \end{aligned}$$

which under the assumption $M_h \ll M_{h-1}$ leads to solutions

$$\lambda_h^* = \prod_{j=h}^{L}\hat{\lambda}_j^*, \qquad \hat{\lambda}_j^* = \frac{\mathrm{Cov}(f_{j+1}, f_j)}{\mathrm{Var}(f_j)}.$$

Test 3. Hierarchical MSCV with . Sod Test.

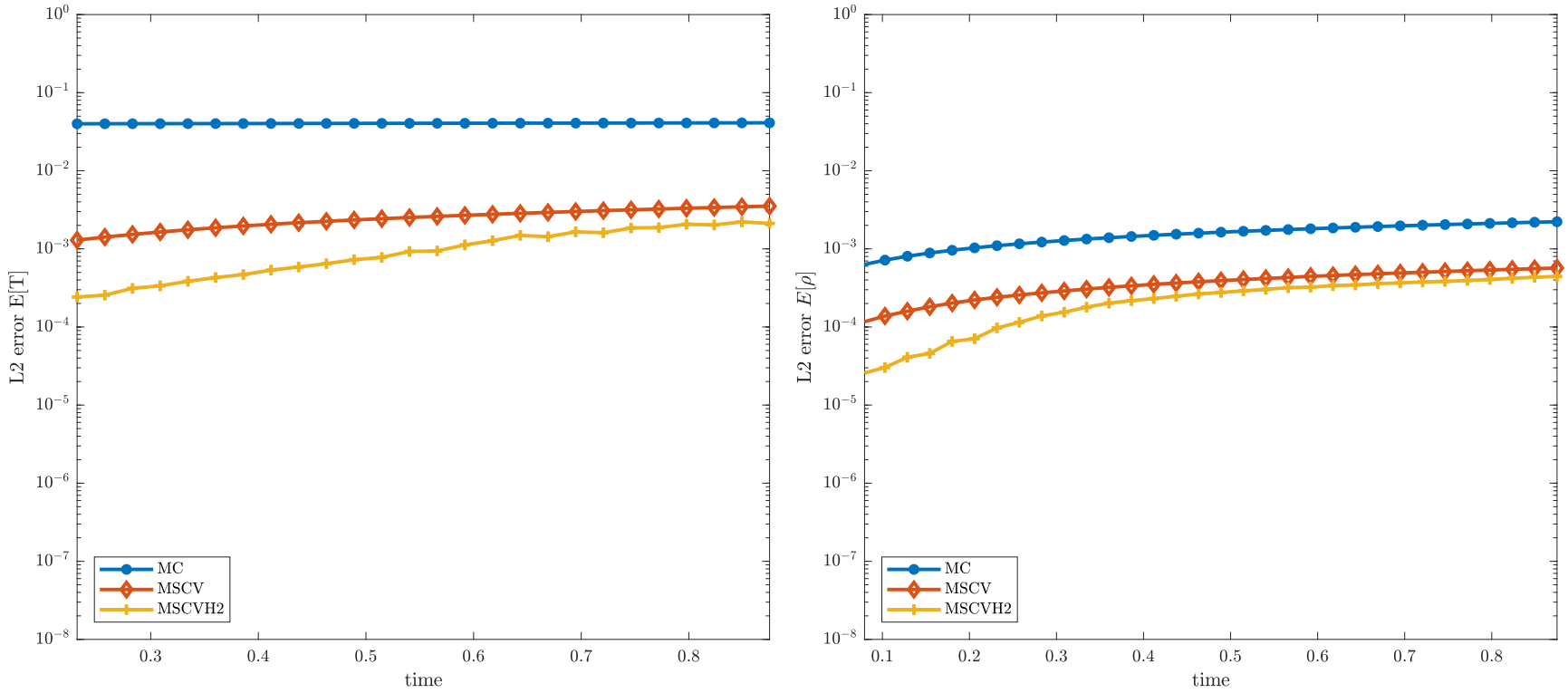


Figure: Boltzmann $M = 10$, BGK $M_{E_1} = 100$, Euler system $M_{E_2} = 10^5$.

Outline

Fokker-Planck models with uncertain parameters

We consider a distribution function $f(z,x,v,t)$, $t \geq 0$, $v \in V \subseteq \mathbb{R}^{d_v}$, $d_x, d_v \geq 1$, and $z \in I_Z \subseteq \mathbb{R}^{d_z}$ a *random vector* with distribution $p(z)$ representing the the density of *particles/agents* whose evolution is given by the following model

$$\frac{\partial}{\partial t} f(z,x,v,t) + v \cdot \nabla_x f(z,x,v,t) = \mathcal{J}(f,f)(z,x,v,t)$$

where $\mathcal{J}(\cdot,\cdot)$ is a *nonlinear* operator describing the interaction:

$$\mathcal{J}(f,f)(z,x,v,t) = \nabla_v \cdot [\mathcal{B}[f](z,v,t) f(z,x,v,t) + \nabla_v (D(z,v) f(z,x,v,t))]$$

where $\mathcal{B}[\cdot]$ is a *nonlocal operator* of the form

$$\mathcal{B}[f](z,v,t) = \int_V P(z,v,v_*)(v - v_*) f(z,v_*,t) dv_*.$$

with $P \geq 0$ whereas $D(z,v) \geq 0$ such that $D(z,v) = 0$ if $v \in \partial V$.

Example 1: The classical Fokker-Planck equation

We focus on the *space homogeneous case* and we take $V = \mathbb{R}^{d_v}$ with

$$D(z,v) = \frac{\sigma(z)^2}{2}, \qquad \text{and} \qquad P(z,v,v_*) = 1$$

$$\mathcal{B}[f](z,v,t) = (v-u),\ u(z) = \int_{\mathbb{R}^{d_v}} v f(z,v,0) dv,$$

with u which is a *conserved quantity* in the system. The *stationary distribution* is the Maxwellian distribution with uncertain momentum and temperature

$$f^\infty(z,v) = \left(\frac{1}{2\pi\sigma(z)^2}\right)^{d_v/2} \exp\left\{-\frac{|v-u(z)|^2}{2\sigma(z)^2}\right\}.$$

The *trends to equilibrium* are then determined in terms of the Boltzmann H-functional $\mathbb{H}(f) = \int_{\mathbb{R}^{d_v}} f \log f dv$, and $f(z,v,t)$ converges in *relative entropy* $\mathbb{K}(f) = \mathbb{H}(f) - \mathbb{H}(f^\infty)$ to the equilibrium $f^\infty(z,v)$ exponentially fast

$$\mathbb{K}(f(z,v,t)) \le e^{-2t/\sigma(z)^2} \mathbb{K}(f(z,v,0)), \qquad t \ge 0,$$

Quasi equilibrium states

- For FP with *nonlocal drift* analytical insights on the large time behavior are often lost together with sharp trends to equilibrium.
- One way to *tackle the problem* consists in considering the solution of the vanishing flux condition

$$D(v)\partial_v f(z,v,t) = (\mathcal{B}[f](z,v,t) + D'(z,v))f(z,v,t),$$

  which is *not in general solvable* due to nonlinearity on the rhs.
- This problem can be faced through the notion of *quasi steady-state* that is

$$f^q(z,v,t) = C_z \exp\left\{ - \int_{-\infty}^{v} \frac{\mathcal{B}[f](z,v_*,t) + D'(z,v_*)}{D(z,v_*)} dv_* \right\},$$

- Therefore, f^q is *not the global in time equilibrium* of the problem but it enjoys the nice property to annihilate the flux for each time $t \geq 0$.
- If $\mathcal{B}[\cdot]$ *does not depend on time* the quasi-equilibrium *coincides* with the steady state distribution of the nonlinear FP of interest.

Preliminaries on stochastic Galerkin methods

Let $(\Omega, \mathcal{F}, P)$ be a *probability space* and let us define a random variable

$$z : (\Omega, \mathcal{F}) \to (I_z, \mathcal{B}_{\mathbb{R}^{d_z}}),$$

with $I_z \subseteq \mathbb{R}^{d_z}$ and $\mathcal{B}_{\mathbb{R}^{d_z}}$ the Borel set.Let $\mathbb{P}_M$ be the *orthogonal polynomial space* of degree M

$$\mathbb{P}_M = \{g : I_z \to \mathbb{R} : g \in \text{span}\{\Phi_h\}_{h=0}^M\},$$

where $\{\Phi_h\}_{h=0}^M$ is a set of polynomials of z with degree up to $M \geq 0$, forming an orthonormal basis of $L^2(\Omega)$, i.e.

$$\int_{I_z} \Phi_h(z)\Phi_k(z)\, p(z)\, dz = \delta_{hk},$$

The *solution* can be represented in $L^2(\Omega)$ as follows

$$f(z, v, t) = \sum_{k=0}^{\infty} \int_{I_z} f(z, v, t)\Phi_k(z)p(z)dz\Phi_k(z) = \sum_{k=0}^{\infty} \hat{f}_k(v, t)\Phi_k(z),$$

Preliminaries on stochastic Galerkin methods II

The generalized polynomial chaos method *approximates the solution* of the differential problem $f(z,v,t)$ in $\mathbb{P}_M$ with the finite sum

$$f_M(z,v,t) = \sum_{k=0}^{M} \hat{f}_k(v,t)\Phi_k(z),$$

solution of the differential problem

$$\frac{\partial}{\partial t} f_M(z,v,t) = \mathcal{J}_M(f_M, f_M)(z,v,t). \tag{1}$$

To *derive* an equation for the coefficients $\hat{f}_h(v,t)$ of the expansion, one multiplies equation (1) by $\Phi_h(z),\ h = 1,..,M$ and integrate over I_z. This gives

$$\partial_t \int_{I_z} f_M(z,v,t)\Phi_h(z)p(z)dz = \int_{I_z} \mathcal{J}(f_M, f_M)(z,v,t)\Phi_h(z)p(z)dz$$

Hence, from the orthogonality of the polynomial basis, one obtains a coupled system of $(M+1)$ equations describing the *evolution of the projection coefficients*.

Preliminaries on stochastic Galerkin methods III

The *stochastic Galerkin approximation* is given by the following coupled system of partial differential equations

$$\partial_t \hat{f}_h(v,t) = \nabla_v \cdot \left[\sum_{k=0}^{M} \mathcal{B}_{hk}[f_M] \hat{f}_k(v,t) + \nabla_v \left(\sum_{k=0}^{M} D_{hk}(v) \hat{f}_k(v,t) \right) \right],$$

where

$$\begin{aligned} \mathcal{B}_{hk}[f_M](v,t) &= \int_{I_z} \mathcal{B}[f_M] \Phi_k(z) \Phi_h(z)\, p(z)\, dz \\ &= \int_{I_z} \int_{V} P(z,v,v_*) |v - v_*| f_M(z,v_*,t) dv_* \Phi_k(z) \Phi_h(z)\, p(z)\, dz. \end{aligned}$$

and

$$D_{hk}(v) = \int_{I_z} D(z,v) \Phi_k(z) \Phi_h(z)\, p(z)\, dz.$$

Or in vector notations as follows

$$\partial_t \hat{\mathbf{f}}(v,t) = \nabla_v \cdot \left[\mathbf{B}[f_M](v,t) \hat{\mathbf{f}}(v,t) + \nabla_v \mathbf{D}(v) \hat{\mathbf{f}}(v,t) \right].$$

Stochastic Galerkin methods: properties

- Indicating now with $\|\hat{\mathbf{f}}\|_{L^2}$ the standard L^2 norm of the vector $\hat{\mathbf{f}}$

$$\|\hat{\mathbf{f}}\|_{L^2} = \left(\int_V \sum_{k=0}^{M} \hat{f}_k^2 dv \right)^{1/2},$$

  we have $\|f_M\|_{L^2(\Omega)} = \|\hat{\mathbf{f}}\|_{L^2}$. Furthermore, **B**, **D** are symmetric.
- The resulting problem can be then solved with *suitable numerical methods*.
- Insights on statistical quantities like the *expected value and variance* of the differential problem are defined in terms of the projections.
- In particular, one gets *spectral approximations* of

$$\mathbb{E}_z[f(z,v,t)] \approx \hat{f}_0(v,t).$$

  and

$$\mathrm{Var}_z[f(z,v,t)] \approx \int_{I_z} \left(\sum_{k=0}^{M} \hat{f}_k(v,t)\Phi_k(z) - \hat{f}_0(v,t) \right)^2 p(z)dz.$$

Stochastic Galerkin methods: properties II

If we consider $d_v = 1$ the following stability result holds true

Theorem

If $\|\partial_v \mathcal{B}_{hk}\|_{L^\infty} \leq C_{\mathbf{B}}$ with $C_{\mathbf{B}} > 0$ for all $h, k = 0, \dots, M$ and if the diffusion functions are such that $D_{hk} \leq C_{\mathbf{D}}$ for all $h, k = 0, \dots, M$, hence we have

$$\|\hat{\mathbf{f}}(t)\|_{L^2}^2 \leq e^{t(C_{\mathbf{B}}+2C_{\mathbf{D}})} \|\hat{\mathbf{f}}(0)\|_{L^2}^2$$

hence stability of the SG expansion.

- We describe the *micro-macro reformulation* by relying on the notion of *quasi-equilibrium* f^q satisfying $J(f^q, f^q)(z, v, t) = 0$.
- The *idea* consists in decomposing the solution of the F-P model in

$$f(z, v, t) = f^q(z, v, t) + g(z, v, t),$$

with $g(z, v, t)$ a distribution such that

Micro-macro stochastic Galerkin scheme

$$\int_{\mathbb{R}^{d_v}} \varphi(v) g(z,v,t) dv = 0 \text{ if } \int_{\mathbb{R}^{d_v}} \varphi(v) f(z,v,t) dv = \int_{\mathbb{R}^{d_v}} \varphi(v) f^q(z,v,t) dv,$$

Proposition 1

Let us consider the nonlinear F-P model. For each time $t \geq 0$ and $z \in I_z$ the operator $\mathcal{J}(f,f)$ can be rewritten as

$$\mathcal{J}(f,f) = \mathcal{J}(g,g) + \mathcal{G}(f^q, g),$$

where $\mathcal{G}(f^q,g)(z,v,t) = \mathcal{B}[f^q](z,v,t)g(z,v,t) + \mathcal{B}[g](z,v,t)f^q(z,v,t)$. Moreover, if $f^q \to f^\infty$ for $t \to +\infty$ the only admissible steady state solution of the problem

$$\partial_t f(z,v,t) = \mathcal{J}(g,g)(z,v,t) + \mathcal{B}[f^q](z,v,t)g(z,v,t) + \mathcal{B}[g](z,v,t)f^q(z,v,t),$$

is $g^\infty(z,v) \equiv 0$.

Micro-macro stochastic Galerkin scheme II

Now, introducing the following decomposition *for all coefficients* $M \geq 0$

$$f_M(z,v,t) = f_M^q(z,v,t) + g_M(z,v,t), \qquad v \in \mathbb{R}^{d_v}, t \geq 0,$$

being

$$f_M^q(z,v,t) = \sum_{k=0}^{M} \widehat{f_k^q}(v,t)\Phi_k(z), \qquad \widehat{f_k^q}(v,t) = \int_{I_z} f^q(z,v,t)\Phi_k(z)p(z)dz,$$

the *resulting gPC-SG* problem reads

$$\begin{aligned}\frac{\partial}{\partial t} f_M(z,v,t) &= \mathcal{J}_M(f_M, f_M) \\ &= \mathcal{J}_M(g_M, g_M) + \mathcal{G}_M(f_M^q, g_M)\end{aligned}$$

and we obtain a *system of coupled PDEs for the evolution of the nonequilibrium part*

Micro-macro stochastic Galerkin scheme III

$$\frac{\partial}{\partial t}\hat{g}_h(v,t) = \nabla_v \cdot \left[\sum_{k=0}^{M} \mathcal{B}_{hk}[g_M]\hat{g}_k + \nabla_v\left(\sum_{k=0}^{M} D_{hk}(v)\hat{g}_k\right)\right]$$
$$+ \nabla_v \cdot \left[\sum_{k=0}^{M} \mathcal{B}_{hk}[f_M^q]\hat{g}_k + \mathcal{B}_{hk}[g_M]\widehat{f^q}_k\right] - \frac{\partial}{\partial t}\widehat{f_k^q}(v,t),$$

where $B_{hk}[g_M]$ and $B_{hk}[f_M^q]$ are defined as

$$\mathcal{B}_{hk}[g_M](v,t) = \int_{I_z} \mathcal{B}[g^M]\Phi_k(z)\Phi_h(z)p(z)dz,$$
$$\mathcal{B}_{hk}[f_M^q](v,t) = \int_{I_z} \mathcal{B}[f_M^q]\Phi_k(z)\Phi_h(z)p(z)dz.$$

Reformulation of the micro-macro stochastic Galerkin scheme

We observe that the *micro-macro stochastic Galerkin* method admits the following equivalent formulation. We have

$$\frac{\partial}{\partial t} f(z,v,t) = \tilde{\mathcal{J}}(f^q + g, f^q + g)(z,v,t),$$

being the operator $\tilde{\mathcal{J}}(\cdot,\cdot)$ defined as follows

$$\tilde{\mathcal{J}}(h_1,h_2) = \frac{1}{2}\left(\nabla_v \cdot \mathcal{B}[h_1]h_2 + \nabla_v \cdot \mathcal{B}[h_2]h_1\right)$$
$$\frac{1}{2}\left(\rho_{h_1}\nabla_v \cdot \nabla_v(D(v)h_1) + \rho_{h_2}\nabla_v \cdot \nabla_v(D(v)h_2)\right),$$

where h_1, h_2 are two densities on $V \subseteq \mathbb{R}$ and $\rho_{h_1} = \int_V h_1 dv$, $\rho_{h_2} = \int_V h_2 dv$. Hence,

$$\tilde{\mathcal{J}}(f^q + g, f^q + g) = \tilde{\mathcal{J}}(g,g) + 2\tilde{\mathcal{J}}(g,f^q) := \mathcal{L}(f^q, g),$$

On a new formulation of micro-macro stochastic Galerkin

Thus, we can write

$$\frac{\partial}{\partial t} g(z,v,t) = \mathcal{L}(f^q, g)(z,v,t) - \frac{\partial}{\partial t} f^q(z,v,t).$$

Now, plugging everything in the *stochastic Galerkin setting* we have

$$\frac{\partial}{\partial t} g_M(z,v,t) = \mathcal{L}_M(f_M^q, g_M) - \frac{\partial}{\partial t} f_M^q(z,v,t).$$

Since $g_M(z,v,t) = f_M(z,v,t) - f_M^q(z,v,t)$ we observe that

$$\mathcal{L}_M(f_M^q, g_M)(z,v,t) = \tilde{\mathcal{J}}_M(f_M, f_M) - \tilde{\mathcal{J}}_M(f_M^q, f_M^q).$$

A convenient reformulation is therefore

$$\frac{\partial}{\partial t} f_M(z,v,t) = \tilde{\mathcal{J}}_M(f_M, f_M) - \tilde{\mathcal{J}}_M(f_M^q, f_M^q).$$[10]

[10] F. Filbet, L. Pareschi, T. Rey '19

Test 1: Explicit steady state in opinion dynamics

We consider a model of opinion formation obeying to a F-P model with

$$\mathcal{B}[f](z,v,t) = \gamma(z)(v-u), \qquad D(z,v) = \frac{\sigma^2(z)}{2}(1-v^2)^2,$$

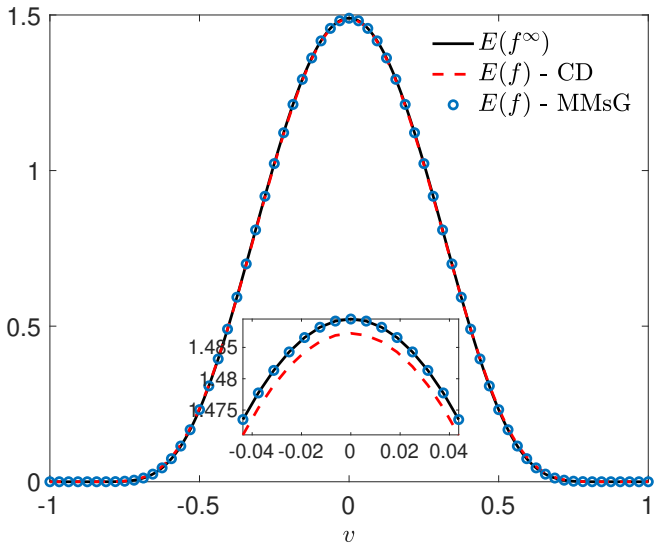
The discretization of the polynomial chaos expansion for $f^M(z,v,t)$ reads

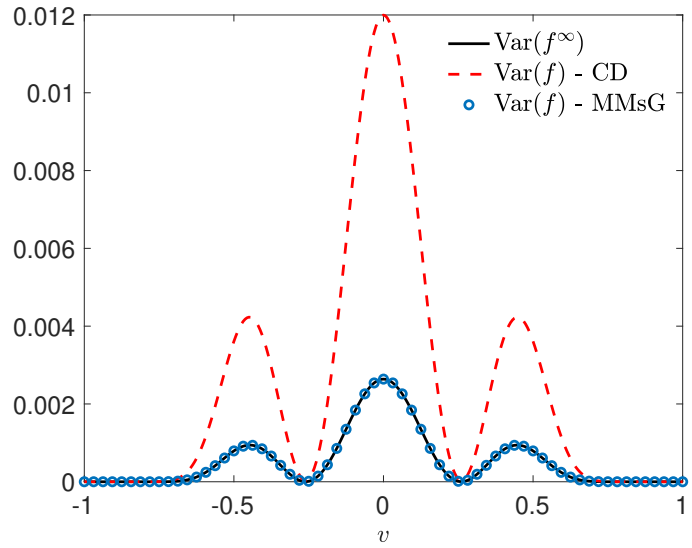
$$\frac{d}{dt}\hat{\mathbf{f}}_i(t) = \frac{\mathbf{B}_{i+1}\hat{\mathbf{f}}_{i+1}(t) - \mathbf{B}_{i-1}\hat{\mathbf{f}}_{i-1}(t)}{2\Delta v} + \frac{\mathbf{D}_{i+1}\hat{\mathbf{f}}_{i+1}(t) - 2\mathbf{D}_i\hat{\mathbf{f}}_i(t) + \mathbf{D}_{i-1}\hat{\mathbf{f}}_{i-1}(t)}{\Delta v^2},$$

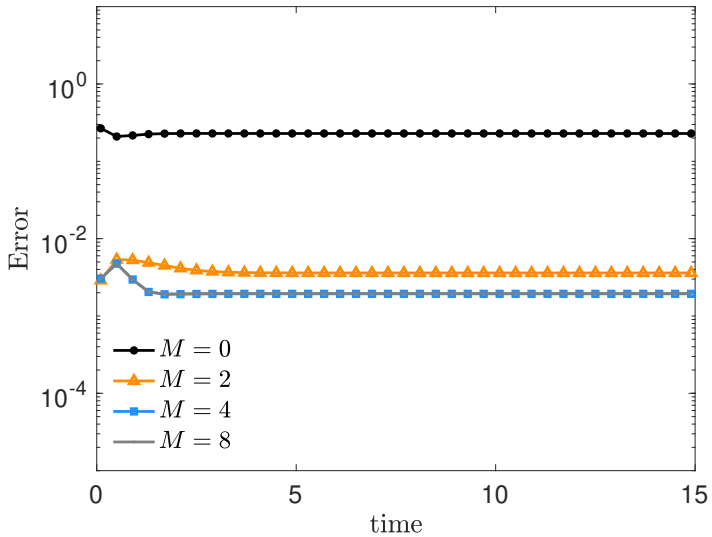
Instead, within the micro-macro formulation we have

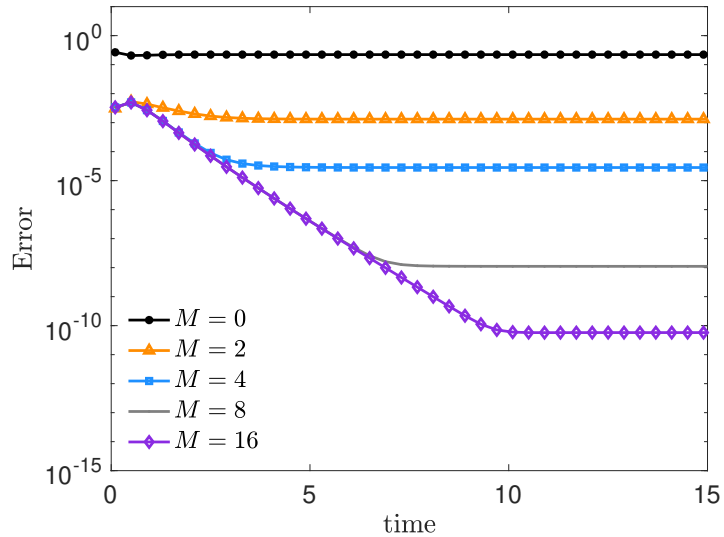
$$\begin{aligned}\frac{d}{dt}\hat{\mathbf{f}}_i(t) =& \frac{\mathbf{B}_{i+1}\hat{\mathbf{f}}_{i+1}(t) - \mathbf{B}_{i-1}\hat{\mathbf{f}}_{i-1}(t)}{2\Delta v} + \frac{\mathbf{D}_{i+1}\hat{\mathbf{f}}_{i+1}(t) - 2\mathbf{D}_i\hat{\mathbf{f}}_i(t) + \mathbf{D}_{i-1}\hat{\mathbf{f}}_{i-1}(t)}{\Delta v^2} \\ &- \left(\frac{\mathbf{B}_{i+1}\hat{\mathbf{f}}^{\infty}_{i+1} - \mathbf{B}_{i-1}\hat{\mathbf{f}}^{\infty}_{i-1}}{2\Delta v} + \frac{\mathbf{D}_{i+1}\hat{\mathbf{f}}^{\infty}_{i+1} - 2\mathbf{D}_i\hat{\mathbf{f}}^{\infty}_{i} + \mathbf{D}_{i-1}\hat{\mathbf{f}}^{\infty}_{i-1}}{\Delta v^2} \right),\end{aligned}$$

Test 1


$E(f^\infty)$
$E(f)$ - CD
$E(f)$ - MMsG
1.5
1
0.5
0
-1
-0.5
0
0.5
1
v
1.485
1.48
1.475
-0.04 -0.02 0 0.02 0.04


$\mathrm{Var}(f^\infty)$
$\mathrm{Var}(f)$ - CD
$\mathrm{Var}(f)$ - MMsG
0.012
0.01
0.008
0.006
0.004
0.002
0
-1
-0.5
0
0.5
1
v


Error
10^0
10^{-2}
10^{-4}
$M = 0$
$M = 2$
$M = 4$
$M = 8$
0
5
10
15
time


Error
10^0
10^{-5}
10^{-10}
10^{-15}
$M = 0$
$M = 2$
$M = 4$
$M = 8$
$M = 16$
0
5
10
15
time

Outline

Conclusions

- In many kinetic models including the effects of *uncertainty* is essential, since at most we can aspire to have statistical informations on some parameters.
- *Monte Carlo methods* are attractive since they are non intrusive and easily parallelizable but they exhibit statistical fluctuations and slow convergence.
- *Multiscale control variate methods* employ the knowledge of some simpler kinetic/fluid models with different fidelity levels to improve the statistical estimate at the various space-time scales.
- *Stochastic Galerkin* methods based on gPC provide spectral accuracy for smooth random fields but contribute to the curse of dimensionality and sometimes are not able to preserve the main physical properties.
- *Micro-Macro Stochastic Galerkin* methods provide spectral accuracy with a reduced number of modes without altering the structure of the original method.

The material for this part is mostly based on the recent survey
G.D., Liu Liu, Lorenzo Pareschi, Xueyu Zhu. Multi-Fidelity Methods for Uncertainty Propagation in Kinetic Equations. Panoramas et synthèses, In press.