Stabilization methods for Saint-Venant equations. Linear and non-linear aspects, treatment of the Coriolis term

The project aims to extend recent work proposed in the framework of the linear Saint-Venant equations with rotation [3]. This work highlighted the failure of the classical Finite Volume methods for this model, and led to the development of a class of numerical methods capable of preserving the geostrophic balance and ensuring stability results in the sense of energy. This failure of the usual Finite Volumes schemes is mainly due to the expression of the diffusion term present in the numerical flows, which distorts the equilibrium of the model (in other words, the equilibrium solutions of the equivalent model are no longer those of the original model). To this end, the objective is to propose appropriate viscosity terms to preserve the structure of the kernel. The strategy combines techniques inspired by the low-Froude schemes [9] as well as other stabilization methods for the rotational model [6, 7]. An appropriate dosage of these terms at the flow level ensures the stability of the discrete kernel, thus ensuring an equivalent with the continuous framework.

The linear case contains most of the dominant dynamics of regimes of operational interest. Based on this observation, this work opens the way to many areas of expansion with high application potential. The proposed topic will be structured around the following points:

1 Extension to the non-linear framework

The main purpose here is to study the non-linear case. In the absence of the rotation term, results have recently been obtained for the multi-layer model in semi-implicit [12] and explicit [8] schemes. These methods are based on stabilization terms very similar to those proposed in the linear case [3], and guarantee stability in the sense of energy in a discrete time and space framework. Here again, the discrete operators obtained are reinterpreted at the continuous level by the presence of stabilization terms on the energy equation. On this basis, the objective is therefore to study the possibility of extending some of these results in the presence of the rotation term. From a numerical point of view, the extension to the non-linear case does not present any particular technical difficulty. The aim would be to study the behaviour of the method, particularly around equilibrium states. In particular, it would be interesting to carry out a comparative study with standard Finite Volumes methods in the vicinity of geostrophic equilibria.

2 Geometrical aspects

Bearing in mind the application perspectives, the issue of extension to unstructured meshes is of primary interest. The main challenge will be to exhibit discrete operators with the mimetic properties required to inherit the results established in the continuous framework. Initially, these aspects could be studied within the simplified framework of regular simple meshes.

From a general point of view, the environment of staggered meshes can also be of multiple interest, particularly from the point of view of the stability of methods. In this respect, most operational simulation platforms are formulated on this type of discretization. We therefore propose to reformulate and analyze these schemes in this type of geometric environment. Again, one of the key points will be the construction of good discrete operators. It should be noted that very recent work has been carried out in the non-linear framework [10] in the absence of rotation terms, based on formalism [1], and could constitute a first working basis.

3 Entropic stability for Saint-Venant equations

The viscosity methods introduced in the linear [3] and non-linear [12] and [8] framework provide a powerful tool to ensure the stability of the schemes in the energy direction. In the context of non-linear Saint-Venant equations, very few finite volume standard schemes are able to provide discrete entropy inequalities in time and space, even in the absence of topography. In recent work [5], these stabilization terms have been used to ensure discrete entropy inequalities for the hydrostatic reconstruction scheme [2], indroduced to ensure lake-type balances at rest for the Saint-Venant equations (well-balanced schemes). It would therefore be a question of applying this strategy to other types of well-balanced approaches for Saint-Venant, such as the one based on the pre-balanced formulation [11] for example. Depending on the level of progress of the project, the extension to 2d on unstructured meshes may also be considered, via the convex combination reconstruction method introduced in [4].

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