Deterministic filtering of non-smooth dynamical systems

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Data assimilation is a set of methods intended to combine different information about a physical system such as 1) a mathematical model and 2) observed data. See for instance (Law, Stuart & Zygalakis 2015). In particular, it includes filtering methods which, allow under appropriate conditions, to infer information about a hidden signal from an observed signal.

Few words on deterministic filtering methods

The hidden state $\{x(t),\ t \geq 0\}$ satisfies a differential equation

$$\dot{x}(t) = f(x(t)) + \sigma(x(t))w(t) \tag{1}$$

where $w(.)$ is a certain state disturbance and the observation $\{y(t),\ t \geq 0\}$ is expressed as

$$y(t) = h(x(t)) + \rho v(t) \tag{2}$$

and $v(.)$ is an observation disturbance. The basic problem of filtering is to find the *best* estimator of the hidden state according to a criterion. First strides in deterministic approaches to filtering where made by Mortensen in the 60's, see (Mortensen 1968). Mortensen's estimator can be described as follows. Given an observation trajectory $\{\hat{y}(t),\ 0 \leq t \leq T\}$, we want to minimize

$$J(x, t; w) \triangleq \phi(x_0) + \frac{1}{2}\int_0^t \{|w(r)|^2 + \rho^{-2}|\hat{y}(r) - h(x(r))|^2\}\mathrm{d}r$$

over of the possible disturbance functions w. Here a final state is prescribed at $x(t) = x$ and Equation (1) is solved backward in time. Following the dynamic programming method, the cost-to-go function

$$V(x, t) \triangleq \inf_{w(.)} J(x, t; w)$$

satisfies a Hamilton Jacobi Bellman equation

$$\frac{\partial V}{\partial t} + f \cdot \nabla V + \frac{1}{2} a \nabla V \cdot \nabla V = \frac{1}{2\rho^2}|y(t) - h(x)|^2$$

with the initial condition $V(0, x) = \phi(x)$. Here $a \triangleq \sigma \cdot \sigma^T$. The quantity $-V(x, t)$ is interpreted as a measure of the likelihood of $x(t) = x$ at time t. A Mortensen estimator of $x(t)$ is defined as $\hat{x}(t) \in \arg\min V(x, t)$. It is not necessarily unique. See (Fleming 1998) for further developpement with sufficiently regular functions f, σ, h.

Objective of the project

The goal of the project is to investigate the Mortensen's filtering problem where Equation (1) is replaced by a differential inclusion of the form

$$\dot{x}(t) + \partial\psi(x(t)) \ni f(x(t)) + \sigma(x(t))w(t), \quad \text{a.e } t \in (0, T) \tag{3}$$

where ψ is a convex function and $\partial\psi$ is its subdifferential operator. We will also seek its counterpart in a discrete-time context (Moireau 2018). This work is motivated by the state estimation of two fundamental non-smooth dynamical systems (a) an elasto-perfectly-plastic oscillator and (b) a dry friction one dimensional model. See for instance (Bastien, Schatzman, Lamarque 2000) for the use of differential inclusions in non-smooth mechanics.

References

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