FROM THE SWELL TO THE BEACH: MODELLING SHALLOW WATER WAVES

D. LANNES

1. General introduction

1.1. Brief overview.

Notation 1. Let us give here several notations that will be used throughout this paper.

- $d = 1, 2$ denotes the horizontal dimension and $X \in \mathbb{R}^d$ the horizontal variables; the vertical variable is denoted z.
- We denote by $\nabla_{X,z}$ the $(d+1)$-dimensional gradient operator, and by ∇ the $\mathbb{R}^d$-dimensional gradient taken with respect to the variable X only. Similar conventions are used for $\Delta_{X,z}$ and Δ.
- The velocity field in the fluid domain is denoted $\mathbf{U} \in \mathbb{R}^{d+1}$. We denote by $V \in \mathbb{R}^d$ and w its horizontal and vertical components respectively. When $d = 1$ we write v instead of V.
- We denote by $Q = \int_{-h_0+b}^{\zeta} V$ the horizontal discharge and by $\overline{V} = Q/h$ $(h = h_0 + \zeta - b)$ the vertically averaged horizontal velocity; in dimension $d = 1$, these quantities are denoted q and $\overline{v}$ respectively.
- We use the notation $f(D)$ for Fourier multipliers defined, when possible, by $\widehat{f(D)u} = \widehat{f}\widehat{u}$, the notation $\widehat{\cdot}$ standing for the Fourier transform on $\mathbb{R}^d$.

1.2. **The free surface Euler equations.**

Denoting by $X \in \mathbb{R}$ $(d = 1, 2)$ the horizontal coordinates and by z the vertical coordinate, we assume that the elevation of the surface of the water above the rest state $z = 0$ is given at time t by the graph of a function $\zeta(t, \cdot)$, and that the bottom is parametrized by a time independent

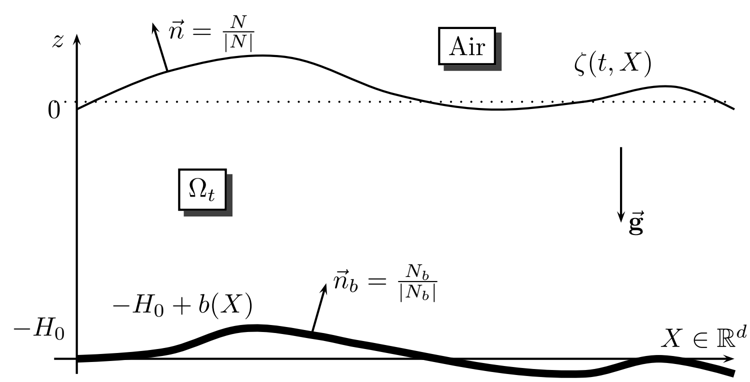

Figure 1. Main notations.

function $-h_0+b$ ($h_0>0$ is a constant); the domain occupied by the fluid at time t is therefore

$$\Omega_t=\{(X,z)\in\mathbb{R}^d\times\mathbb{R},-h_0+b(X)<z<\zeta(t,X)\}.$$

We also denote by $\mathbf{U}(t,X,z)\in\mathbb{R}^{d+1}$ the velocity of a fluid particle located at (X,z) at time t, and by $V(t,X,z)\in\mathbb{R}^d$ and $w(t,X,z)$ its horizontal and vertical component respectively. For a non viscous fluid of constant density ρ, the balance of forces in the fluid domain is given by the Euler equations

$$\partial_t\mathbf{U}+\mathbf{U}\cdot\nabla_{X,z}\mathbf{U}=-\frac{1}{\rho}\nabla_{X,z}P-g\mathbf{e}_z \quad \text{in} \quad \Omega_t, \tag{1}$$

where g is the acceleration of gravity and $\mathbf{e}_z$ is the unit upwards vertical vector. Incompressibility then takes the form

$$\nabla_{X,z}\cdot\mathbf{U}=0 \quad \text{in} \quad \Omega_t, \tag{2}$$

and we also assume that the flow is irrotational

$$\nabla_{X,z}\times\mathbf{U}=0 \quad \text{in} \quad \Omega_t; \tag{3}$$

we discuss in Section 3 how to remove this latter assumption.

In addition to the equations (1)-(3) which are given in the fluid domain Ω_t, we need boundary conditions. Two of them are given at the surface: the first one is the so-called kinematic boundary condition and expressed the fact that fluid particles do not cross the surface

$$\partial_t\zeta-\underline{U}\cdot N=0 \tag{4}$$

with the notations

$$\underline{U}(t,X)=\mathbf{U}(t,X,\zeta(t,X)) \quad \text{and} \quad N=\begin{pmatrix}-\nabla\zeta\\ 1\end{pmatrix};$$

the second boundary condition at the surface is the so-called dynamic boundary condition

$$P=P_{\rm atm}=\text{constant} \quad \text{on} \quad \{z=\zeta(t,X)\}. \tag{5}$$

Remark 1. The condition (6) means that surface tension is neglected, which is relevant for applications to coastal oceanography where the scales involved are significantly larger than the capillary scale; see for instance [Lan13] and references therein for generalizations including surface tension.
Inversely, the scales considered in coastal oceanography are in general small enough to neglect the variations of the atmospheric pressure. In some specific cases such as storms or meteotsunamis for instance, it is however relevant to consider a variable surface pressure [Ben15].

Finally, a last boundary condition is needed at the bottom, assumed to be impermeable

$$U_{\rm b}\cdot N_{\rm b}=0, \tag{6}$$

with the notations

$$U_{\rm b}(t,X)=\mathbf{U}(t,X,-h_0+b(t,X)) \quad \text{and} \quad N_{\rm b}=\begin{pmatrix}-\nabla b\\ 1\end{pmatrix}.$$

The question of solving equations (1)-(6) is a free surface problem in the sense that the equations are cast on a domain which is itself one of the unknowns (as Ω_t is determined by $\zeta(t\cdot)$). In order to solve it, it is necessary to find an equivalent formulation in which the equations are cast in a fixed domain. To mention only the local Cauchy problem, several equivalent formulations have been used: a Lagrangian formulation of the free surface in the pioneering work [Nal74] that solved the problem when $d = 1$ and for small data, as well as in [Wu97, Wu99] where the assumption of small data was removed and the result extended to the two dimensional case $d = 2$; a variational and geometrical approach based on Arnlold's remark that the motion of an inviscid incompressible fluid can be viewed as the geodesic flow on the infinite-dimensional manifold of volume-preserving diffeomorphisms [SZ08]; a full Lagrangian formulation of Euler's equations [Lin05, CS07], etc. We describe below to other formulations: one is Zakharov's Hamiltonian formulation [Zak68] whose well-posedness was proved in [Lan05] (and [ABZ14] for the low regularity Cauchy problem and [ASL08a, Igu09] for uniform bounds in several asymptotic regimes), as well as a formulation in (ζ, Q), where Q is the horizontal discharge, that proves very useful to derive and understand the mechanism at stakes in shallow water asymptotic models. For other recent mathematical advances on the water waves equations, such as long time/global existence, we refer to the surveys [IP17, Del18].

1.3. **The Zakharov-Craig-Sulem formulation.** From the irrotationality assumption, there exists a velocity potential Φ such that $\mathbf{U} = \nabla_{X,z}\Phi$. The Euler equation (1) reduces therefore to the Bernoulli equation

$$\partial_t \Phi + \frac{1}{2}|\nabla_{X,z}\Phi|^2 + gz = -\frac{P - P_{\rm atm}}{\rho}. \tag{7}$$

From the incompressibility condition (2) and the bottom boundary condition (6), we also know that $\Delta_{X,z}\Phi = 0$ in Ω_t and that $N_b \cdot \nabla_{X,z}\Phi = 0$ at the bottom. It follows that Φ (and therefore the velocity field $\mathbf{U}$) is fully determined by the knowledge of its trace ψ at the surface, $\psi(t, X) = \Phi(t, X, \zeta(t, X))$. The full water waves equations (1)-(6) can therefore be reduced to a set of two evolution equations on ζ and ψ. The equation for ζ is furnished by the kinematic equation (4) while the equation on ψ is obtained by taking the trace of the Bernoulli equation (7) at the surface. Zakharov remarked in [Zak68] that these equations can be put in canonical Hamiltonian form,

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \zeta} \\ \frac{\delta H}{\delta \psi} \end{pmatrix} = 0,$$

where the Hamiltonian is given by the mechanical (potential+kinetic) energy,

$$H(\zeta, \psi) = \frac{1}{2} \int_{\mathbb{R}^d} g\zeta^2 + \int_{\mathbb{R}^d} \int_{-h_0+b}^{\zeta} |\nabla_{X,z}\Phi|^2.$$

Introducing the Dirichlet-Neumann operator $G[\zeta, b]$ defined by

$$G[\zeta, b]\psi = N \cdot \nabla_{X,z}\Phi_{|_{z=\zeta}} \quad \text{where} \quad \begin{cases} \Delta_{X,z}\Phi = 0 & \text{in } \Omega \\ \Phi_{|_{z=\zeta}} = \psi, & N_b \cdot \nabla_{X,z}\Phi_{|_{z=-h_0+b}} = 0, \end{cases}$$

Craig and Sulem [CSS92, CS93] wrote the equation on ζ and ψ in explicit form

$$\begin{cases} \partial_t \zeta - G[\zeta, b]\psi = 0, \\ \partial_t \psi + g\zeta + \frac{1}{2}|\nabla\psi|^2 - \frac{1}{2}\frac{(G[\zeta, b]\psi + \nabla\zeta \cdot \nabla\psi)^2}{1 + |\nabla\zeta|^2} = 0. \end{cases} \tag{8}$$

The local well posedness of this formulation was proved in [Lan05]. Not to mention other related issues such as global well posedness for small data, this local existence result has been extended in two different directions: low regularity in [ABZ14] and uniform bounds in shallow water [ASL08a, Igu09]. These two extensions go somehow in two opposite directions as low regularity focuses on the behavior at high frequencies, while the shallow water limit, considered throughout these notes, is essentially a low frequency asymptotic.

1.4. **The (ζ, Q) formulation.** The Zakharov-Craig-Sulem equations are a set of evolution equations on two functions, ζ and ψ, that do not depend on the vertical variable z. Another way of getting rid of the vertical variable is to integrate vertically the free surface Euler equations. Denoting by V and w the horizontal and vertical components of the velocity field $\mathbf{U}$, this leads to the introduction of the *horizontal discharge* Q,

$$Q(t,X) := \int_{-h_0+b(X)}^{\zeta(t,X)} V(t,X,z)\mathrm{d}z; \tag{9}$$

integrating the horizontal component of the Euler equation (1) and using the boundary conditions (4) and (6), this gives

$$\begin{cases} \partial_t\zeta + \nabla\cdot Q = 0, \\ \partial_t Q + \nabla\cdot\Big(\int_{-h_0+b}^{\zeta} V\otimes V\Big) + \frac{1}{\rho}\int_{-h_0+b}^{\zeta}\nabla P = 0. \end{cases} \tag{10}$$

The next step is to decompose the pressure term. A special solution to the free surface Euler equations (1)-(6) corresponds to the rest state $\zeta = 0$, $U = 0$; the vertical component of the Euler equation (1) and the boundary condition (6) then give the following ODE for P,

$$-\frac{1}{\rho}\partial_z P - g = 0, \qquad P_{|z=0} = P_{\mathrm{atm}},$$

and the solution, $P = P_{\mathrm{atm}} - \rho g z$ is called *hydrostatic* pressure. When the fluid is not at rest, the solution to the ODE

$$-\frac{1}{\rho}\partial_z P - g = 0, \qquad P_{|z=\zeta} = P_{\mathrm{atm}},$$

namely, $P_{\mathrm{H}} = P_{\mathrm{atm}} - \rho g(z-\zeta)$ is still called hydrostatic and it is often convenient to decompose the pressure field P into its hydrostatic and non-hydrostatic components,

$$P = P_{\mathrm{atm}} + \rho g(\zeta - z) + P_{\mathrm{NH}};$$

integrating the vertical component of (1) from z to ζ and taking into account the boundary condition (6), one readily derives the following expression for the non-hydrostatic pressure,

$$P_{\mathrm{NH}}(t,X,z) = \rho\int_z^{\zeta(t,X)} (\partial_t w + \mathbf{U}\cdot\nabla_{X,z} w). \tag{11}$$

The evolution equations on ζ and Q can then be written under the form

$$\begin{cases} \partial_t\zeta + \nabla\cdot Q = 0, \\ \partial_t Q + \nabla\cdot\Big(\int_{-h_0+b}^{\zeta} V\otimes V\Big) + gh\nabla\zeta + \frac{1}{\rho}\int_{-h_0+b}^{\zeta}\nabla P_{\mathrm{NH}} = 0, \end{cases}$$

where h is the water height, $h = h_0 + \zeta - b$. The quadratic term in the second equation shows the importance of measuring the vertical dependance of the horizontal velocity V; this dependance is considered as a variation with respect to the vertical average of V. More precisely, we decompose the horizontal velocity field as

$$V(t, X, z) = \overline{V}(t, X) + V^*(t, X, z)$$

where for any function $f(t, \cdot)$ defined on the fluid domain Ω_t, we use the notation

$$\overline{f}(t, X) = \frac{1}{h}\int_{-h_0+b}^{\zeta} f(t, X, z)\mathrm{d}z \quad \text{and} \quad f^*(t, X, z) = f(t, X, s) - \overline{f}(t, X).$$

We can therefore write

$$\int_{-h_0+b}^{\zeta} V \otimes V = \frac{1}{h} Q \otimes Q + \mathbf{R} \quad \text{with} \quad \mathbf{R} = \int_{-h_0+b}^{\zeta} V^* \otimes V^* \tag{12}$$

so that the equations take the form

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + \nabla \cdot (\frac{1}{h} Q \otimes Q) + gh\nabla\zeta + \nabla \cdot \mathbf{R} + \frac{1}{\rho}\int_{-h_0+b}^{\zeta} \nabla P_{\mathrm{NH}} = 0. \end{cases} \tag{13}$$

Remark 2. The average horizontal velocity $\overline{V}$ and the horizontal discharge Q are related through $Q = h\overline{V}$. Instead of (13), one can therefore equivalently write a system of equations on the variables ζ and $\overline{V}$, namely,

$$\begin{cases} \partial_t \zeta + \nabla \cdot (h\overline{V}) = 0, \\ \partial_t \overline{V} + \overline{V} \cdot \nabla \overline{V} + g\nabla\zeta + \frac{1}{h}\nabla \cdot \mathbf{R} + \frac{1}{\rho h}\int_{-h_0+b}^{\zeta} \nabla P_{\mathrm{NH}} = 0. \end{cases} \tag{14}$$

Obviously, the last two terms of the second equation in (13) are the most complicated ones. To begin with, they are defined through (11) and (12) in terms of the velocity field $\mathbf{U}(t, X, z)$ and not in terms of ζ and Q. We can however state the following result.

Proposoition 1. *The equations* (13) *form a closed set of equations in* ζ *and* V. *More precisely, if we denote*

$$L^2_b(\Omega, \mathrm{div}, \mathrm{curl}\ ) := \{\mathbf{U} \in L^2(\Omega)^{d+1}, \mathrm{div}\mathbf{U} = 0, \mathrm{curl}\ \mathbf{U} = 0 \quad \textit{and} \quad U_b \cdot N_b = 0\},$$

then the discharge and reconstruction mappings respectively defined by

$$\mathfrak{D}[\zeta] : \begin{array}{lcl} L^2_b(\Omega, \mathrm{div}, \mathrm{curl}\ ) & \to & H^{1/2}(\mathbb{R}^d)^d \\ \mathbf{U} = \begin{pmatrix} V \\ w \end{pmatrix} & \mapsto & Q := \int_{-h_0+b}^{\zeta} V \end{array}$$

and

$$\mathfrak{R}[\zeta] : \begin{array}{lcl} H^{1/2}(\mathbb{R}^d)^d & \to & L^2_b(\Omega, \mathrm{div}, \mathrm{curl}\ ) \\ Q & \mapsto & \nabla_{X,z}\Phi \end{array} \quad \textit{with} \quad \begin{cases} \Delta_{X,z}\Phi = 0 \quad \textit{in } \Omega, \\ N \cdot \nabla_{X,z}\Phi_{|_{z=\zeta}} = -\nabla \cdot Q \\ N_b \cdot \nabla_{X,z}\Phi_{|_{z=-h_0+b}} = 0 \end{cases}$$

are well defined and $\mathfrak{R}[\zeta]$ *is a left-inverse to* $\mathfrak{D}[\zeta]$.

We refer to [Lan17] for the proof, which relies on the key observation that

$$N \cdot \nabla_{X,z}\Phi_{|_{z=\zeta}} = -\nabla \cdot \big(\int_{-h_0+b}^{\zeta} V \big).$$

As a consequence of Proposition 1, the last two terms in (13) are (non explicit, non local, non linear) functions of ζ and Q:

- Since $V^* = V - \overline{V}$ denotes the fluctuation of the horizontal velocity V with respect to its vertical average $\overline{V}$, of the horizontal velocity field. The tensor $\mathbf{R} = \int_{-h_0+b}^{\zeta} V^* \otimes V^*$ measures the contribution to the momentum equation of these fluctuations. It is therefore reminiscent of the Reynolds stress tensor in turbulence.
- The non-hydrostatic pressure contains nonlinear but also linear terms; as we shall see, it contains in particular the linear dispersive effects that are important for a good description of wave propagation.

These terms are very complex, but is possible to derive relatively simple asymptotic expansions in terms of ζ and Q in some particular regimes. In deep water, asymptotic models can be derived for waves of small steepness (see for instance [Mat92, Mat93, Cho95, CGH$^+$06, LB09, Lan13]), but we shall focus throughout these notes on *shallow water* models.

1.5. **Nondimensionalization of the equations.** In order to study the asymptotic behavior of the solutions to the water waves equations, it is convenient to introduce non-dimensionalized quantities based on the typical scales of the problem, namely: the typical depth h_0, the order of the surface variation $a_{\rm surf}$, the order of the bottom variations $a_{\rm bott}$ and the typical horizontal scale L. We can therefore form three dimensionless parameters

$$\mu = \frac{h_0^2}{L^2}, \qquad \varepsilon = \frac{a_{\rm surf}}{h_0}, \qquad \beta = \frac{a_{\rm bott}}{h_0}.$$

The first one is the *shallowness parameter*, the second the *amplitude parameters*, and the third the *topography parameter*. We are interested throughout this article in *shallow water configurations*, in the sense that μ is assumed to be small.

Remark 3. Another parameter, the *steepness* $\epsilon = \frac{a}{L} = \varepsilon\sqrt{\mu}$ is also found in the literature, but its main relevance is in intermediate to deep water, and it will therefore not been used in these notes.

Dimensionless quantities are defined as follows,

$$\widetilde{X} = \frac{X}{L}, \qquad \widetilde{z} = \frac{z}{h_0}, \qquad \widetilde{t} = \frac{t}{L/\sqrt{gh_0}},$$

$$\widetilde{\zeta} = \frac{\zeta}{a_{\rm surf}}, \qquad \widetilde{b} = \frac{b}{a_{\rm bott}}, \qquad \widetilde{Q} = \frac{Q}{a_{\rm surf}\sqrt{gh_0}}, \qquad \widetilde{w} = .\frac{w}{aL/h_0\sqrt{g/h_0}}.$$

Plugging into (13) then yields the dimensionless form of the equations. Omitting the tildes for the sake of clarity, they read

$$(15) \qquad \begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + \varepsilon \nabla \cdot (\frac{1}{h} Q \otimes Q) + h\nabla\zeta + \varepsilon \nabla \cdot \mathbf{R} + \frac{1}{\varepsilon}\int_{-1}^{\varepsilon\zeta} \nabla P_{\rm NH} = 0, \end{cases}$$

where the dimensionless water height is $h = 1 + \varepsilon\zeta - \beta b$ and the dimensionless "turbulent" tensor $\mathbf{R}$ and non-hydrostatic pressure are

$$\mathbf{R} = \int_{-1+\beta b}^{\varepsilon\zeta} V^* \otimes V^* \quad \text{and} \quad \frac{1}{\varepsilon} P_{\rm NH} = \int_z^{\varepsilon\zeta} \big(\partial_t w + \varepsilon V \cdot \nabla w + \frac{\varepsilon}{\mu} w \partial_z w\big), \tag{16}$$

with, in their dimensionless version,

$$\overline{V} = \frac{1}{h}\int_{-1+\beta b}^{\varepsilon\zeta} V(t,X,z)\mathrm{d}z \quad \text{and} \quad V^*(t,X,z) = V(t,X,z) - \overline{V}(t,X).$$

The equations (15) can equivalently be written in $(\zeta, \overline{V})$ variables (recall that $Q = h\overline{V}$),

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t\overline{V} + \varepsilon\overline{V}\cdot\nabla\overline{V} + \nabla\zeta + \varepsilon\frac{1}{h}\nabla\cdot\mathbf{R} + \frac{1}{\varepsilon h}\int_{-1}^{\varepsilon\zeta}\nabla P_{\rm NH} = 0. \end{cases} \tag{17}$$

Remark 4. Similarly, one can derive a dimensionless version of the Zakharov-Craig-Sulem formulation (8),

$$\begin{cases} \partial_t\zeta - \dfrac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b]\psi = 0, \\ \partial_t\psi + \zeta + \varepsilon\dfrac{1}{2}|\nabla\psi|^2 - \dfrac{1}{2}\varepsilon\mu\dfrac{(\frac{1}{\mu}G_\mu[\varepsilon\zeta,\beta b]\psi + \varepsilon\nabla\zeta\cdot\nabla\psi)^2}{1+\varepsilon^2|\nabla\zeta|^2} = 0. \end{cases} \tag{18}$$

where $G_\mu[\varepsilon\zeta,\beta b]\psi = \big(\partial_z\Phi - \varepsilon\mu\nabla\zeta\cdot\nabla\Phi\big)_{|z=\varepsilon\zeta}$ and

$$\begin{cases} (\partial_z^2 + \mu\Delta)\Phi = 0 & \text{for } -1+\beta b < z < \varepsilon\zeta \\ \Phi_{|z=\varepsilon\zeta} = \psi, & \big(\partial_z\Phi - \beta\mu\nabla b\cdot\nabla\Phi\big)_{|z=-1+\beta b} = 0. \end{cases}$$

Setting $\varepsilon = \beta = 0$, one gets the linearized water waves equations for a flat bottom. In this case, the equation for Φ can be explicitly solved and the Dirichlet-Neumann operator becomes a simple Fourier multiplier $G_\mu[0,0] = \sqrt{\mu}|D|\tanh(\sqrt{\mu}|D|)$. In particular, the linear dispersion relation for the water waves equations is

$$\omega_{\rm WW}^2 = k^2\frac{\tanh(\sqrt{\mu}k)}{\sqrt{\mu}k},$$

where $\mathbf{k}$ is a wave number of a plane wave solution of the linearized equations, $k = |\mathbf{k}|$ and $\omega_{\rm WW}$ the associated frequency.

2. The nonlinear shallow water equations and higher order approximation for irrotational flows

We derive and comment in this section several shallow water asymptotic models. In the dimensionless version of the water waves equations (15) there are the nonlocal "turbulent" and non-hydrostatic components. These two terms involve the velocity and pressure fields inside the fluid domain and if one wants to study their asymptotic behavior in shallow water it is therefore necessary to describe the inner structure of the velocity and pressure fields; this is performed in §2.1 and §2.2 respectively. The first model obtained in the shallow water asymptotics is the nonlinear shallow water (NSW) system; it is derived in §2.3 where its mathematical properties and several open problems are also reviewed. We then address in §2.4 the Boussinesq equations which furnish a second order approximation with respect to the shallowness parameter μ, but under a smallness assumption on the amplitude of the waves (weak nonlinearity). Removing this smallness assuption, one obtains the more general but more complex Serre-Green-Naghdi equations (SGN), which are derived and commented in §2.5. We thun turn in §2.7 to investigate one directional waves that are interesting because they can be described by a single scalar equation easier to analyse.

2.1. **The inner structure of the velocity field.** It is possible to describe the inner structure of the velocity field in shallow water by using the incompressibility and irrotationality conditions (2) and (3), as well as the bottom boundary condition (6). In their dimensionless version, these conditions become

$$\begin{cases} \mu\nabla\cdot V + \partial_z w = 0, \\ \partial_z V - \nabla w = 0, \\ \nabla^\perp \cdot V = 0, \\ w_b - \beta\mu\nabla b\cdot V_b = 0. \end{cases} \tag{19}$$

The first and last equations can be used to obtain

$$w = -\mu\nabla\cdot\big[(1+z-\beta b)\overline{V}\big] - \mu\nabla\cdot\int_{-1+\beta b}^{z} V^*.$$

and with the second equation this yields

$$\begin{aligned} V^* &= -\Big(\int_z^{\varepsilon\zeta}\nabla w\Big)^* \\ &= \mu\Big(\int_z^{\varepsilon\zeta}\nabla\nabla\cdot\big[(1+z'-\beta b)\overline{V}\big]\mathrm{d}z'\Big)^* + \mu\Big(\int_z^{\varepsilon\zeta}\nabla\nabla\cdot\int_{-1+\beta b}^{z} V^*\Big)^*. \end{aligned}$$

It is therefore natural to introduce the operator $\mathbf{T}[\varepsilon\zeta,\beta b]$ and $\mathbf{T}^*[\varepsilon\zeta,\beta b]$ acting on $\mathbb{R}^d$-valued functions defined on the fluid domain Ω and defined as

$$\mathbf{T}[\varepsilon\zeta,\beta b]W = \int_z^{\varepsilon\zeta}\nabla\nabla\cdot\int_{-1+\beta b}^{z'} W \quad\text{and}\quad \mathbf{T}^*[\varepsilon\zeta,\beta b]W = \big(\mathbf{T}[\varepsilon\zeta,\beta b]W\big)^*. \tag{20}$$

The above expression for V^* can then be written under the form

$$(1-\mu\mathbf{T}^*)V^* = \mu\mathbf{T}^*\overline{V}$$

so that

$$V^* = \mu\mathbf{T}^*\overline{V} + O(\mu^2).$$

Since $\overline{V}$ does not depend on z, the quantity $\mathbf{T}^*\overline{V}$ can be computed explicitly, leading to a shallow water expansion of the inner velocity field in terms of ζ and $\overline{V}$. When the bottom is flat ($b=0$), this expansion reads

$$\begin{cases} V = \overline{V} - \frac{1}{2}\mu\big((1+z)^2 - \frac{1}{3}h^2\big)\nabla\nabla\cdot\overline{V} + O(\mu^2), \\ w = -\mu(1+z)\nabla\cdot\overline{V} + O(\mu^2); \end{cases} \tag{21}$$

for the sake of clarity, the generalization in the presence of topography is given in (75) in Appendix A.

2.2. **The inner structure of the pressure field.** As already seen, the pressure field can be written as the sum of the hydrostatic pressure and a non-hydrostatic correction. In dimensionless variables, this reads

$$P = (\zeta - z) + P_{\rm NH} \quad \text{with} \quad \frac{1}{\varepsilon} P_{\rm NH} = \int_z^{\varepsilon\zeta} \big(\partial_t w + \varepsilon V \cdot \nabla w + \frac{\varepsilon}{\mu} w \partial_z w\big).$$

From the asymptotic expansion (21), we deduce that, when the bottom is flat,

$$\frac{1}{\varepsilon} P_{\rm NH} = -\mu \big[\frac{h^2}{2} - \frac{(1+z)^2}{2}\big] \big(\partial_t + \varepsilon \overline{V} \cdot \nabla - \varepsilon \nabla \cdot \overline{V}\big) \nabla \cdot \overline{V} + O(\mu^2); \tag{22}$$

we refer to (76) for the generalization of this formula when the bottom is not flat.

It follows that if one knows ζ and $\overline{V}$ (from experimental measurement or, approximately, by solving one of the asymptotic models derived below) then it is possible to reconstruct the pressure field in the fluid domain. An interesting problem for applications to coastal oceanography is the inverse problem: is it possible to reconstruct the surface elevation ζ by pressure measurements at the bottom (through pressure sensors lying on the sea bed). In the case of progressive waves (solitary or cnoidal waves), it is possible to do so (see for instance [OVDH12, CC13]) but the situation is more complex for general non progressive waves. Indeed, as many inverse problems, this reconstruction is an ill-posed problem (one roughly has to solve a Laplace equation in the fluid domain with no boundary condition at the surface and double Dirichlet and Neumann condition at the bottom). An heuristic formula was proposed in [VO17] and a weakly nonlinear reconstruction was derived in [BL17] (and experimentally validated with in situ measurements [BLMM18, MBL+19]) using an additional argument of nonsecular growth to circumvent this ill-posedness.

2.3. **First order approximation: the nonlinear shallow water equations.** The nonlinear shallow water equations are an approximation of order $O(\mu)$ of the water waves equations (15) in the sense that terms of order $O(\mu)$ are dropped. The main point consists therefore in studying the dependence of the "turbulent" and non-hydrostatic terms on μ.

From the results of §2.1 and §2.2, and recalling the definition (16) of $\mathbf{R}$ and $P_{\rm NH}$, we easily get that

$$\nabla \cdot \mathbf{R} = O(\mu^2) \quad \text{and} \quad \frac{1}{\varepsilon} \int_{-1+\beta b}^{\varepsilon\zeta} \nabla P_{\rm NH} = O(\mu).$$

Neglecting the $O(\mu)$ terms in the (ζ, Q) formulation of the water waves equations (15), one obtains the nonlinear shallow water equations (NSW),

$$\begin{cases} \partial_t \zeta + \nabla \cdot Q = 0, \\ \partial_t Q + \varepsilon \nabla \cdot (\frac{1}{h} Q \otimes Q) + h \nabla \zeta = 0, \end{cases} \qquad \text{for } t \geq 0, \quad x \in \mathbb{R}^d, \tag{23}$$

with $h = 1 + \varepsilon\zeta - \beta b$ (see (24) below for an equivalent formulation in $(\zeta, \overline{V})$ variables). This is a hyperbolic system of equations that furnishes a quite rough but very robust approximation for shallow water waves. We review below several known results and open problems related to the NSW model.

2.3.1. *The initial value (or Cauchy) problem for strong solutions to the NSW equations.* The NSW equations (23) can be equivalently written in $(\zeta,\overline{V})$ variables (recall that $Q=h\overline{V}$),

$$\begin{cases}\partial_t\zeta+\nabla\cdot(h\overline{V})=0,\\ \partial_t\overline{V}+\varepsilon\overline{V}\cdot\nabla\overline{V}+\nabla\zeta=0,\end{cases}\qquad \text{for } t\geq 0,\quad x\in\mathbb{R}^d \tag{24}$$

with $h=1+\varepsilon\zeta-\beta b$ and with initial condition

$$(\zeta,\overline{V})_{|t=0}=(\zeta^{\rm in},\overline{V}^{\rm in}). \tag{25}$$

There is local conservation of energy for the NSW equations,

$$\partial_t\mathfrak{e}_{\rm NSW}+\nabla\cdot\mathfrak{F}_{\rm NSW}=0, \tag{26}$$

with energy density and energy flux given by

$$\mathfrak{e}_{\rm NSW}=\frac{1}{2}\big[\zeta^2+h|\overline{V}|^2\big]\quad\text{ and }\quad\mathfrak{F}_{\rm NSW}=\big(\zeta+\varepsilon\frac{1}{2}|\overline{V}|^2\big)h\overline{V};$$

in particular, this yields conservation of the mechanical energy,

$$\frac{d}{dt}E_{\rm NSW}=0\quad\text{ with }\quad E_{\rm NSW}=\int_{\mathbb{R}^d}\mathfrak{e}_{\rm NSW}.$$

Under the non vanishing depth condition,

$$\exists h_{\min}>0,\qquad \sup_{X\in\mathbb{R}^d}h(t,X)\geq h_{\min}. \tag{27}$$

the conservation of $E_{\rm NSW}$ therefore furnishes a control of the L^2-norm of $(\zeta,\overline{V})$. The nonvanishing depth condition actually ensures that the NSW equations form a Friedrich symmetrizable hyperbolic system. It follows therefore from the general theory of Friedrich symmetrizable hyperbolic systems (see for instance [AG91, Tay97, BGS07]) that the initial value problem is locally well posed for times of order $O(1/\varepsilon)$ if the initial data $(\zeta^{\rm in},\overline{V}^{\rm in})$ belongs to $H^s(\mathbb{R}^d)^{1+d}$ with $s>1+d/2$ and satisfies the non vanishing depth condition (27). Note that the $O(1/\varepsilon)$ time scale for the life span of the solutions is optimal in dimension $d=1$ since shocks are known to develop at this time scale. Finally, let us mention that if the nonvanishing depth condition is relaxed, then the problem becomes a much more complex free boundary system of equations (see below).

2.3.2. *Weak solutions.* In the case of a flat topography ($b=0$) the NSW equations coincide with the isentropic Euler equations gor compressible gases, with h playing the role of the density and wih pressure law $\mathcal{P}(\rho)=\frac{1}{2}g\rho^2$, and it is therefore possible to use the construction of weak-entropy solutions following the dense literature on compressible gases, such as [DiP83, LPS96, CP12]; these solutions are obtained as the inviscid limit of viscous generalization of the NSW equations. We refer to [Bre09] for a review on these topics. Uniqueness remains an open problem.
The situation for the two-dimensional case is even more complicated, and almost nothing is known. As stated by Lax [Lax08],

> There is no theory for the initial value problem for compressible flows in two space dimensions once shocks show up, much less in three space dimensions. This is a scientific scandal and a challenge.

Fortunately,

> Just because we cannot prove that compressible flows with prescribed initial values exist doesnt mean that we cannot compute them

and indeed, shocks are computed for the NSW in many applications; in coastal oceanography for instance, shocks are relevant because they are used to describe broken waves. The mathematical entropy coincides for the NSW equations with the energy; the dissipation of entropy associated to weak entropy solutions is therefore a dissipation of energy that corresponds with a pretty good accuracy to the energy actually dissipated by wave breaking [BBC10].

2.3.3. *Initial Boundary value problems.* The equations (24) are cast on $\mathbb{R}^d$ but the equations must sometimes be considered in a domain with a boundary. This boundary can be physical (e.g. a wall) or artificial: for instance, for numerical simulations, one has to consider a bounded domain whose boundary has no physicial relevance. For the sake of clarity, let us discuss first the one-dimensional case $d = 1$, on a finite interval $[0, L]$,

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{v}) = 0, \\ \partial_t \overline{v} + \varepsilon \overline{v} \cdot \partial_x \overline{v} + \partial_x \zeta = 0, \end{cases} \qquad \text{for } t \geq 0, \quad x \in (0, L) \tag{28}$$

with $h = 1 + \varepsilon\zeta - \beta b$ and with initial condition

$$(\zeta, \overline{v})_{|t=0} = (\zeta^{\rm in}, \overline{v}^{\rm in}) \quad \text{on} \quad [0, L]. \tag{29}$$

In addition, boundary conditions must be imposed at $x = 0$ and $x = L$. Some examples of boundary conditions are

- Generating boundary conditions. The water elevation is knwon (from buoy measurements for instance) at the entrance of the domain and prescribed as a boundary data,
$$\zeta(t, 0) = f(t);$$
in this case, the boundary $x = 0$ is non physical.
- Wall. There is a fixed wall located at $x = L$, on which the waves bounces back. In this case the boundary $x = L$ is physical and the corresponding boundary condition is
$$\overline{v}(t, L) = 0.$$
- Transparent conditions. Such boundary conditions are very important for numerical simulations in the cases where there is no physical boundary condition at $x = L$ and one wants to impose a boundary condition that does not create any artificial reflexion. In the particular case of the NSW in dimension $d = 1$, a simple analysis of the Riemann invariants shows that such a condition is given by
$$R_-(\zeta, \overline{v}) := 2(\sqrt{h} - 1) - \varepsilon \overline{v} = 0 \qquad \text{at } x = L,$$
where R_- is the left going Riemann invariant (see §2.7.1 below for more details).

Initial boundary value problems for hyperbolic systems have been considered quite intensively [Maj83a, Maj83b, Maj12, Mét01, Mét12, Fre98, BGS07, Cou03]; we refer to [ILar] for a sharp general theory in dimension $d = 1$ showing that such problems are locally well-posed in H^m ($m \geq 2$) under suitable compatibility conditions. In the particular case of 2×2 systems, an analysis based on Riemann invariants can

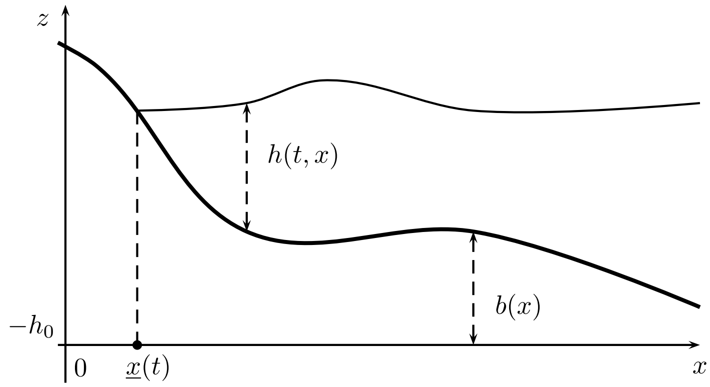

FIGURE 2. The shoreline problem in dimension $d = 1$

also be performed [LY85], and proves very useful for numerical implementation (see for instance [Mar05, LW]). In dimension $d = 2$, the "wall" boundary condition $\overline{V} \cdot \mathbf{n} = 0$ can be deduced from classical works on the compressible Euler equations [Sch86] but other types of boundary conditions are much more delicate and remain an open problem.

2.3.4. *A free boundary problem: the shoreline problem.* The nonvanishing depth condition (27) is of course a serious restriction for applications to coastal oceanography, where one typically has to deal with beaches. Let us consider the case for instance where the shoreline is at time t the graph of some function $y \in \mathbb{R} \mapsto \underline{X}(t,u)$ if $d = 2$ (and a single point $\underline{x}(t)$ if $d = 1$) and that the sea is, say, on the right part of the shoreline (see Figure 2). The initial value problem is then much more difficult since it is now a free boundary problem: one must solve the NSW equations on $\Omega_t = \{X = (x,y) \in \mathbb{R}^2, x > \underline{X}(t,y)\}$ (or $\Omega_t = \{x \in \mathbb{R}, x > \underline{x}(t,)\}$ if $d = 1$) whose boundary, the shoreline (or more accurately, its projection on the horizontal plane) evolves according to the kinematic equation

$$\partial_t \underline{X} = \overline{V}_{|x=\underline{X}(t,y)} \cdot \begin{pmatrix} 1 \\ \partial_y \underline{X} \end{pmatrix} \qquad (\text{or } \underline{x}'(t) = u(t, \underline{x}(t)) \text{ if } d = 1), \tag{30}$$

which involves the trace at the boundary of the velocity. A reasonable assumption to solve this free boundary problem is to assume that the surface of the water is transverse to the bottom topography at the shoreline in the following sense

$$\partial_\nu h < 0 \text{ on } \{X = \underline{X}(t,y)\}, \tag{31}$$

where ν is the outwards unit normal to Ω_t (if $d = 1$ this condition reduces to $h'(t, \underline{x}(t)) > 0$). Proving that the shoreline problem is well-posed consists in proving that there exists a smooth enough family of mapping $t \mapsto \underline{X}(t, \cdot)$ (or simply $t \mapsto \underline{x}(t)$) on some time interval $[0,T]$ and a family of smooth enough functions ζ and $\overline{V}$ solving the nonlinear shallow water on Ω_t and the kinematic equation (30). In dimension $d = 1$, such a result can be found in [LM18] as a particular case of a more general result for the Green-Naghdi equations, but the dispersive terms of this latter make the analysis more complicated than necessary, and the proof could certainly be simplified considerably if one is only interested in the nonlinear shallow water equations. Let us also mention that the isentropic Euler equations for compressible gases has been solved in [JM09] and [CS11] for $d = 1$ and [JM15] and [CS12] for $d \geq 2$ under the assumption that of a physical boundary condition at the interface with vacuum (using the terminology of [LY00]), namely,

$$-\infty < \partial_\nu c^2 < 0 \text{ at the inferface with vacuum,}$$

where $c = (\mathcal{P}'(\rho))^{1/2}$ is the sound speed. Using the analogy mentioned in §2.3.2, the vacuum problem with physical boundary condition exactly coincides with the shoreline problem with transversality condition (31) in the case of a flat topography ($b = 0$); an extension of the techniques of the above references to the case of a nonflat topography looks feasable and could be done to cover the two-dimensional case $d = 2$.

2.4. **Weakly nonlinear second order approximations: the Boussinesq equations.** Compared to the NSW equations, the Boussinesq equations have a better precision, namely, $O(\mu^2)$ instead of $O(\mu)$, but require an additional assumption of *weak nonlinearity* that can be formulated as a smallness condition on ε,

$$\text{Weak nonlinearity:} \qquad \varepsilon = O(\mu). \tag{32}$$

Traditionnally (but not always as we shall see below for the Boussinesq-Peregrine model), an assumption on the smallness of the topography variations is also made,

$$\text{Small topography variations:} \qquad \beta = O(\mu). \tag{33}$$

Under these two assumptions, terms of size $O(\varepsilon\mu)$ and $O(\beta\mu)$ can be treated as $O(\mu^2)$ terms, and the results of §2.1 and §2.2 yield the following approximations on the turbulent and non-hydrostatic terms $\mathbf{R}$ and $P_{\rm NH}$ defined in (16),

$$\begin{aligned}
\nabla \cdot \mathbf{R} &= O(\mu^2) \\
\frac{1}{\varepsilon}\int_{-1}^{\varepsilon\zeta} \nabla P_{\rm NH} &= -\mu\frac{1}{3}\nabla\nabla\cdot\partial_t\overline{V} + O(\mu^2) \\
&= -\mu\frac{1}{3}\Delta\partial_t\overline{V} + O(\mu^2),
\end{aligned}$$

the last identity stemming from the third equation in (19) and (21). Plugging these approximations into (17) and dropping the $O(\mu^2)$ terms, one obtains the following Boussinesq equations

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ (1-\mu\frac{1}{3}\Delta)\partial_t\overline{V} + \varepsilon(\overline{V}\cdot\nabla\overline{V}) + \nabla\zeta = 0. \end{cases} \tag{34}$$

Remark 5. The irrotationality assumption has been used to replace $(1-\mu\frac{1}{3}\nabla\nabla^{\rm T})\partial_t\overline{V}$ by the simpler term $(1-\mu\frac{1}{3}\Delta)\partial_t\overline{V}$. In the presence of vorticity, it is in general not possible to do so (see §3.5.2 below).

There is actually not a single Boussinesq model, but a whole family. There are various reasons why many formally equivalent Boussinesq models have been derived, such as their mathematical structure (well-posedness, conservation of energy, integrability, solitary waves, etc.) or their physical properties. Among the latters, the linear dispersive properties of these models is a central question. The linear dispersion associated to (34) is

$$\omega^2 = \frac{k^2}{1+\frac{1}{3}\mu k^2}$$

where $\mathbf{k}$ is a wave number, $k = |\mathbf{k}|$ and ω the associated frequency. This dispersion relation is as expected a $O(\mu^2)$ approximation of the linear dispersion relation of the full water waves equations (see Remark 4),

$$\omega_{\rm WW}^2 = k^2 \frac{\tanh(\sqrt{\mu}k)}{\sqrt{\mu}k},$$

but the two formulas differ significantly when $\sqrt{\mu}k$ is not very small (i.e. for shorter waves and/or larger depth). It is possible to derive Boussinesq models with better dispersive properties and that differ from (34) by $O(\mu^2)$ terms, and therefore keep the same overall $O(\mu^2)$ precision. These new Boussinesq systems depend on several parameters. The first one can be introduced using the so-called BBM trick [BBM72] that is based on the observation that

$$\begin{aligned}\partial_t \overline{V} &= -\nabla\zeta + O(\mu),\\ &= \alpha\partial_t\overline{V} - (1-\alpha)\nabla\zeta + O(\mu),\end{aligned}$$

for any real number α. This substitution can be made in the dispersive term in the second equation of (34),

$$-\mu\frac{1}{3}\Delta\partial_t\overline{V} = -\mu\alpha\frac{1}{3}\Delta\partial_t\zeta + \mu\frac{1}{3}(1-\alpha)\Delta\nabla\zeta + O(\mu^2)$$

and induces only a $O(\mu^2)$ modification of (34); the resulting model therefore keeps the overall $O(\mu^2)$ precision of (34). Other parameters can be introcuced, following an idea of Nwogu [Nwo93], by making a change of of unknown for the velocity. More precisely, we introduce the velocity $V_{\theta,\delta}$ by

$$V_{\theta,\delta} = (1-\mu\theta\frac{1}{3}\Delta)^{-1}(1-\mu\delta\frac{1}{3}\Delta)\overline{V} \tag{35}$$

(this new quantity $V_{\theta,\delta}$ is an approximation of the velocity field at some level line in the fluid domain, see for instance [Lan13]). Finally, a fourth parameter λ can be introduced by remarking that since we have $\partial_t\zeta = -\nabla\cdot\overline{V}_{\theta,\delta} + O(\mu)$ from the first equation, it is possible to add $-\mu\frac{\lambda}{3}(\Delta\partial_t\zeta - \Delta\nabla\cdot V_{\theta,\delta})$ to the first equation (this is a variant of the BBM trick used above). One finally obtains the so called *abcd* Boussinesq systems [BCS02, BCS04, BCL05],

$$\begin{cases}(1-\mu\mathbf{b}\Delta)\partial_t\zeta + \nabla\cdot(hV) + \mu\mathbf{a}\Delta\nabla\cdot V + 0,\\ (1-\mu\mathbf{d}\Delta)\partial_t V + \nabla\zeta + \varepsilon(V\cdot\nabla)V + \mu\mathbf{c}\Delta\nabla\zeta = 0,\end{cases} \tag{36}$$

where $h = 1 + \varepsilon\zeta - \beta b$, V stands for $V_{\theta,\delta}$ and

$$\mathbf{a} = -\frac{\theta+\lambda}{3},\quad \mathbf{b} = \frac{\delta+\lambda}{3},\quad \mathbf{c} = -\frac{\alpha+\delta-1}{3},\quad \mathbf{d} = \frac{\alpha+\theta}{3}$$

(so that $\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d} = \frac{1}{3}$). This family of approximation can be extended by changing the structure of the nonlinearity [BCL05, Cha07].

Remark 6. For the NSW equations, the (ζ, Q) formulation (23) and the $(\zeta, \overline{V})$ formulation (23) are totally equivalent for smooth solutions, and this will also prove true for the Serre-Green-Naghdi equations. However, such an equivalence does not hold for the Boussinesq systems. We derived the *abcd* family of Boussinesq systems (36) from the $(\zeta, \overline{V})$ formulation (17) of the water waves equation; the same procedure applied to the (ζ, Q) formulation (15) leads to slightly different models; we refer to [FBCR15] for an analysis of the slight differences between these models.

Let us conclude this small survey on Boussinesq systems by considering what happens if the assumption (33) of small topogrpahy variations is not made. Since β must now be considered as a $O(1)$ rather than $O(\mu)$ quantity, the expansion given above for the non-hydrostatic term must be revisited. We now get from §2.1 and §2.2 that

$$\frac{1}{h}\frac{1}{\varepsilon}\int_{-1}^{\varepsilon\zeta} \nabla P_{\rm NH} = \mu \mathcal{T}_b \partial_t \overline{V} + O(\mu^2),$$

where

$$\mathcal{T}_b V = -\frac{1}{3h_b}\nabla\cdot(h_b^3\nabla\cdot V) + \frac{\beta}{2h_b}\big[\nabla(h_b^2\nabla b\cdot V) - h_b^2\nabla b\cdot\nabla\cdot V\big] + \beta^2\nabla b\nabla b\cdot V$$

(notice that $h_b\mathcal{T}_b$ is a positive second order elliptic operator). Plugging this approximation into (17) and dropping the $O(\mu^2)$ terms, one obtains the Boussinesq-Peregrine [Per67] system

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ (1+\mu\mathcal{T}_b)\partial_t\overline{V} + \nabla\zeta + \varepsilon(\overline{V}\cdot\nabla)\overline{V} = 0; \end{cases} \tag{37}$$

a generalization of the *abcd* systems for large topography variations can be derived from (37) by adapting the above procedure (see [Lan13]).

Let us now describe some of mathematical results and open problems dealing with the Boussinesq models derived in this section.

2.4.1. *The initial value problem for strong solutions.* The (hyperbolic) NSW equations (24) are locally well posed in Sobolev spaces over a $O(1/\varepsilon)$ time scale and this is sharp because shocks occur for such times. The Boussinesq systems being a dispersive perturbation of the NSW equations, one expects that solutions to locally well posed Boussinesq models should exist on a time scale which is at least $O(1/\varepsilon)$. One may expect dispersion to help, but methods based on dispersive estimates only yield an existence time of order $O(1/\sqrt{\varepsilon})$ [LPS12]. A convenient and easy option to reach the $O(1/\varepsilon)$ time scale is to work with *abcd* systems with a symmetrized nonlinearity [BCL05, Cha07, Lan13]; this time scale has finally beed proved for the original *abcd* systems in a series of papers [MSZ12, SX12, Bur16b, Bur16a, SWX17] for all the linearly well posed *abcd* systems, except for the case $\mathbf{b} = \mathbf{d} = 0$ and $\mathbf{a} = \mathbf{c} > 0$ which remains open.

The above references (except [Cha07]) deal with a flat topography but, as remarked in [SX12], it is not difficult to extend them to the case of a non flat topography satisfying the assumption (33) of small topography variations. Proving existence over $O(1/\varepsilon)$ times is much more difficult for Boussinesq models with large topography variations (i.e. without assumption (33)) such as the Boussinesq-Peregrine model (37). Such a result has only been proved [MG17] for a variant of the Boussinesq-Peregrine model (37) taylored to allow the implementation of techniques developed in [BM10] for the lake equations.

There are surprisingly few results regarding global existence. This has been proved for the "standard" Boussinesq system (34) in [Sch81, Ami84], where a weak solution is constructed using a parabolic regularization of the mass conservation equation and mimicking the hyperbolic theory; the solution is then proved to be regular and unique. For the general *abcd* systems (36), global well posedness have been proved in some specific cases using the particular structure of the equations, such as the Bona-Smith system ($\mathbf{a} = -1/3$, $\mathbf{b} = 0$, $\mathbf{c} = -1/3$, $\mathbf{d} = 1/3$) [BS76] and the Hamiltonian cases ($\mathbf{b} = \mathbf{d} > 0$, $\mathbf{a} \leq 0$, $\mathbf{c} < 0$) [BCS04]. When $\mathbf{b} = \mathbf{d} < 0$ refined stattering results in the energy space have also been proved [KMPP19, KM19].

2.4.2. *Initial boundary value problems.* The problem of initial boundary value problems is extremely important for applications to coastal oceanography and several numerical solutions have been proposed, such as the source function method [WKS99] for instance; these methods however are not fully satisfactory and require a significant increase of computational time.

In contrast with hyperbolic systems of equations for which the initial boundary value problem has been intensively studied, there is almost no theoretical result if a dispersive perturbation is added to the equations, as this is the case for the Boussinesq equations. There are only some results concerning the one dimensional case, particular examples of the *abcd* damily (36) and/or specific boundary conditions: homogeneous boundary conditions as in [Xue08], or [BC98, ADM09] for the Bona-Smith system, where the regularizing dispersive term of the first equation (due to the fact that $\mathbf{b} > 0$) plays a central role. In [LW], *generating* boundary conditions (see §2.3.3) have been considered for the Boussinesq-Abott system, a dispersive perturbation of the NSW equations written in (ζ, q) variables (23). This latter reference is based on the concept of dispersive boundary layer introduced in [BLM19] for the analysis of a wave-structure interaction problem; it provides a local well-posedness of the initial boundary value problem. However, as the other local well-posedness results given in the above references, the existence time thus obtained is far from the $O(1/\varepsilon)$ time scale which, as seen above, is the relevant one. Reaching such a time-scale is considerably more difficult and requires a precise analysis of the dispersive boundary layer; to this day such an analysis has only been performed in [BLM19].

Another relevant issue is the convergence towards the initial boundary value problem for the NSW equations as the dispersive (or shallowness) parameter μ tends to zero; here again, the analysis of the dispersive boundary layer should be a key point (such a convergence has been proved in [BLM19]).

For *transparent* boundary conditions (which allow waves to cross the boundary of the computational domain without reflexion, see §2.3.3), the situation looks even more complicated. There are some results for the linear problem: for scalar equations (linear KdV or BBM for instance) [BMGN18, BNS17] and for the linearization of (34) around the rest state [KN17]. The nonlinear case remains open.

2.5. **Second order approximation: the Serre-Green-Naghdi equations and variants.** The Serre-Green-Naghdi (SGN) equations are an approximation of order $O(\mu^2)$ of the water waves equations (15) in the sense that the terms of order $O(\mu)$ that were neglected in the nonlinear shallow water equations are kept, and only terms of order $O(\mu^2)$ are dropped. The precision of this model is therefore the same as the precision of the Boussinesq models investigated in §2.4, but they have a wider range of application since they do not require the weak nonlinearity assumption (32) nor the weak topography assumption (33). The price to pay is that the $O(\varepsilon\mu)$ and $O(\beta\mu)$ terms must be kept in the model, making it more complicated than the Boussinesq systems (36). For the sake of clarity, we consider here the case of a flat bottom only ($b = 0$) and refer to Appendix A for the equations with a non flat topography.

The "turbulent" and non-hydrostatic terms in (15) can be expended as follows, following the results of §2.1 and §2.2,

$$\nabla \cdot \mathbf{R} = O(\mu^2)$$
$$\frac{1}{\varepsilon}\int_{-1}^{\varepsilon\zeta} \nabla P_{\rm NH} = \mu h \mathcal{T}\big[\partial_t \overline{V} + \varepsilon \nabla \cdot \big(h\overline{V}\otimes\overline{V}\big)\big] + \mu\varepsilon h \mathcal{Q}_1\big(\zeta,\overline{V}\big) + O(\mu^2)$$

where

$$\mathcal{T}V = -\frac{1}{3h}\nabla\big(h^3\nabla\cdot V\big),$$
$$\mathcal{Q}_1(\zeta,V) = \frac{2}{3h}\nabla\big[h^3\big(\partial_x V\cdot\partial_y V^{\perp} + (\nabla\cdot V)^2\big)\big].$$

Therefore, even in a fully nonlinear regime and with the higher $O(\mu^2)$ precision of the SGN equations, the contribution of the "turbulent" term $\varepsilon\nabla\cdot\mathbf{R}$ remains too small to be relevant and can be neglected. All the additional terms of the SGN equations with respect to the NSW equation are therefore due to the non hydrostatic pressure. Plugging the above expansions into (15) and dropping the $O(\mu^2)$ terms, one obtains the SGN equations,

$$\begin{cases}\partial_t\zeta + \nabla\cdot Q = 0,\\ (1+\mu\mathbf{T})\big[\partial_t Q + \varepsilon\nabla\cdot\big(\frac{1}{h}Q\otimes Q\big)\big] + h\nabla\zeta + \varepsilon\mu h\mathcal{Q}_1(h,\frac{Q}{h}) = 0,\end{cases} \tag{38}$$

where $\mathbf{T} = h\mathcal{T}\frac{1}{h}$. We refer to (77) for the generalization of these equations when the topography is not flat. These equations are actually known under different names, such as Serre [Ser53, SG69], Green-Naghdi [GN76, KBEW01], or fully nonlinear Boussinesq [WKGS95].

Remark 7. Replacing $Q = h\overline{V}$ in (38), one obtains the following equivalent formulation (as far as smooth solutions are concerned) in $(\zeta,\overline{V})$,

$$\begin{cases}\partial_t\zeta + \nabla\cdot(h\overline{V}) = 0,\\ (1+\mu\mathcal{T})\big[\partial_t\overline{V} + \varepsilon\nabla\cdot\big(h\overline{V}\otimes\overline{V}\big)\big] + \nabla\zeta + \varepsilon\mu\mathcal{Q}_1(h,\overline{V}) = 0.\end{cases} \tag{39}$$

As in the weakly nonlinear regime with the Boussinesq equations, it is possible to derive formally equivalent systems using similar procedures (the "BBM-trick" and a change of unknown for the velocity); a family of SGN equations generalizing the *abcd* Boussinesq systems (36) can be derived [CLM11, Lan13]. In a similar vein, it is possible to derive equivalent systems (i.e. systems that differ formally from (38) by $O(\mu^2)$ terms) that have a better mathematical structure [Isr10a, Isr11] or that are more adapted to numerical computations [LM15].

2.5.1. *Known results and open problems.* We review here several known results and open problems about the SGN equations.
- *Initial value problems and singularity formation.* There is local conservation of energy for the SGN equations,

$$\partial_t\mathfrak{e}_{\rm SGN} + \nabla\cdot\mathfrak{F}_{\rm SGN} = 0, \tag{40}$$

with energy density and energy flux given (when the bottom is flat, see [CL14] for the generalization to non flat bottoms) by

$$\mathfrak{e}_{\rm SGN} = \frac{1}{2}\big[\zeta^2 + h|\overline{V}|^2 + \mu\frac{1}{6}h^3|\nabla\cdot\overline{V}|\big], \tag{41}$$

$$\mathfrak{F}_{\rm SGN} = \big[\zeta + \frac{1}{2}|\overline{V}|^2 + \mu\frac{1}{6}h^2|\nabla\cdot\overline{V}| - \mu\frac{1}{3}h(\partial_t + \varepsilon\overline{V}\cdot\nabla)(h\nabla\cdot\overline{V})\big]h\overline{V} \tag{42}$$

Integrating over $\mathbb{R}^d$, this yields conservation of the mechanical energy,

$$\frac{d}{dt}E_{\rm SGN} = 0 \quad \text{with} \quad E_{\rm SGN} = \int_{\mathbb{R}^d}\mathfrak{e}_{\rm SGN}$$

In addition to the control of the L^2-norm of $(\zeta, \overline{V})$ that we had for the NSW equations, we now have a control of $\sqrt{\mu}\nabla\cdot\overline{V}$ provided that the nonvanishing conditions (27) is satisfied. This extra control allows one to control the extra nonlinear terms $\varepsilon\mu\mathcal{Q}_1(h, \overline{V})$ in (39) which has therefore a semi-linear structure. Local existence was proved in [Li06] for small times, and in [ASL08b, Isr11] for times of order $O(1/\varepsilon)$, uniformly with respect to $\mu \in (0, 1)$. Contrary to the NSW equations, the SGN equations contain third order dispersive term that play a regularizing role. The question of global well posedness therefore becomes relevant, and one could conjecture in dimension $d = 1$ a scenario similar to the one observed for the Camassa-Holm equation which is somehow the "unidirectional version" of the SGN equations (see below), namely: one has global existence for some data and wave breaking for others (i.e., the L^∞-norm is bounded but the derivative of the velocity and/or the surface elevation blows up in finite time). This scenario is supported showing that there exist shocks relating a constant state to a periodic wave train, and that, at least numerically, such shocks can be dynamically obtained [GNST].

- *Initial boundary value problems.* With respect to the NSW equations, the new dispersive and nonlinear terms of the SGN equation render the analysis much more complicated in the presence of a boundary. The case of a wall boundary condition $\overline{V}\cdot\mathbf{n} = 0$ is the simplest one since the boundary terms in the energy estimates vanish. In the particular one dimensional case $d = 1$, the result can be adapted from [LM18] but considerable simplifications could be made using the nonvanishing depth condition (27). Even in dimension $d = 1$, other types of boundary conditions (e.g. generating and transparent) are much more complex and remain open. The case of transparent boundary conditions for the linearized SGN equations around the rest state (which are actually the same as the linearized Boussinesq equations around the rest state) has been addressed in [KN17].

In view of the difficulty of the nonlinear case, an alternative has been proposed, consisting in implementing a perfectly matched layer (PML) approach for a hyperbolic relaxation of the Green-Naghdi equations [Kaz18]. This approach can also be used to deal with generating boundary conditions but the size of the layer in which the PML approach is implemented is typically of two wavelength, which for applications to coastal oceanography can typically represent an increase of 100% of the computational domain. Other methods such as the source function method [WKS99] also require a significant increase of computational time.

- *Free boundary problems: the shoreline problem.* As for the NSW equation, it is natural to remove the nonvanishing depth condition (27) and to consideer the shoreline problem (see above). This problem has been solved in dimension $d = 1$ in [LM18], but the two dimensional case remains open.

2.6. **Other models.** A word on full dispersion models:
Multilayer [ZS08, MSK12, FNPPSM18]
Isobe Kakinuma [Igu18a, NI18, Igu18b]

2.7. **Scalar models.** Intuitively, in the one dimensional case $d = 1$, waves can be decomposed into components that "go to the left" or "go to the right". It is therefore not a surprise that waves are then governed by a system of two scalar evolution equations. The idea behing scalar asymptotic models is that if we want to describe only waves that go mainly, say, "to the right", then a single scalar equation should be enough. We make this idea more precise in this section.
Throughout this section, we shall focus on the case of a flat topography $b = 0$. We refer for instance to [Joh73, Mil79, vGP93, Isr10b] for generalizations to a non flat topography.

In dimension $d = 1$, the SGN equations (39) reduce at leading order in ε and μ to the linear wave equation

$$\begin{cases} \partial_t \zeta + \partial_x \overline{v} = 0, \\ \partial_t \overline{v} + \partial_x \zeta = 0, \end{cases}$$

so that any perturbation of the rest state can be decomposed into a left-going and a right-going wave. Purely right-going waves are obtained when $\zeta = u$ and are therefore determined by

$$(\partial_t + \partial_x)\zeta = 0 \quad \text{and} \quad u = \zeta. \tag{43}$$

The scalar models that are described below generalize this approach to more complex asymptotic models than the linear wave equation.

2.7.1. *A fully nonlinear, nondispersive model.* Let us consider here the NSW equations which is fully nonlinear (no smallness assumption on ε) but neglects all the terms of order $O(\mu)$ (where the dispersive terms are, as shown above); this is equivalent to taking $\mu = 0$ in (39),

$$\begin{cases} \partial_t \zeta + \partial_x (h\overline{v}) = 0, \\ \partial_t \overline{v} + \varepsilon \overline{v} \partial_x \overline{v} + \partial_x \zeta = 0. \end{cases} \qquad (h = 1 + \varepsilon\zeta) \tag{44}$$

In the subcritical case, i.e. when $h - \varepsilon^2 \overline{v}^2 > 0$, this hyperbolic system can be diagonalized using the Riemann invariants. More precisely, introducing

$$R_\pm(\zeta, \overline{v}) = 2\big(\sqrt{h} - 1\big) \pm \varepsilon \overline{v} \quad \text{and} \quad \lambda_\pm(\zeta, \overline{v}) = \pm\varepsilon\overline{v} + \sqrt{h},$$

the NSW equation can be diagonalized into two coupled transport equations

$$(\partial_t \pm \lambda_\pm \partial_x) R_\pm = 0.$$

Purely right-going waves are therefore obtained if $R_- = 0$ and therefore characterized by

$$\partial_t \zeta + \partial_x \zeta + 3\varepsilon \frac{\zeta}{1 + \sqrt{1 + \varepsilon\zeta}} \partial_x \zeta = 0, \quad \text{and} \quad \overline{v} = \frac{2}{\varepsilon}\big(\sqrt{1 + \varepsilon\zeta} - 1\big); \tag{45}$$

as expected, this is a $O(\varepsilon)$ perturbation of the relations defining right-going waves for the linear waves equations. The equation for ζ is a non-viscous Burgers equations whose solutions form shocks at the time scale $O(1/\varepsilon)$. Note that solutions to the scalar model (45) are *exact* solutions to the NSW system (44), sometimes called simple waves.

Remark 8. In (45), ζ is determined through the resolution of a scalar evolution equation, and $\overline{v}$ is given by an algebraic expression in terms of z. It is of course possible to switch the roles of ζ and $\overline{v}$, leading to another kind of simple wave,

$$\partial_t\overline{v} + \partial_x\overline{v} + \varepsilon\frac{3}{2}\overline{v}\partial_x\overline{v} = 0 \quad \text{and} \quad \zeta = \overline{v} + \varepsilon\frac{1}{4}\overline{v}^2. \tag{46}$$

2.7.2. *A fully dispersive, linear model.* The symmetric case compared with the Burgers model (45) is to neglect all the nonlinearities ($\varepsilon = 0$) and to keep all the terms in μ (the validity of the resulting model is therefore not restricted to shallow water regimes). For such an approximation, it is more convenient to work with the ZCS formulation (18). The linear model thus obtained is

$$\begin{cases} \partial_t\zeta - \omega_{\rm WW}(D)^2\psi = 0, \\ \partial_t\psi + \zeta = 0. \end{cases} \tag{47}$$

where the symbol $\omega_{\rm WW}(k)$ of the Fourier multiplier $\omega_{\rm WW}(D)$ is given by

$$\omega_{\rm ww}(k) = k\Big(\frac{\tanh(\sqrt{\mu}k)}{\sqrt{\mu}k}\Big)^{1/2} =: kc_{\rm WW}(k).$$

The above system can therefore be diagonalized into to scalar uncoupled nonlocal equations

$$\begin{cases} \partial_t\big(\zeta + c_{\rm WW}(D)\partial_x\psi\big) + c_{\rm WW}(D)\partial_x\big(\zeta + c_{\rm WW}(D)\partial_x\psi\big) = 0, \\ \partial_t\big(\zeta - c_{\rm WW}(D)\partial_x\psi\big) - ic_{\rm WW}(D)\partial_x\big(\zeta - c_{\rm WW}(D)\partial_x\psi\big) = 0. \end{cases}$$

Right-going waves correspond to waves with a positive group velocity and are therefore obtained when the equations are reduced to the first of these two scalar equations, i.e. when

$$\partial_t\zeta + c_{\rm WW}(D)\partial_x\zeta = 0 \quad \text{and} \quad \overline{v} = c_{\rm WW}(D)\zeta. \tag{48}$$

where for the second relation, we used the identity $D\frac{\tanh(\sqrt{\mu}D)}{\sqrt{\mu}}\psi = -\partial_x\overline{v}$, which is exact when $\varepsilon = \beta = 0$. One can check that, as expected, (48) is a formal $O(\varepsilon, \mu)$ perturbation of (43).

As in Remark 8, one can alternatively derive an equation on $\overline{v}$ and express ζ in terms of $\overline{v}$; one obtains

$$\partial_t\overline{v} + c_{\rm WW}(D)\partial_x\overline{v} = 0 \quad \text{and} \quad \zeta = c_{\rm WW}(D)^{-1}\overline{v}. \tag{49}$$

Here again, solution to the scalar approximations (48) or (49) furnish *exact* solutions to the underlying system (47).

2.7.3. *The Whitham equation(s).* We have so far obtained a fully nonlinear, nondispersive approximation ($\mu = 0$, full dependence on ε) and a fully dispersive, linear, approximation ($\varepsilon = 0$, full dependence on μ). These approximations are given respectively by (45) and (48). Combining both models, a $O(\varepsilon\mu)$ approximation is obtained, namely

$$\partial_t\zeta + c_{\rm WW}(D)\partial_x\zeta + 3\varepsilon\frac{\zeta}{1+\sqrt{1+\varepsilon\zeta}}\partial_x\zeta = 0 \tag{50}$$

and

$$\overline{v} = c_{\rm WW}(D)\zeta + \frac{2}{\varepsilon}\big(\sqrt{1+\varepsilon\zeta} - 1 - \frac{1}{2}\varepsilon\big).$$

Taking $\overline{v}$ instead of ζ as reference to build the scalar approximation, as in Remark 8, one obtains the following approximation

$$\partial_t\overline{v} + c_{\rm WW}(D)\partial_x\zeta + \frac{3}{2}\varepsilon\overline{v}\partial_x\overline{v} = 0 \quad \text{and} \quad \zeta = c_{\rm WW}^{-1}(D)\overline{v} + \frac{\varepsilon}{4}\overline{v}^2. \tag{51}$$

This latter equation is known as the Whitham equation, proposed in [Whi67, Whi99] as an alternative to the KdV equation with weaker (and better) dispersive properties, and able to reproduce peaking and wave breaking. This equations has been intensively studied in recent years. For instance, existence and stability of solitary waves has been proved in [EGW12] and their peaking towards cusped solutions in [EWar], and wave breaking has been rigorously established in [NS94, CE98, Hur17] as a manifestation of the general rule that weakly dispersive perturbations to the Burgers equation lead to the formation of singularities [CCG10]. A natural question is to ask wether such results still hold for the alternative Whitham equation (50) on the surface elevation.

2.7.4. *The KdV and BBM equations.* The KdV and BBM equations are the scalar equations associated to the Boussinesq equations (34), which, we recall, are a $O(\mu^2)$ approximation of the water waves equations (17) under the weak nonlinearity assumption (32), namely, $\varepsilon = O(\mu)$. These equations can be derived from the Boussinesq equations (34) along a procedure similar to the one used below to derive the Camassa-Holm equation from the SGN equation. As we show now, it can also be derived directly from the Whitham equation (50).

Indeed, under the weak nonlinearity approximation, the Whitham equation (50) furnishes a $O(\varepsilon\mu) = O(\mu^2)$ approximation of the water waves equation (17). This will remain true if we replace the non local dispersive term of the Whitham equation by a $O(\mu^2)$ approximation and the nonlinear term by a $O(\varepsilon^2) = O(\mu^2)$ approximation. Since

$$c_{\rm WW}(D)\partial_x\zeta = \partial_x\zeta + \frac{1}{6}\mu\partial_x^3\zeta + O(\mu^2) \quad \text{and} \quad \frac{3\varepsilon\zeta}{1+\sqrt{1+\varepsilon\zeta}}\partial_x\zeta = \frac{3}{2}\varepsilon\zeta\partial_x\zeta + O(\varepsilon^2),$$

we obtain the KdV equation

$$\partial_t\zeta + \partial_x\zeta + \frac{1}{6}\mu\partial_x^3\zeta + \frac{3}{2}\varepsilon\zeta\partial_x\zeta = 0; \tag{52}$$

it is notable that one arrives at the same equation if we make similar approximations on the Whitham equation (51) on the velocity $\overline{v}$ instead of the surface elevation ζ.

We have seen in §2.4 that there is a whole family of Boussinesq systems, the *abcd* systems (36) that all furnish a $O(\mu^2)$ approximation to the water waves equations under the weak nonlinearity assumption (32). One of the arguments used to derive the *abcd* system from the Boussinesq system (34) is the so-called BBM trick that was introduced to derive the BBM equation from the KdV equation [BBM72]. It consists in remarking that owing to (52) and the weak nonlinearity assumption, one has $\partial_t\zeta = -\partial_x\zeta + O(\mu)$, so that $\mu\partial_x^3\zeta = -\mu\partial_x^2\partial_t\zeta + O(\mu^2)$. Without damaging the $O(\mu^2)$ precision of the KdV approximation, one can use instead the BBM equation,

$$(1 - \frac{1}{6}\mu\partial_x^2)\partial_t\zeta + \partial_x\zeta + \frac{3}{2}\varepsilon\zeta\partial_x\zeta = 0 \tag{53}$$

or more generally, any member of the KdV/BBM family

$$\big(1+(p-\frac{1}{6})\mu\partial_x^2\big)\partial_t\zeta+\partial_x\zeta+\mu p\partial_x^3\zeta+\frac{3}{2}\varepsilon\partial_x\zeta=0 \qquad (p\leq\frac{1}{6}). \tag{54}$$

2.7.5. *The Camassa-Holm equation.* The equations from the KdV/BBM family (54) are all globally well posed in reasonable Sobolev spaces and therefore unable to reproduce the wave breaking phenomenon. The reason of this is that dispersion balances the nonlinearity. There are two possibilities to avoid such a situation. The first one is to work with a model with weaker dispersion: this corresponds to the Whitham equations (50) and (51) which, under the weak nonlinarity assumption, furnish an approximation of the same precision as the KdV/BBM family. And indeed, as we have seen, the Whitham equation (the classical one (51) at least) can lead to wave breaking. The second possibility to obtain wave breaking is to work with a model having stronger nonlinearities. In order to do so while keeping the $O(\mu^2)$ precision of the KdV/BBM family, one can relax the weak nonlinearity assumption (32) and replace it by

$$\text{Moderate nonlinearity:} \qquad \varepsilon=O(\sqrt{\mu}). \tag{55}$$

Under this assumption, one must keep the $O(\varepsilon\mu)$ terms in order to keep the same $O(\mu^2)$ precision as for the KdV-BBM family. Among the asymptotic systems derived above, the only one that takes this terms into account is the SGN model. In dimension $d=1$, this model can be written under the form

$$\begin{cases}\partial_t\zeta+\partial_x(h\overline{v})=0,\\ \partial_t\overline{v}+\varepsilon\overline{v}\partial_x\overline{v}+\partial_x\zeta=\frac{\mu}{3}\frac{1}{h}\partial_x\big[h^3(\partial_x\partial_t\overline{v}+\varepsilon\overline{v}\partial_x^2\overline{v}-\varepsilon(\partial_x\overline{v})^2)\big].\end{cases} \tag{56}$$

Let us for instance seek an equation on $\overline{v}$ and an algebraic expression for ζ in terms of $\overline{v}$. Since the SGN equations are a $O(\mu)$ perturbation of the NSW equations, right going waves are expected to be a $O(\mu)$ perturbation of the Burgers equation (46), that is,

$$\partial_t\overline{v}+\partial_x\overline{v}+\varepsilon\frac{3}{2}\overline{v}\partial_x\overline{v}+\mu P=0 \quad\text{and}\quad \zeta=\overline{v}+\varepsilon\frac{1}{4}\overline{v}^2+\mu R.$$

where P and a function of $\overline{v}$ and its derivatives. Plugging the ansaz for ζ in the second equation of (56), one gets

$$\partial_t\overline{v}+\partial_x\overline{v}+\varepsilon\frac{3}{2}\overline{v}\partial_x\overline{v}+\mu\partial_x R=\frac{1}{3}\mu(1-\varepsilon\overline{v})\partial_x^2\partial_t\overline{v}+\frac{1}{3}\mu\varepsilon\partial_x\big[3\overline{v}\partial_x\partial_t\overline{v}+\overline{v}\partial_x^2\overline{v}-(\partial_x\overline{v})^2\big]$$

or equivalently, using the ansatz for the scalar equation for $\overline{v}$,

$$P=\partial_x R-\frac{1}{3}(1-\varepsilon\overline{v})\partial_x^2\partial_t\overline{v}-\frac{1}{3}\varepsilon\partial_x\big[3\overline{v}\partial_x\partial_t\overline{v}+\overline{v}\partial_x^2\overline{v}-(\partial_x\overline{v})^2\big].$$

This last equation gives P in terms of R and we therefore just have to find an expression for this latter quantity in terms of $\overline{v}$. In order to do so, we plug the ansatz for ζ in the first equation of (56). This yields an evolution equation for $\overline{v}$ that should of course be the same as our anzatz. By identification, this yields an expression for R, from which we deduce P. We refer to [CL09] or [Lan13] for the details of the computations; the final outcome is a family of Camassa-Holm equations that generalizes the above KdV/BBM family,

$$\partial_t\overline{v}+\partial_x\overline{v}+\varepsilon\frac{3}{2}\overline{v}\partial_x\overline{v}+\mu\big(\mathbf{a}\partial_x^3\overline{v}+\mathbf{b}\partial_x^2\partial_t\overline{v}\big)=\varepsilon\mu\big(\mathbf{c}\overline{v}\partial_x^3\overline{v}+\mathbf{d}\partial_x\overline{v}\partial_x^2\overline{v}\big), \tag{57}$$

where

$$\mathbf{a} = p, \quad \mathbf{b} = p - \frac{1}{6}, \quad \mathbf{c} = -\frac{3}{2}p - \frac{1}{6}, \quad \mathbf{d} = -\frac{9}{2}p - \frac{23}{24},$$

the parameter p coming, as for the KdV-BBM family, from using the BBM trick. Note that a wider range of parameter can be achieved by performing a change of variable on the velocity of the same kind as (35) in the derivation of the *abcd* systems (34). The equation one would obtain for ζ is [Lan13]

$$\partial_t \zeta + \partial_x \zeta + 3\varepsilon \frac{\zeta}{1+\sqrt{1+\varepsilon\zeta}} \partial_x \zeta + \mu\big(\mathbf{a}\partial_x^3 \zeta + \mathbf{b}\partial_x^2 \partial_t \zeta\big) = \varepsilon\mu\big(\mathbf{c}\zeta\partial_x^3\zeta + \mathbf{d}\partial_x\zeta\partial_x^2\zeta\big); \tag{58}$$

expanding the nonlinear terms into powers of ε up to $O(\varepsilon^4)$ (recall that under the assumption of moderate nonlinearity, one has $O(\varepsilon^4) = O(\mu^2)$ so that the corresponding terms can be neglected), one gets [CL09]

$$\begin{aligned} \partial_t \zeta + \partial_x \zeta + \frac{3}{2}\varepsilon\zeta\partial_x\zeta - \frac{3}{8}\varepsilon^2\zeta^2\partial_x\zeta + \frac{3}{16}\varepsilon^3\zeta^3\partial_x\zeta + \mu\big(\mathbf{a}\partial_x^3\zeta + \mathbf{b}\partial_x^2\partial_t\zeta\big) \\ = \varepsilon\mu\big(\mathbf{c}\zeta\partial_x^3\zeta + \mathbf{d}\partial_x\zeta\partial_x^2\zeta\big). \end{aligned} \tag{59}$$

Compared to the KdV-BBM family, the inclusion of new nonlinear terms of size $O(\varepsilon\mu)$ (as well as $O(\varepsilon^2)$ and $O(\varepsilon^3)$ in (59)) restores the possibility of wave breaking, as shown in [CL09] for (57) and (59) (and this could likely be extended to (59)); we recall that wave breaking for $\overline{v}$ means that $\overline{v}$ remains bounded but that $\partial_x\overline{v}$ blows up in finite time (and a similar definition holds of course for ζ). This wave breaking is shown to occur on a $O(1/\varepsilon)$ time scale, which is the same as for the Burgers equations (45) and (46).

Let us mention finally that (57) can be related, up to some rescaling, to the Camassa-Holm equation [FF81, CH93]

$$\partial_t U + \widehat{\kappa}\partial_x U + 3U\partial_x U - \partial_t\partial_x^2 U = 2\partial_x U\partial_x^2 U + U\partial_x^3 U \qquad (\widehat{\kappa} \neq 0)$$

provided that $\mathbf{b} < 0$, $\mathbf{a} \neq \mathbf{b}$, $\mathbf{b} = -2\mathbf{c}$, $\mathbf{d} = 2\mathbf{c}$ or to the Degasperis-Procesi equation [DP99]

$$\partial_t U + \widehat{\kappa}\partial_x U + 4U\partial_x U - \partial_t\partial_x^2 U = 3\partial_x U\partial_x^2 U + U\partial_x^3 U \qquad (\widehat{\kappa} \neq 0)$$

provided that $\mathbf{b} < 0$, $\mathbf{a} \neq \mathbf{b}$, $\mathbf{b} = -\frac{8}{3}\mathbf{c}$, $\mathbf{d} = 3c$. There is a huge literature devoted to these two equations and which can be used to get some insight on the behavior of (57) (note however that the case $\widehat{\kappa} = 0$, which has a very rich mathematical structure, *cannot* be related to a one directional wave propagation model). A natural question is therefore to ask which of these properties remain true for other ranges of the parameters in (57) and for the equations (58) and (59) on the surface elevation.

2.8. **Justification procedure.**

3. Extension to rotational flows

The goal of this section is to show how to generalize the results of Section 2 when non zero vorticity is allowed, that is, when assumption (3) is removed from the basic equations. We first show how to generalize the Zakharov-Craig-Sulem formulation of the water waves equation (8) as well as the (ζ, Q) formulation (13) when vorticity is present. In order to introduce the dimensionless version of these equations, the notion of *strength* of the vorticity needs to be introduced. For most of this section, we consider a strength $\alpha = 1/2$ which is the largest one for which we have rigorous bounds that allow the justification of the asymptotic models along a procedure similar to the one described in §2.8 for the irrotational case. As in the irrotational case, an asymptotic description of the velocity and pressure field in the fluid domain is needed in order to inderstand the contribution of the turbulent and non-hydrostatic components in the averaged Euler equations (17); this analysis is performed in §3.2. The incidence on the NSW and SGN models is then discussed in §3.3; it is in particular shown that the SGN equations must be extended by a third equation on some turbulent tensor. This extended model can serve as a basis for the modelling of wave breaking, provided that some ad hoc mechanism is added to the equations; this is done in §3.4.

3.1. **The water waves equations with vorticity.** If one wants to take vorticity effects into account, it is necessary to remove the assumption (3) from the water waves equations (1)-(6). The vorticity $\omega := \operatorname{curl} \mathbf{U}$ is therefore not identically equal to zero and satisfies instead the vorticity equation

$$\partial_t \boldsymbol{\omega} + \mathbf{U} \cdot \nabla_{X,z} \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla_{X,z} \mathbf{U} \tag{60}$$

(we treat here the case $d = 2$, the adaptation to the case $d = 1$ being straightforward). We show here how to generalize, in the presence of vorticity, the two formulations of the water waves equations considered in these notes, namely, the Zakharov-Craig-Sulem formulation (8) and the (ζ, Q) formulation (13). The dimensionless version of these equations is then given and the notion of *strength* of the vorticity is introduced.

3.1.1. *The generalized Zakharov-Craig-Sulem formulation in the presence of vorticity.* The Zakharov-Craig-Sulem (ZCS) equations (8) are a system of two evolution equations on ζ and ψ, this latter quantity being defined as the trace at the surface of the velocity potential Φ defined by the relation $\mathbf{U} = \nabla_{X,z}\Phi$. This relation being a reformulation of the irrotationality assumption (3), there is no direct generalization of the (ZCS) equations in the presence of vorticity.

Instead, it was noticed in [CL15] that, in the irrotational framework, one has

$$\nabla\psi = \underline{V} + \underline{w}\nabla\zeta,$$

where $\underline{V}$ and $\underline{w}$ respectively denote the horizontal and vertical component of the velocity field evaluated at the surface of the fluid domain. We can therefore seek directly an equation on

$$\begin{aligned} U_\parallel &:= \underline{V} + \underline{w}\nabla\zeta \\ &= \big(\underline{U} \times N\big)_{\mathrm{h}}, \end{aligned}$$

the subscript h denoting the horizontal component. Taking the trace of the Euler equation (1) at the surface and taking the horizontal component of the cross product of the resulting equation with N, one arrives after some computations at

$$\partial_t U_\parallel + g\nabla\zeta + \frac{1}{2}\nabla|U_\parallel|^2 - \frac{1}{2}\nabla\big((1+|\nabla\zeta|^2)\underline{w}^2\big) = -\underline{\omega}\cdot N\underline{V}^\perp,$$

where we also used the fact that since the pressure P is constant at the surface, $(\nabla_{X,z}P)_{|_{z=\zeta}} \times N = 0$. Denoting by Π and $\Pi_\perp$ the orthogonal projectors onto gradient and orthogonal gradient vector fields,

$$\Pi = -\frac{\nabla\nabla^{\mathrm{T}}}{\Delta}, \qquad \Pi_\perp = -\frac{\nabla^\perp(\nabla^\perp)^{\mathrm{T}}}{\Delta},$$

we can decompose $U_\parallel$ as

$$U_\parallel = \Pi U_\parallel + \Pi_\perp U_\parallel = \nabla\psi + \nabla\widetilde{\psi}$$

for some scalar functions ψ and $\widetilde{\psi}$. Remarking that $\Delta\widetilde{\psi} = \underline{\omega}\cdot N$, there is no need to derive an equation for $\widetilde{\psi}$. For ψ however, such an equation is necessary, and it is obtained by applying Π to the above evolution equation on $U_\parallel$. We can now state the extended Zakharov-Craig-Sulem formulation in the presence of vorticity

$$\begin{cases} \partial_t\zeta + \underline{V}\cdot\nabla\zeta - \underline{w} = 0, \\ \partial_t\psi + g\zeta + \frac{1}{2}|U_\parallel|^2 - \frac{1}{2}\big((1+|\nabla\zeta|^2)\underline{w}^2\big) = \frac{\nabla^{\mathrm{T}}}{\Delta}(\underline{\omega}\cdot N\underline{V}^\perp), \\ \partial_t\boldsymbol{\omega} + \mathbf{U}\cdot\nabla_{X,z}\boldsymbol{\omega} = \boldsymbol{\omega}\cdot\nabla_{X,z}\mathbf{U}, \end{cases} \tag{61}$$

which is a closed system of equations in $(\zeta,\psi,\boldsymbol{\omega})$ in the sense that it is possible to reconstruct the full velocity field $\mathbf{U}$ (and a fortiori its trace $\underline{U}$ at the surface) in terms of these three quantities. The derivation and mathematical analysis (local well-posedness, Hamiltonian structure, shallow water asymptotics, etc.) of this formulation can be found in [CL15]. A generalization of this formulation in the presence of a Coriolis force can also be found in [Mel17].

3.1.2. *The generalized (ζ, Q) formulation in the presence of vorticity.* The derivation of the averaged Euler equations (13) did not require the irrotationality assumption (3) and are therefore still valid in the presence of vorticity,

$$\begin{cases} \partial_t\zeta + \nabla\cdot Q = 0, \\ \partial_t Q + \nabla\cdot(\dfrac{1}{h}Q\otimes Q) + gh\nabla\zeta + \nabla\cdot\mathbf{R} + \dfrac{1}{\rho}\displaystyle\int_{-h_0+b}^{\zeta}\nabla P_{\mathrm{NH}} = 0, \end{cases} \tag{62}$$

with P_{NH} and $\mathbf{R}$ still defined by (11) and (12) respectively. The difference with the irrotational case is that there is no such thing as Proposition 1, i.e., these equations do not form a closed set of evolution equations in (ζ, Q). A possible generalization would be to prove that (62) and the vorticity equation (60) form a closed set of equations in $(\zeta, Q, \boldsymbol{\omega})$. From the definition of P_{NH} and $\mathbf{R}$, this would require to generalize the reconstruction mapping of Proposition 1 by a mapping $\mathfrak{R}[\zeta] : (Q,\boldsymbol{\omega}) \mapsto \mathbf{U}$ where $\mathbf{U} = (V^{\mathrm{T}}, w)^{\mathrm{T}}$ satisfies

$$\int_{-h_0+b}^{\zeta} V = Q, \qquad N_{\mathrm{b}}\cdot U_{\mathrm{b}} = 0, \qquad \mathrm{curl}\,\mathbf{U} = \boldsymbol{\omega}, \qquad \mathrm{div}\,\mathbf{U} = 0.$$

3.1.3. *Nondimensionalized equations and well-posedness of the equations.* Proceeding as in §1.5, and with the same notations, it is possible to derive a dimensionless version of (61) and (62) provided that we define the *strength of the vorticity*. An imoprtant effect of the vorticity is that it induces a vertical shear; recalling that the vertical variable is scaled by h_0 the horizontal velocity V is scaled by $a\sqrt{g/h_0}$, a typical scale to measure this shear is the natural scale of $\partial_z V$, namely $\Omega_0 = a/h_0\sqrt{g/h_0}$. This motivates the following definition

$$\text{The vorticity is of strength } \alpha > 0 \text{ if } \Omega_0^{-1}\mathrm{curl}\,\mathbf{U} = O(\mu^\alpha). \tag{63}$$

Omitting the tildes for dimensionless quantities and defining

$$\boldsymbol{\omega}_\mu = \begin{pmatrix} \mu^{-\alpha}\big(\partial_z V^\perp - \nabla^\perp w\big) \\ -\mu^{1/2-\alpha}\nabla\cdot V^\perp \end{pmatrix} \tag{64}$$

this means that $\boldsymbol{\omega}_\mu$ is a $O(1)$ quantity with respect to μ. The time evolution of $\boldsymbol{\omega}_\mu$ is directly given by the non dimensionalization of (60),

$$\partial_t \boldsymbol{\omega}_\mu + \frac{\varepsilon}{\mu}\mathbf{U}^\mu \cdot \nabla^\mu \boldsymbol{\omega}_\mu = \frac{\varepsilon}{\mu}\boldsymbol{\omega}_\mu \cdot \nabla^\mu \mathbf{U}^\mu \tag{65}$$

where $\mathbf{U}^\mu = \begin{pmatrix} \sqrt{\mu}V \\ w \end{pmatrix}$ and $\nabla^\mu = \begin{pmatrix} \sqrt{\mu}\nabla \\ \partial_z \end{pmatrix}$ –note in particular that $\boldsymbol{\omega}^\mu = \mu^{3/2-\alpha}\nabla^\mu \times \mathbf{U}^\mu$.

We shall mainly consider throughout these notes the case $\alpha = 1/2$, which is the smallest value of α (and therefore the strongest vorticity) for which it is known that the nondimensionalized generalized ZCS equations (61) are well-posed over a time $O(1/\varepsilon)$ and uniformly with respect to $\mu \leq 1$. This result, proved in [CL15], ensures that all the asymptotic expansions performed in §3.2 and §3.3 are justified.

Extending such a result to larger vorticities (i.e. to smaller values of α) is still an open problem, but it is however possible to derive some asymptotic models in such regimes, as shall be done in §3.5.

3.2. **The inner structure of the velocity and pressure fields in the presence of vorticity.** *For the reasons explained above, we consider here a vorticity of strength* $\alpha = 1/2$, in the sense of (63). As in §2.1 for the irrotational case, it is possible to describe the inner structure of the velocity field in shallow water in the presence of vorticity. With the nondimensionlization (65), the relations (19) that were used in the irrotational case must be replaced by

$$\begin{cases} \mu\nabla\cdot V + \partial_z w = 0, \\ \sqrt{\mu}\partial_z V - \sqrt{\mu}\nabla w = -\mu\boldsymbol{\omega}_{\mu,\mathrm{h}}^\perp, \\ \mu\nabla^\perp\cdot V = \mu\boldsymbol{\omega}_{\mu,\mathrm{v}}, \\ w_b - \beta\mu\nabla b\cdot V_b = 0. \end{cases} \tag{66}$$

where $\boldsymbol{\omega}$ is as defined in (64) (with $\alpha = 1/2$). The first and last equations can be used to obtain

$$w = -\mu\nabla\cdot\big[(1+z-\beta b)\overline{V}\big] - \mu\nabla\cdot\int_{-1+\beta b}^{z} V^*,$$

which is the same relation as in the irrotational case. The influence of the vorticity appears when we plug this relation into the second equation, leading to

$$\begin{aligned} V^* =& \mu\Big(\int_z^{\varepsilon\zeta}\nabla\nabla\cdot\big[(1+z'-\beta b)\overline{V}\big]\mathrm{d}z'\Big)^* + \mu\Big(\int_z^{\varepsilon\zeta}\nabla\nabla\cdot\int_{-1+\beta b}^{z} V^*\Big)^* \\ &+ \sqrt{\mu}\Big(\int_z^{\varepsilon\zeta}\boldsymbol{\omega}_{\mu,\mathrm{h}}^\perp\Big)^* \end{aligned}$$

(this expression differs from the corresponding irrotational one by the presence of the last term). Defining $\mathbf{T}[\varepsilon\zeta,\beta b]$ and $\mathbf{T}^*[\varepsilon\zeta,\beta b]$ as in (20), we can write in condensed form

$$(1-\mu\mathbf{T}^*)V^* = \sqrt{\mu}V_{\mathrm{sh}}^* + \mu\mathbf{T}^*\overline{V}$$

where V_{sh} is the shear velocity created by the vorticity,

$$V_{\mathrm{sh}} = \int_z^{\varepsilon\zeta}\boldsymbol{\omega}_{\mu,\mathrm{h}}^\perp.$$

so that

$$V^* = \sqrt{\mu}V_{\rm sh}^* + \mu\mathbf{T}^*\overline{V} + \mu^{3/2}T^*V_{\rm sh}^* + O(\mu^2).$$

This shows that *the fluctuation of the horizontal velocity arount its average is mainly due to the influence of the vorticity*, which contributes at order $O(1/\sqrt{\mu})$ while the dispersion associated to the (irrotational) nonlocal effects only contributes at order $O(1/\mu)$. The shallow water expansion of the velocity field in the presence of vorticity is therefore given when the bottom is flat by

$$\begin{cases} V = \overline{V} + \sqrt{\mu}V_{\rm sh}^* - \frac{1}{2}\mu\big((1+z)^2 - \frac{1}{3}h^2\big)\nabla\nabla\cdot\overline{V} + \mu^{3/2}T^*V_{\rm sh}^* + O(\mu^2), \\ w = -\mu(1+z)\nabla\cdot\overline{V} - \mu^{3/2}\nabla\cdot\int_{-1}^{z}V_{\rm sh}^* + O(\mu^2); \end{cases} \tag{67}$$

the generalization to non flat bottoms is given in (78). Note that contrary to what happens for the horizontal velocity, the contribution of the vorticity to the vertical velocity is smaller than the irrotational contribution.

As in §2.2, plugging these approximations into the formula for the non hydrostatic pressure, namely,

$$\frac{1}{\varepsilon}P_{\rm NH} = \int_z^{\varepsilon\zeta}\big(\partial_t w + \varepsilon V\cdot\nabla w + \frac{\varepsilon}{\mu}w\partial_z w\big)$$

to obtain an asymptotic expression of the non hydrostatic pressure field in the fluid domain. One easily checks that the new vorticity terms contribute to order $O(\mu^{3/2})$, so that the expansion (22) derived in the irrotational framework remains valid, but with a residual term of order $O(\mu^{3/2})$ instead of $O(\mu^2)$,

$$\frac{1}{\varepsilon}P_{\rm NH} = -\mu\big[\frac{h^2}{2} - \frac{(1+z)^2}{2}\big]\big(\partial_t + \varepsilon\overline{V}\cdot\nabla - \varepsilon\nabla\cdot\overline{V}\big)\nabla\cdot\overline{V} + O(\mu^{3/2}); \tag{68}$$

(similarly, when the bottom is not flat, (76) still holds, but with a residual of order $O(\mu^{3/2})$ instead of $O(\mu^2)$).

Remark 9. Of course, plugging (67) into the above formula for $P_{\rm NH}$, one can get a more precise expansion, up to order $O(\mu^2)$. The additional terms are quite complicated however, and for the sake of clarity, we chose here to limit our analysis to a $O(\mu^{3/2})$ precision; we refer to [CL14] for the full $O(\mu^2)$ expansion.

Note finally that even though the vorticity does not appear in (68), it plays a role in the evolution of ζ and $\overline{V}$. It is therefore not surprising that the reconstruction of the surface elevation from pressure measurements is more complex in the presence of vorticity, and has been done only in some particular cases such as solitary waves [Hen13] and linear plane waves [HT17].

3.3. **The NSW and SGN equations in the presence of vorticity.** *We remind that we consider here a vorticity of strength* $\alpha = 1/2$, in the sense of (63). The "turbulent" and non-hydrostatic terms in (15) can be expended as follows, following the results of §3.2,

$$\varepsilon\nabla\cdot\mathbf{R} = \varepsilon\mu\nabla\cdot\mathbf{E} + O(\varepsilon\mu^{3/2})$$

$$\frac{1}{\varepsilon}\int_{-1}^{\varepsilon\zeta}\nabla P_{\rm NH} = \mu h\mathcal{T}\big[\partial_t\overline{V} + \nabla\cdot\big(h\overline{V}\otimes\overline{V}\big)\big] + \mu\varepsilon h\mathcal{Q}_1\big(\zeta,\overline{V}\big) + O(\varepsilon\mu^{3/2}),$$

where the symmetric tensor $\mathbf{E}$ measures the quadratic self interaction of the fluctuation $V_{\rm sh}^*$ of the shear velocity $V_{\rm sh}$ created by the vorticity,

$$\mathbf{E} = \int_{-1}^{\varepsilon\zeta} V_{\rm sh}^* \otimes V_{\rm sh}^*.$$

Therefore, for large amplitude waves $\varepsilon = O(1)$, the contribution of the vorticity term to the averaged Euler equations due to the "turbulent" term $\varepsilon\nabla\cdot\mathbf{E}$, which is of size $O(\varepsilon\mu)$, is larger than the rotational part of the non hydrostatic pressure, which is of size $O(\mu^{3/2})$. In the weakly nonlinear regime (32), i.e. if $\varepsilon = O(\mu)$, this is the opposite situation. Both contribution are of equal order in the medium amplitude regime $\varepsilon = O(\sqrt{\mu})$.

Remark 10. As explained above, we work here with a $O(\mu^{3/2})$ precision instead of the $O(\mu^2)$ precision used for the SGN equation in the irrotational case. The computations are pushed further in [CL14] to keep the $O(\mu^2)$ precision. It is in particular shown that new turbulent terms appear at order $O(\varepsilon\mu^{3/2})$, and the $O(\mu^{3/2})$ terms of the non hydrostatic pressure are also computed explicitly.

Let us now consider the consequences of this new "turbulent" term on the Nonlinear Shallow Water and Serre-Green-Naghdi models.

3.3.1. *The NSW equations in the presence of vorticity.* As seen above, the first contribution of the rotational terms to the averaged Euler equations is of size $O(\varepsilon\mu)$, which is below the $O(\mu)$ precision of the NSW equations (24). Therefore, *in the presence of vorticity, the NSW equations* (24) *still furnish a* $O(\mu)$ *approximation to the (rotational) water waves equations.*

If the dynamics of the surface elevation ζ and of the average velocity $\overline{V}$ are not affected by the vorticity, this does not mean that there are no rotational effects at all. For instance, in the irrotational setting, the horizontal velocity is independent of the vertical coordinate, see (21), so that the horizontal velocity at the surface is well approximated by the average velocity,

$$\underline{V}(t,x) = \overline{V}(t,X) + O(\mu) \quad \text{where} \quad \underline{V}(t,x) := V\big(t,X,\varepsilon\zeta(t,X)\big).$$

As shown by (67), a corrective term must be added to this approximation if one wants to keep the same precision, namely,

$$\underline{V}(t,x) = \overline{V}(t,X) - \sqrt{\mu}\frac{1}{h}\int_{-1}^{\varepsilon\zeta}\int_{z}^{\varepsilon\zeta} \boldsymbol{\omega}_{\mu,\rm sh}^{\perp} + O(\mu)$$

(the corrective term being equal to $V_{\rm sh}^*$ evaluated at the surface); if one is interested, say, in the motion of the drifters at the surface in a zone with background currents, this corrective term should be added to the velocity furnished by the NSW equations.

3.3.2. *The SGN equations in the presence of vorticity.* Plugging the above expansions into the averaged Euler equations (15) and dropping the $O(\mu^{3/2})$ terms, one obtains the same SGN equations as in (38) but with the additional "turbulent" term $\varepsilon\mu\nabla\cdot\mathbf{E}$ in the momentum equation (or $\varepsilon\mu\frac{1}{h}\nabla\cdot\mathbf{E}$ if we work with the formulation in $(\zeta,\overline{V})$ variables (39)). The difficulty here is that $\mathbf{E}$ is not a function of ζ and Q but of the horizontal component of the vorticity $\boldsymbol{\omega}_{\mu,\rm h}$ (through $V_{\rm sh}$). In order to compute it, it seems therefore necessary to solve the vorticity equation (65) which is an equation cast in the fluid domain which is $d+1$ dimensional (while the SGN equation are cast on $\mathbb{R}^d$); solving this equation would be essentially as challenging numerically as solving the full free surface Euler equations. Fortunately, it happens that it is possible to derive an equation satisfied by $\mathbf{E}$ on $\mathbb{R}^d$; after some computations (see [CL14]), one gets that up to $O(\varepsilon\sqrt{\mu})$ terms, the symmetric tensor $\mathbf{E}$ solves the equation

$$\partial_t \mathbf{E} + \varepsilon\overline{V}\cdot\nabla\mathbf{E} + \varepsilon\nabla\cdot\overline{V}\mathbf{E} + \varepsilon\nabla\overline{V}^{\mathrm{T}}\mathbf{E} + \varepsilon\mathbf{E}\nabla\overline{V} = 0. \tag{69}$$

The conclusion is that *the presence of the vorticity can be taken into account in the SGN equations without having to solve the vorticity equation* (65) but by extending the SGN equations by a third coupled evolution equation on $\mathbf{E}$. The SGN equations in $(\zeta,\overline{V})$ variables (39) then become

$$\begin{cases} \partial_t\zeta + \nabla\cdot(h\overline{V}) = 0, \\ (1+\mu\mathcal{T})\big[\partial_t\overline{V} + \varepsilon\overline{V}\cdot\nabla\overline{V}\big] + \nabla\zeta + \varepsilon\mu\mathcal{Q}_1(h,\overline{V}) + \varepsilon\mu\frac{1}{h}\nabla\cdot\mathbf{E} = 0, \\ \partial_t\mathbf{E} + \varepsilon\overline{V}\cdot\nabla\mathbf{E} + \varepsilon\nabla\cdot\overline{V}\mathbf{E} + \varepsilon\nabla\overline{V}^{\mathrm{T}}\mathbf{E} + \varepsilon\mathbf{E}\nabla\overline{V} = 0. \end{cases} \tag{70}$$

Remark 11. Contrary to (39) which are precise up to $O(\mu^2)$ terms, the above equations are precise up to $O(\varepsilon\mu^{3/2})$ terms only. The $O(\mu^2)$ precision is reached in [CL14], but the equations (70) must be further extended by two other coupled evolution equations: one is the third order turbulent tensor $\mathbf{F}$ and the other one is the second order momentum $V^\sharp$ of the fluctuation of the shear velocity,

$$\mathbf{F} = \int_{-1}^{\varepsilon\zeta} V_{\mathrm{sh}}^*\otimes V_{\mathrm{sh}}^*\otimes V_{\mathrm{sh}}^* \quad \text{and} \quad V^\sharp = \frac{12}{h^3}\int_{-1}^{\varepsilon\zeta}(z+1)^2 V_{\mathrm{sh}}^*.$$

We also refer to [CL14] for generalization to non flat topographies.

As shown in [CL14], the equations (70) also admit a local conservation of energy, the energy density being here the sum of the energy density associated to the irrotational SGN equations and of a rotation (or turbulent) energy $\mathfrak{e}_{\mathrm{rot}}$; a similar correction must also be made for the energy flux, so that (26) becomes

$$\partial_t\big(\mathfrak{e}_{\mathrm{NSW}} + \mathfrak{e}_{\mathrm{rot}}\big) + \nabla\cdot\big(\mathfrak{F}_{\mathrm{NSW}} + \mathfrak{F}_{\mathrm{rot}}\big) = 0, \tag{71}$$

where

$$\mathfrak{e}_{\mathrm{rot}} = \frac{1}{2}\mathrm{Tr}\mathbf{E} \quad \text{and} \quad \mathfrak{F}_{\mathrm{rot}} = \frac{1}{2}\mathrm{Tr}\mathbf{E}\overline{V} + \mathbf{E}\overline{V}.$$

Therefore, there is local conservation of the total energy, which is the sum of the irrotational one $\mathfrak{e}_{\mathrm{SGN}}$ and of the the rotational one $\mathfrak{e}_{\mathrm{rot}}$. There can therefore be a transfer of energy between both quantities. It is therefore tempting to try to model wave breaking –during which the mechanical energy (i.e. the sum of the potential and kinetic energies) of the waves is dissipated– by a mechanism that would ensure such a transfer to the turbulent energy at wave breaking.

3.4. **Wave breaking and enstrophy creation.** The derivation of (70) is rigorously justified by the uniform bounds derived in [CL15] on the solution of (61). This rigorous approach breaks down when singularities form, and in particular when wave breaking occurs. The models proposed below are therefore far from being mathematically justified and comparison with experimental observations is at this day the best way to validate them.

Various formal approaches have been proposed to extend the range of application of SGN types models to realistic physical configurations in coastal oceanography, where wave breaking obviously has to be taken into account. It has for instance been proposed to switch locally (in the vicinity of wave breaking) from the SGN equations to the NSW equation [TP11, BCL+11, FKR16, DM17], and to treat wave breaking as shocks (see §2.3.2), a difficulty being to find good "breaking criteria" telling us and to to switch to and back the NSW equation [TBM+12]. Another common approach (see for instance [SMD93, SYV97, KCKD00]) is to model wave breaking by the addition of an eddy viscosity near wave breaking. Here again, one needs a "breaking criterion" to tell us when and where to add this eddy viscosity, and one must also propose an expression for this eddy viscosity, which can for instance be based on hyperbolic shock wave theory [GPP11] or other physical considerations [MSV05]. We refer to [Bro13, KR18] for surveys on these questions.

There is an intense research activity around these topics and at this day, no conclusive solution has been found. A seductive approach based on a series of works [RG12, RG13, RG15, KR19, RDF19] is based on the idea mentioned above of a transfer mechanism between mechanical and turbulent energy. We describe this approach (and more specifically [KR19, RDF19]) below with the formalism developed throughout these notes. For the sake of clarity, we stick here to the one dimensional case and a flat bottom.

To start with, let us rewrite (70) in dimension $d = 1$; the turbulent tensor $\mathbf{E}$ is then a scalar, denoted E and it is convenient to introduce, as in [RG12, RG13] the enstrophy $\varphi = \frac{1}{h^3}E$, so that (70) can be written

$$\begin{cases} \partial_t\zeta + \partial_x(h\overline{v}) = 0, \\ (1+\mu\mathcal{T})\big[\partial_t\overline{v} + \varepsilon\overline{v}\partial_x\overline{v}\big] + \partial_x\zeta + \varepsilon\mu\mathcal{Q}_1(h,\overline{v}) + \varepsilon\mu\frac{1}{h}\partial_x(h^3\varphi) = 0, \\ \partial_t(h\varphi) + \varepsilon\partial_x(\overline{v}\varphi) = 0. \end{cases}$$

The first step proposed in [KR19] is to add an eddy viscosity term in the last term of the momentum equation

$$(1+\mu\mathcal{T})\big[\partial_t\overline{v} + \varepsilon\overline{v}\partial_x\overline{v}\big] + \partial_x\zeta + \varepsilon\mu\mathcal{Q}_1(h,\overline{v}) + \varepsilon\mu\frac{1}{h}\partial_x(h^3\varphi - \nu_T h\partial_x\overline{v}) = 0,$$

where the eddy viscosity coefficient ν_T is discussed below and is a source of energy dissipation, as illustrated by the fact the the energy conservation law (71) becomes

$$\partial_t\big(\mathfrak{e}_{\rm NSW} + \mathfrak{e}_{\rm rot}\big) + \nabla\cdot\big(\mathfrak{F}_{\rm NSW} + \mathfrak{F}_{\rm rot}\big) = -\varepsilon\mu\nu_T h(\partial_x\overline{v})^2;$$

there is therefore a dissipation of the total energy while we rather want, at first order, a conservation of this total energy, and a transfer from the mechanical energy $\mathfrak{e}_{SGN}$ to the turbulent energy $\mathfrak{e}_{\rm rot}$. This can only be achieved through the creation of a corresponding source term in the equation for the enstrophy, namely,

$$\partial_t(h\varphi) + \varepsilon\partial_x(\overline{v}\varphi) = 2\varepsilon\mu\nu_T\frac{1}{h}(\partial_x\overline{v})^2;$$

quite obviously, enstrophy (or, equivalently, turbulent energy), is created in the vicinity of wave breaking, where the gradient of the velocity becomes important; the mechanical energy of the wave is consequently decreased. This mechanism restores the local conservation of the total energy (71). However, in a second step, the smale scale dissipation of the total energy must be taken into account; there should therefore be a dissipation mechanism $\mathcal{D}$ such that

$$\partial_t\big(\mathfrak{e}_{\rm NSW} + \mathfrak{e}_{\rm rot}\big) + \nabla\cdot\big(\mathfrak{F}_{\rm NSW} + \mathfrak{F}_{\rm rot}\big) = -\mathcal{D}.$$

Assuming that this dissipation mechanisms acts at the level of the turbulent energy, one must consequently modify the enstrophy equation that becomes

$$\partial_t(h\varphi) + \varepsilon\partial_x(\overline{v}\varphi) = \varepsilon\mu\nu_T \frac{2}{h}(\partial_x\overline{v})^2 - \frac{2}{h}\mathcal{D}.$$

The final equations then become

$$\begin{cases} \partial_t\zeta + \partial_x(h\overline{v}) = 0, \\ (1+\mu\mathcal{T})\big[\partial_t\overline{v} + \varepsilon\overline{v}\partial_x\overline{v}\big] + \partial_x\zeta + \varepsilon\mu\mathcal{Q}_1(h,\overline{v}) + \varepsilon\mu\frac{1}{h}\partial_x(h^3\varphi - \nu_T h\partial_x\overline{v}) = 0, \\ \partial_t(h\varphi) + \varepsilon\partial_x(\overline{v}\varphi) = \varepsilon\mu\nu_T\frac{2}{h}(\partial_x\overline{v})^2 - \frac{2}{h}\mathcal{D}. \end{cases} \tag{72}$$

Remark 12. The derivation of (72) relies on quite sound physical arguments but a good amount of physical modelling is still required to propose expressions for the eddy viscosity ν_T and the dissipation term $\mathcal{D}$. There is still no consensus regarding what these terms should be. For instance, $\nu_T = C_\nu h\sqrt{gh}$ is proposed in [MSV05] while [KR19] suggests expressions based on the enstrophy,

$$\nu_T = C_p h^2\sqrt{\varphi} \quad \text{and} \quad \mathcal{D} = \frac{1}{2}C_r h^2\varphi^{3/2},$$

with C_p and C_r dimensionless numerical coefficients. A drawback of this last choice is that the enstrophy (or turbulent energy) stays equal to zero if it is initially zero, but good matching with experimental data are observed in many cases [KR19, RDF19].

3.5. **What about larger vorticities?** We considered in the previous section SGN type models derived under the assumption of a vorticity strength $\alpha = 1/2$, where we recall that the vortex strength is defined in (63). This is the strongest vorticity for which bounds on the solutions to the rotational water waves equations (61) have been established uniformly with respect to $\mu \in (0,1)$ and for times of order $O(1/\varepsilon)$ [CL15]. Owing to these uniform bounds, the asymptotic expansions of §3.2 and §3.3 are rigorously justified. In this section, we consider flows with a larger vorticity strength $0 < \alpha < 1/2$, not covered therefore by the theoretical bounds of [CL15]. The derivation of the models derived below is therefore only a formal one.

The first step is to generalize the expansion (67) of the inner velocity field to the case of a vorticity strength $0 < \alpha < 1/2$; by simple computations, one finds,

$$\begin{cases} V = \overline{V} + \mu^\alpha V_{\rm sh}^* - \frac{1}{2}\mu\big((1+z)^2 - \frac{1}{3}h^2\big)\nabla\nabla\cdot\overline{V} + O(\mu^{1+\alpha}), \\ w = -\mu(1+z)\nabla\cdot\overline{V} - \mu^{1+\alpha}\nabla\cdot\int_{-1}^z V_{\rm sh}^* + O(\mu^2); \end{cases} \tag{73}$$

it follows that the turbulent and non hydrostatic components of the averaged Euler equation satisfy

$$\varepsilon\nabla\cdot\mathbf{R} = \varepsilon\mu^{2\alpha}\nabla\cdot\mathbf{E} + O(\varepsilon\mu^{1+\alpha})$$
$$\frac{1}{\varepsilon}\int_{-1}^{\varepsilon\zeta}\nabla P_{\rm NH} = \mu h\mathcal{T}\big[\partial_t\overline{V} + \nabla\cdot\big(h\overline{V}\otimes\overline{V}\big)\big] + \mu\varepsilon h\mathcal{Q}_1\big(\zeta,\overline{V}\big) + O(\varepsilon\mu^{1+\alpha}).$$

We show below how the NSW and Boussinesq models, which were not affected by the presence of a vorticity of strength $\alpha = 1/2$, have to be modified in the presence of a stronger vorticity.

3.5.1. *The NSW equations with a large vorticity.* Of particular interest is the analysis of the rotational effects on the NSW equations when $0 < \alpha < 1/2$. Indeed, plugging the above expansion into the averaged Euler equations (14) and neglecting the $O(\mu)$ terms, one finds

$$\begin{cases} \partial_t h + \nabla\cdot(h\overline{V}) = 0, \\ \partial_t \overline{V} + \varepsilon \overline{V}\cdot\nabla\overline{V} + \nabla\zeta + \varepsilon\mu^{2\alpha}\frac{1}{h}\nabla\cdot\mathbf{E} = 0, \\ \partial_t\mathbf{E} + \varepsilon\overline{V}\cdot\nabla\mathbf{E} + \varepsilon\nabla\cdot\overline{V}\mathbf{E} + \varepsilon\nabla\overline{V}^{\mathrm{T}}\mathbf{E} + \varepsilon\mathbf{E}\nabla\overline{V} = 0. \end{cases}$$

which are the equations derived in [GG12] to describe the conservative motion of compressible fluids. In its one dimensional version, it is also the first model on which a mechanism of creation of entropy has been added to model wave breaking [RG12, RG13].

3.5.2. *The Boussinesq equations with a large vorticity.* Under the assumption (32) of weak nonlinearity, we can plug the above expansions into (14) and neglect the $O(\mu^2)$ terms to obtain

$$\begin{cases} \partial_t \zeta + \nabla\cdot(h\overline{V}) = 0, \\ (1-\mu\frac{1}{3}\nabla\nabla^{\mathrm{T}})\partial_t \overline{V} + \varepsilon \overline{V}\cdot\nabla\overline{V} + \nabla\zeta + \varepsilon\mu^{2\alpha}\nabla\cdot\mathbf{E} = 0, \\ \partial_t\mathbf{E} + \varepsilon\overline{V}\cdot\nabla\mathbf{E} + \varepsilon\nabla\cdot\overline{V}\mathbf{E} + \varepsilon\nabla\overline{V}^{\mathrm{T}}\mathbf{E} + \varepsilon\mathbf{E}\nabla\overline{V} = 0. \end{cases} \tag{74}$$

Remark 13. Contrary to what has been done in §2.4 in the irrotational case, it is not possible here to replace $(1-\mu\frac{1}{3}\nabla\nabla^{\mathrm{T}})\partial_t\overline{V}$ in the second equation by the simpler epxression $(1-\mu\frac{1}{3}\Delta)\partial_t\overline{V}$. Indeed, the quantity $\nabla^{\perp}\cdot\overline{V}$ is no longer small enough to perform such a substitution.

Appendix A. Generalized formula when the topography is not flat

For the sake of clarity, in many cases, we provided in the main text formulas for a flat topography. We give here the generalization of these formulas when the bottom is not flat.

First, in the presence of a non flat topography, the expansion (21) for an irrotational flow must be replaced by

$$\begin{cases} V = & \overline{V} - \frac{1}{2}\mu\big((1+z-\beta b)^2 - \frac{1}{3}h^2\big)\nabla\nabla\cdot\overline{V} \\ & +\beta\big(z-\varepsilon\zeta+\frac{1}{2}h\big)\big[\nabla b\cdot\nabla\overline{V} + \nabla(\nabla b\cdot\overline{V})\big] + O(\mu^2), \\ w = & -\mu\nabla\cdot\big[(1+z-\beta b)\overline{V}\big] + O(\mu^2); \end{cases} \tag{75}$$

and, similarly, for the description of the pressure field in the fluid domain we now have

$$\begin{aligned} \frac{1}{\varepsilon}P_{\mathrm{NH}} = & -\mu\Big[\frac{h^2}{2} - \frac{(1+z-\beta b)^2}{2}\Big]\big(\partial_t + \varepsilon\overline{V}\cdot\nabla - \varepsilon\nabla\cdot\overline{V}\big)\nabla\cdot\overline{V} \\ & + \mu(\varepsilon\zeta - z)h(\partial_t + \varepsilon\overline{V}\cdot\nabla)(\beta\nabla b\cdot\overline{V}) + O(\mu^2). \end{aligned} \tag{76}$$

The same procedure as in §2.5 then leads to the following SGN equations in the presence of topography,

$$\begin{cases} \partial_t\zeta + \nabla\cdot Q = 0, \\ (1+\mu\mathbf{T})\big[\partial_t Q + \nabla\cdot\big(\frac{1}{h}Q\otimes Q\big)\big] + h\nabla\zeta + h\mathcal{Q}_1(h,\frac{Q}{h}) = 0, \end{cases} \tag{77}$$

where $\mathbf{T} = h\mathcal{T}\frac{1}{h}$ and

$$\mathcal{T}V = -\frac{1}{3h}\nabla\big(h^3\nabla\cdot V\big) + \beta\frac{1}{2h}\big[\nabla\big(h^2\nabla b\cdot V\big) - h^2\nabla b\nabla\cdot V\big] + \beta^2\nabla b\nabla b\cdot V,$$

while

$$\mathcal{Q}_1(V) = -2\mathcal{R}_1\big(\partial_x V\cdot\partial_y V^\perp + (\nabla\cdot V)^2\big) + \beta\mathcal{R}_2\big(V\cdot(V\cdot\nabla)\nabla b\big)$$

and

$$\mathcal{R}_1 w = -\frac{1}{3h}\nabla(h^3 w) - \beta\frac{h}{2}w\nabla b, \qquad \mathcal{R}_2 w = \frac{1}{2h}\nabla(h^2 w) + \beta w\nabla b.$$

Finally, in the presence of a vorticity of strength $\alpha = 1/2$, the expansion of the velocity field is

$$\begin{cases} V = & \overline{V} + \sqrt{\mu}V^*_{\rm sh} - \frac{1}{2}\mu\big((1+z-\beta b)^2 - \frac{1}{3}h^2\big)\nabla\nabla\cdot\overline{V} \\ & +\beta\big(z-\varepsilon\zeta+\frac{1}{2}h\big)\big[\nabla b\cdot\nabla\overline{V} + \nabla(\nabla b\cdot\overline{V})\big] + O(\mu^2), \\ w = & -\mu\nabla\cdot\big[(1+z-\beta b)\overline{V}\big] - \mu^{3/2}\nabla\cdot\int_{-1+\beta b}^{z} V^*_{\rm sh} + O(\mu^2). \end{cases} \tag{78}$$

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