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Numerical boundary conditions for transport equations

The implementation of numerical boundary conditions is of crucial importance for the simulation of transport and other evolution phenomena. Yet a complete analysis of stability and/or accuracy issues depending on the type of boundary (inflow vs outflow) and the type of numerical schemes (e.g., finite difference schemes with one or more time levels) is missing. The aim of this project is to give a complete and rigorous description of the treatment of inflow boundary conditions for finite difference schemes with one time level.

More precisely, the aim of the project is to give some insight on high order (and possibly compact) discretizations of the rather elementary transport problem :

$$\begin{cases} \partial_t u + a\,\partial_x u \;=\; 0\,, & t \geq 0\,,\; x \in (0,L)\,, \\ u(0,x) \;=\; u_0(x)\,, & x \in (0,L)\,, \\ u(t,0) \;=\; g(t)\,, & t \geq 0\,. \end{cases}$$

Here $L > 0$ represents the length of the considered space interval and the transport velocity a is assumed to be postive. The initial condition u_0 and the boundary source term g are assumed to be smooth enough. Smoothness of the exact solution u necessitates the verification of compatibility conditions between u_0 and g at the time-space corner $t = x = 0$.

High order, stable and convergent discretizations for the above problem have been analyzed in [CL18] in the case of *homogeneous* incoming boundary conditions ($g \equiv 0$, in that case smoothness of the solution u is equivalent to u_0 being *flat* at 0). The goal of the project is to extend this result to the case of *non-homogeneous* incoming boundary conditions by developing a systematic construction and stability analysis for numerical boundary conditions arising from the so-called inverse Lax-Wendroff method, see e.g. [TS10,FY13,VS15]. Of course, analyzing particular examples may be useful before dealing with the general framework.

If time allows, convergence and stability results could be obtained for nonlinear problems, at least in the framework of monotone schemes.

References

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