Simulation of complex free surface flows

Settings

This project focuses on the simulation of incompressible free surface flows. While the Euler system is installed in a mobile domain (the free surface being an unknown of the problem), models of reduced complexity have been constructed by considering averaged variables along the water column to reduce the size of the problem. Historically, shallow water models have proven their worth in a number of applications. The Saint-Venant equations [7] provide relevant information for long wavelength phenomena.

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0, \\ \partial_t(h\bar{u}) + \partial_x\left(h\bar{u}^2 + g\dfrac{h^2}{2}\right) = -gh\partial_x z_b. \end{cases} \tag{1}$$

To go further in the description of these phenomena, models integrating the non-hydrostatic effects of the pressure field were considered (Boussinesq, Peregrine, ... - see [12] for example), as in the Serre – Green-Naghdi model [11, 14]

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0, \\ \partial_t(h\bar{u}) + \partial_x\left(h\bar{u}^2 + g\dfrac{h^2}{2} + hq\right) = -(q_b + gh)\partial_x z_b, \\ \partial_t(h\bar{w}) + \partial_x(h\bar{u}\bar{w}) = q_b, \\ \partial_t(h\sigma) + \partial_x(h\sigma\bar{u}) = 2\sqrt{3}\left(q - \dfrac{q_b}{2}\right), \\ 2\sqrt{3}\sigma + h\partial_x\bar{u} = 0, \\ \bar{w} - \sqrt{3}\sigma - \bar{u}\partial_x z_b = 0, \end{cases} \tag{2}$$

or the family of dispersive models [1]

$$\begin{cases} \partial_t h + \partial_x(h\bar{u}) = 0, \\ \partial_t(h\bar{u}) + \partial_x\left(h\bar{u}^2 + g\dfrac{h^2}{2} + hq\right) = -(\dfrac{\alpha^2}{2}q + gh)\partial_x z_b, \\ \partial_t(h\bar{w}) + \partial_x(h\bar{u}\bar{w}) = \alpha q, \\ \alpha\bar{w} + h\partial_x\bar{u} - \dfrac{\alpha^2}{2}\bar{u}\partial_x z_b = 0. \end{cases} \tag{3}$$

At the wave propagation speed given by the Saint-Venant model, these dispersive models allow a better description of their shape. The classic Dingemans test case [8] highlights these findings. The experimental data provided illustrate the accuracy of each of the model types (see Fig. 1).

However, these simplified models cannot capture the full richness of the simulated phenomena. They are constructed under simplifying assumptions (frequency and amplitude dispersion regimes) by vertical averaging procedures. This $\mathbb{P}_0$ approximation for velocity variables is legitimate for shallow water flows but is not for general flows. Without going back to Euler's equations, multi-layer models have been proposed to extend the range of applications. Figure 2 describes the notations: the flow is discretized along the vertical axis in layers (mesh cells) of height $h_\alpha = \frac{h}{L}$ for a given $L > 0$. In [9], a formalism is described to construct a hierarchy of models according to the degree of approximation of the variables u, w and q. For example, we are interested in the model composed of the global conservation equation

$$\partial_t h + \partial_x(h\bar{u}) = 0 \qquad \bar{u} = \frac{1}{L}\sum_{\alpha=1}^{L} \bar{u}_\alpha \tag{4a}$$

and of the following equations in each layer $\alpha \in \{1, \dots, L\}$

$$\begin{aligned}\partial_t(h_\alpha, \bar{u}_\alpha) + \partial_x(h_\alpha \bar{u}_\alpha^2 + h_\alpha \bar{q}_\alpha) + \bar{u}_{\alpha+1/2}\Upsilon_{\alpha+1/2} - \partial_x z_{\alpha+1/2} q_{\alpha+1/2} \\ - \bar{u}_{\alpha-1/2}\Upsilon_{\alpha-1/2} + \partial_x z_{\alpha-1/2} q_{\alpha-1/2} = -g h_\alpha \partial_x \eta\end{aligned} \tag{4b}$$

$$\partial_t(h_\alpha \bar{w}_\alpha) + \partial_x(h_\alpha \bar{u}_\alpha \bar{w}_\alpha) + \bar{w}_{\alpha+1/2}\Upsilon_{\alpha+1/2} + q_{\alpha+1/2} - \bar{w}_{\alpha-1/2}\Upsilon_{\alpha-1/2} - q_{\alpha-1/2} = 0 \tag{4c}$$

$$\begin{aligned}\partial_t(h_\alpha \sigma_\alpha) + \partial_x(h_\alpha \sigma_\alpha \bar{u}_\alpha) = 2\sqrt{3}\Bigg[\bar{q}_\alpha - \frac{q_{\alpha+1/2} + q_{\alpha-1/2}}{2} \\ -\Upsilon_{\alpha+1/2}\left(\frac{h_\alpha \partial_x \bar{u}_\alpha}{12} + \frac{\bar{w}_{\alpha+1/2} - \bar{w}_\alpha}{2}\right) + \Upsilon_{\alpha-1/2}\left(\frac{h_\alpha \partial_x \bar{u}_\alpha}{12} + \frac{\bar{w}_\alpha - \bar{w}_{\alpha-1/2}}{2}\right)\Bigg]\end{aligned} \tag{4d}$$

combined with diagnostic equations that express the divergence constraint:

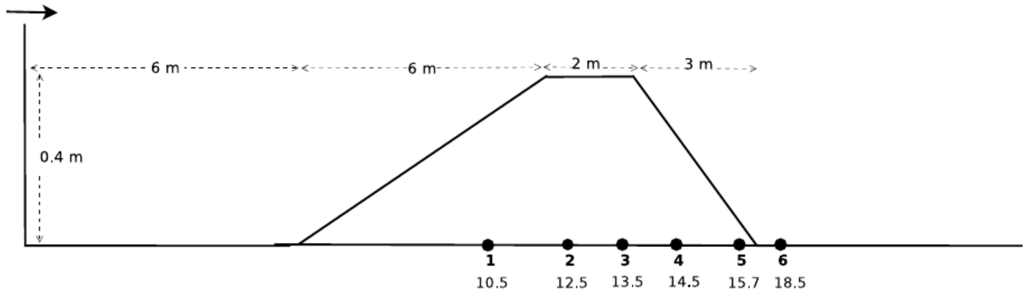
$$2\sqrt{3}\sigma_\alpha + h_\alpha \partial_x \bar{u}_\alpha = 0, \qquad \bar{w}_{\alpha+1} - \bar{w}_\alpha - (\bar{u}_{\alpha+1} - \bar{u}_\alpha)\partial_x z_{\alpha+1/2} - \sqrt{3}(\sigma_{\alpha+1} + \sigma_\alpha) = 0, \tag{4e}$$

$$w_1 - u_1 \partial_x z_b - \sqrt{3}\sigma_1 = 0, \qquad q_{L+1/2} = 0 \tag{4f}$$

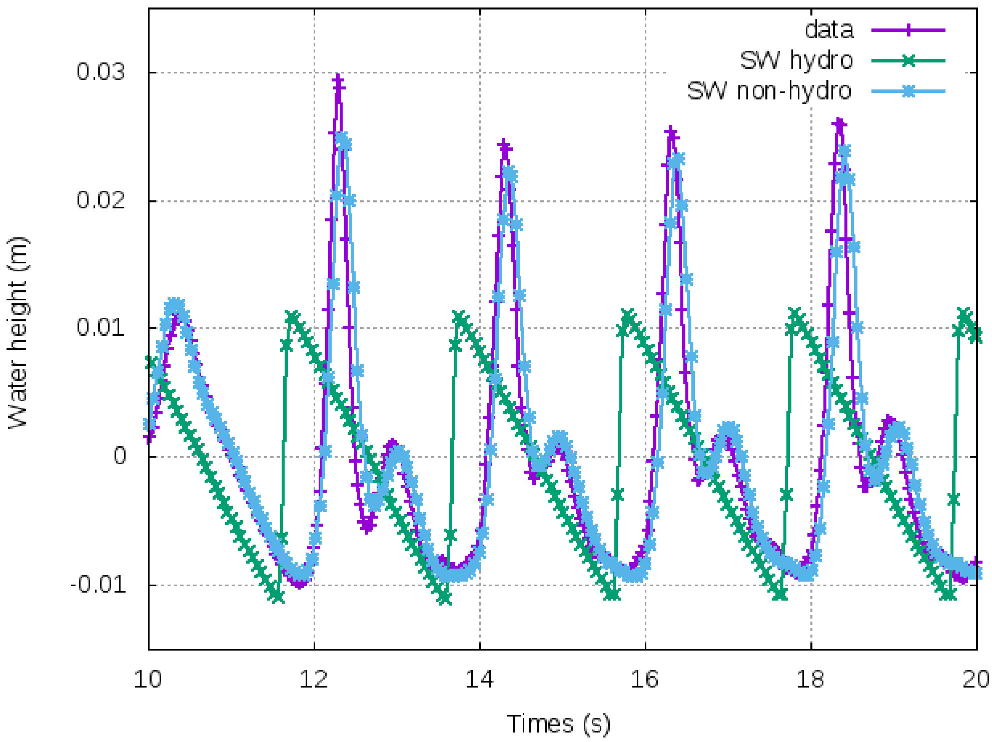
The mass transfert terms between two layers are given by

$$\Upsilon_{\alpha+1/2} = \sum_{\beta=\alpha+1}^{L} \partial_x(h_\beta(u_\beta - \bar{u})).$$

This model degenerates toward 2 for $L = 1$.



(a) Dinguemans's experiment



(b) Comparison between an hydrostatic model [10] and a non-hydrostatic one [2]

Figure 1: Highlighting of dispersive effects

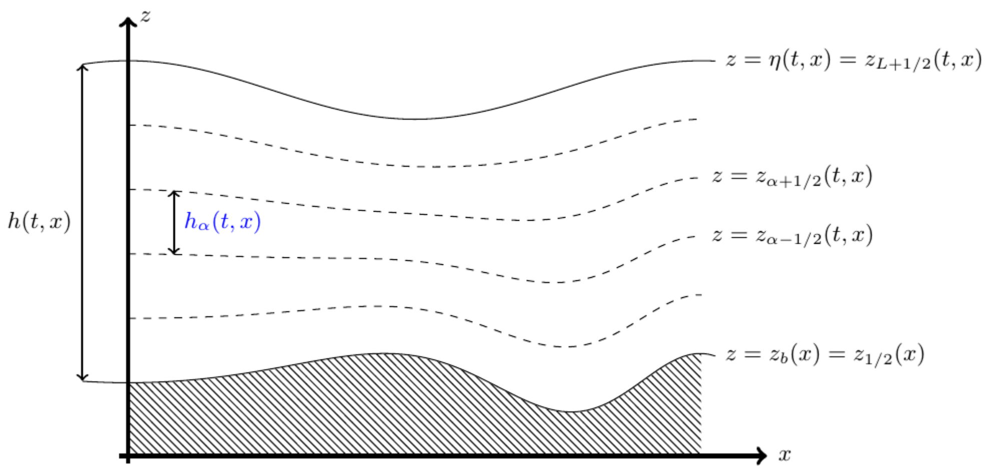


Figure 2: Multilayer notations

The project

Conventional numerical methods for processing dispersive models consist of using a splitting between the standard terms of hydrostatic models and then a correction to verify the divergence constraints. Typically [1, 4], in model 3, all terms present in Saint-Venant's equations 1 are processed using standard solvers, while additional terms are processed by an elliptic problem solver with an operator that is not standard.

In collaboration with researchers from the Seville and Cordoba (Spain) laboratories, a new method was applied to the model 2 with an Uzawa algorithm applied to the mixed velocity-pressure problem. Encouraging results were obtained in dimension 1.

Two orientations are proposed within the framework of this project.

Extension to dimension 2. A first work step consists in extending the algorithm described above to dimension 2 for the monolayer model 2. These results can be compared with the approaches described in [3] or [13].

Extension to the multilayer model. A second work step of the project aims to adapt the methods applied to monolayer models to the multilayer model 4.

Applications. To test the simulation codes obtained, particular attention will be given to the modelling of *Favre secondary waves* described in [6] and simulated in [5]. They occur during sudden changes in flow, for example when a lock is opened.

References

[1] N. Aïssiouene, M.-O. Bristeau, E. Godlewski, A. Mangeney, C. Parés, and J. Sainte-Marie. A two-dimensional method for a family of dispersive shallow water model. 2019.

[2] N. Aïssiouene, M.-O. Bristeau, E. Godlewski, and J. Sainte-Marie. A combined finite volume-finite element scheme for a dispersive shallow water system. Netw. Heterog. Media, 11(1) :1–27, 2016.

[3] N. Aïssiouene, M.-O. Bristeau, E. Godlewski, and J. Sainte-Marie. A robust and stable numerical scheme for a depth-averaged Euler system. hal-01162109, submitted.

[4] M.-O. Bristeau, B. Di Martino, C. Guichard, and J. Sainte-Marie. Layer-averaged Euler and Navier-Stokes equations. Commun. Math. Sci., 15(5) :1221–1246, 2017.

[5] M.-O. Bristeau, N. Goutal, and J. Sainte-Marie. Numerical simulations of a non-hydrostatic shallow water model. Computers & fluids, 47(1) :51–64, 2011.

[6] L. Caudron. Contribution à l'étude des ondes de favre. Technical report, Technical Report HC. 032-E230, Laboratoire National d'Hydraulique, 1968.

[7] A. Barré de Saint-Venant. Théorie du mouvement non permanent des eaux, avec application aux crues des rivières et à l'introduction des marées dans leurs lits. C. R. Acad. Sci., 73 :237–240, 1871.

[8] M.W. Dingemans. Water wave propagation over uneven bottoms, volume 13. World Scientific, 1997.

[9] E.D. Fernandez-Nieto, M. Parisot, Y. Penel, and J. Sainte-Marie. Layer-averaged approximation of Euler equations for free surface flows with a non-hydrostatic pressure. Commun. Math. Sci., 16(05) :1169–1202, 2018.

[10] J.-F. Gerbeau and B. Perthame. Derivation of viscous Saint-Venant system for laminar shallow water; numerical validation. Discrete Contin. Dyn. Syst. Ser. B, 1(1) :89–102, 2001.

[11] A.E. Green and P.M. Naghdi. A derivation of equations for wave propagation in water of variable depth. J. Fluid Mech., 78(02) :237–246, 1976.

[12] D. Lannes. The Water Waves Problem: Mathematical Analysis and Asymptotics, volume 188. Amer. Math. Soc., 2013.

[13] M. Parisot. Entropy-satisfying scheme for a hierarchy of dispersive reduced models of free surface flow. `https://hal.archives-ouvertes.fr/hal-01242128`, 2019.

[14] F. Serre. Contribution à l'étude des écoulements permanents et variables dans les canaux. La Houille Blanche, (6) :830–872, 1953.