Internal wave dynamics in the atmosphere

take-home messages

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Sound-Proof Models

Compressible flow equations $L \sim h_{\text{sc}}$

$$
\begin{aligned}
\boldsymbol{\rho_t} + \nabla \cdot (\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{u})_t + \nabla \cdot (\rho \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi &= 0 \\
(\rho w)_t \ + \nabla \cdot (\rho \boldsymbol{v} w) + P \pi_z &= -\rho g \\
\boldsymbol{P_t} + \nabla \cdot (P \boldsymbol{v}) &= 0
\end{aligned}
$$

drop term for:

anelastic[†] (approx.)

pseudo-incompressible[*]

$$
P = p^{\frac{1}{\gamma}} = \rho\theta \,, \qquad \pi = p/\Gamma P \,, \qquad \Gamma = c_p/R \,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k} \quad (\boldsymbol{u} \cdot \boldsymbol{k} \equiv 0)
$$

Parameter range & length and time scales of asymptotic validity ?

[†] e.g. Lipps & Hemler, JAS, **29**, 2192–2210 (1982)

[*] Durran, JAS, **46**, 1453–1461 (1989)

Regimes of Validity ... Design Regime

Characteristic inverse time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon^{\nu}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Realistic regime with three time scales

$$\overline{\theta} = 1 + \varepsilon^{\mu}\widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^{\mu}) \qquad (\nu = 1 - \mu/2)$$

Analysis of internal wave spectra

$$-\frac{d}{dz}\left(\frac{1}{1-\boxed{\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^2}}}\,\frac{1}{\bar{\theta}\,\bar{P}}\,\frac{dW}{dz}\right)+\frac{\lambda^2}{\bar{\theta}\,\bar{P}}W \;=\; \frac{1}{\omega^2}\,\frac{\lambda^2 N^2}{\bar{\theta}\,\bar{P}}W$$

Internal wave modes $\left(\frac{\omega^2/\lambda^2}{\bar{c}^2} = O(1)\right)$

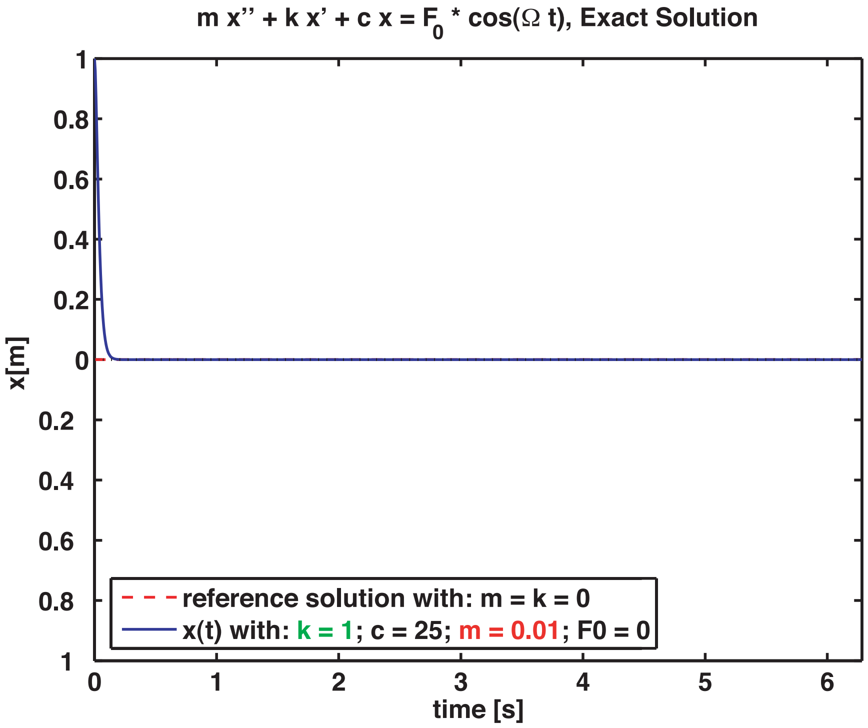
- pseudo-incompressible modes/EVals $=$ compressible modes/EVals $+\ O(\varepsilon^{\mu})\ \dagger$
- phase errors remain small ***over advection time scales*** for $\mu > \frac{2}{3}$

Anelastic and pseudo-incompressible models remain relevant for stratifications

$$\frac{1}{\bar{\theta}}\frac{d\bar{\theta}}{dz} < O(\varepsilon^{2/3}) \qquad \Rightarrow \qquad \Delta\theta\big|_0^{h_{\mathrm{sc}}} \lesssim 40\ \mathrm{K}$$

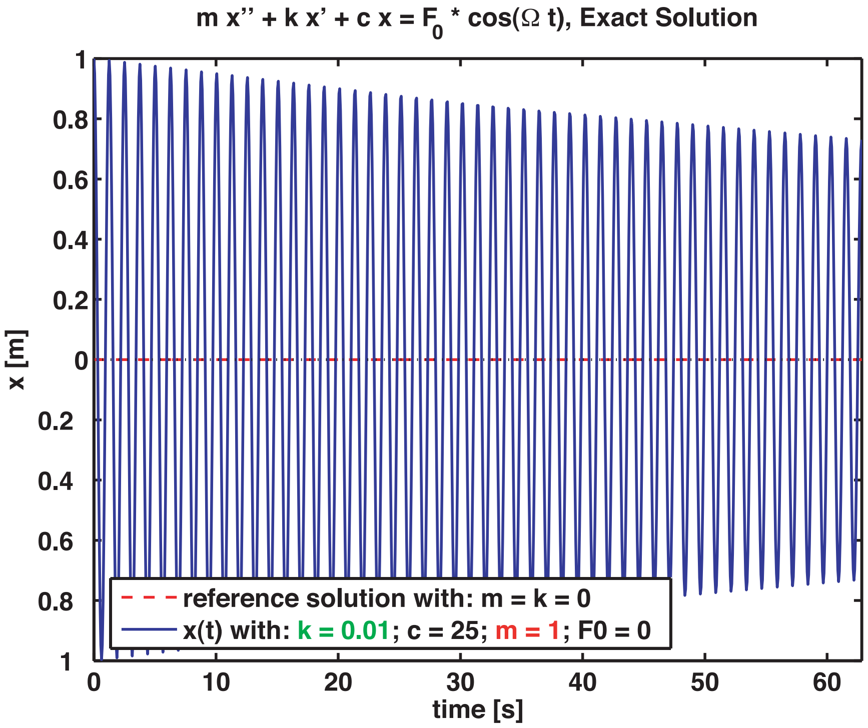
not merely up to $O(\varepsilon^2)$ as in Ogura-Phillips (1962)

$$\varepsilon\, y'' + \delta\, y' + y = \cos(\tau)$$


m x'' + k x' + c x = F$_0$ * cos(Ω t), Exact Solution

$\varepsilon = 0.0004$

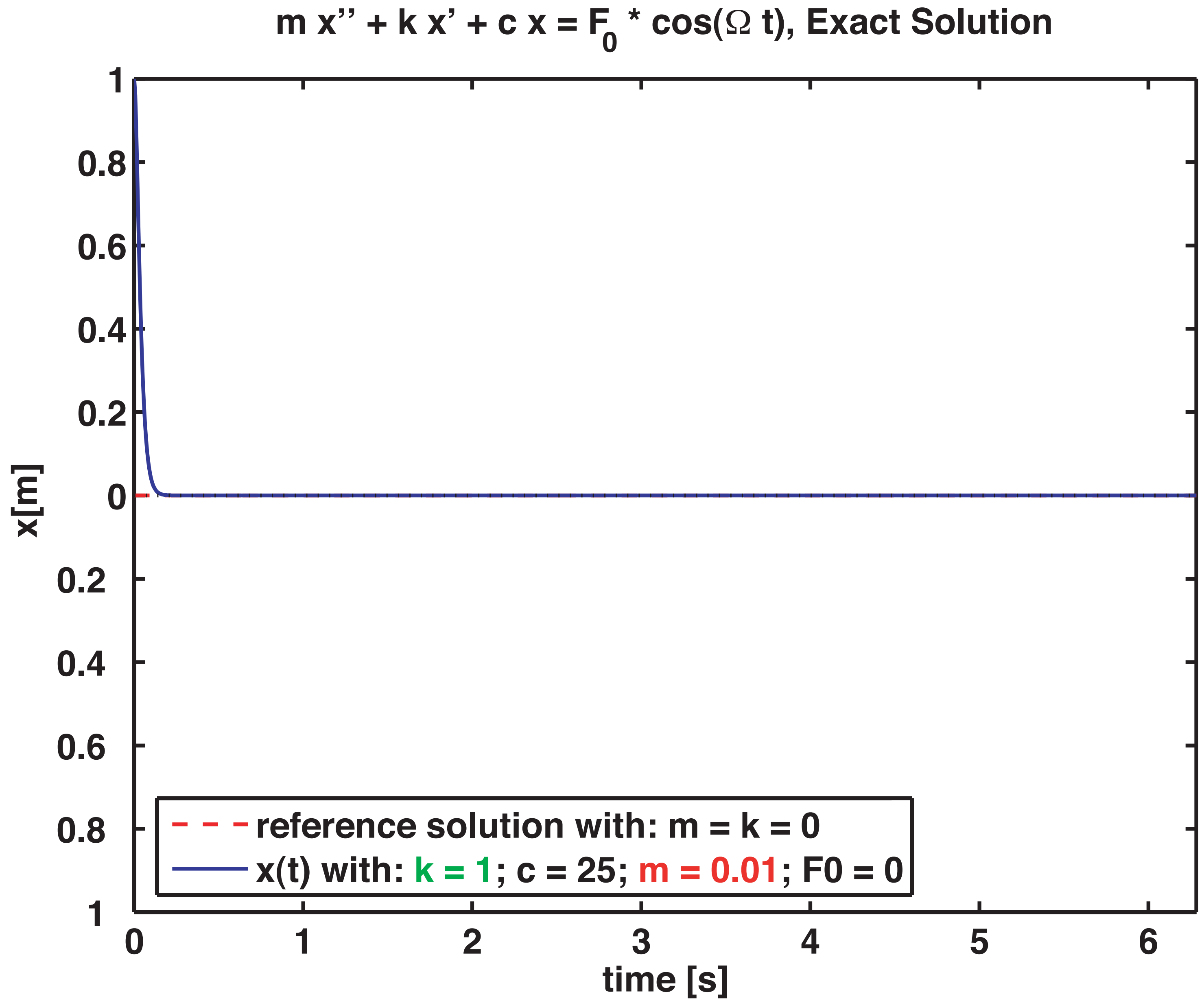
$\delta = 0.04$


m x'' + k x' + c x = F$_0$ * cos(Ω t), Exact Solution

$\varepsilon = 0.04$

$\delta = 0.0004$

The limit is path-dependent!



Matched asymptotic expansions ?

Strongly tilted atmospheric Vortices

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Eileen Päschke	(Deutscher Wetterdienst, Lindenberg)
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Patrick Marschalik	(Fritz Haber Institute, Berlin)
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Roger Smith	(Ludwig-Maximilians Univ., München)

Deutsche
Forschungsgemeinschaft

MetStröm

DFG

ECMWF

CRC 1114
Scaling Cascades in Complex Systems

Motivation

Structure of atmospheric vortices I: two scales

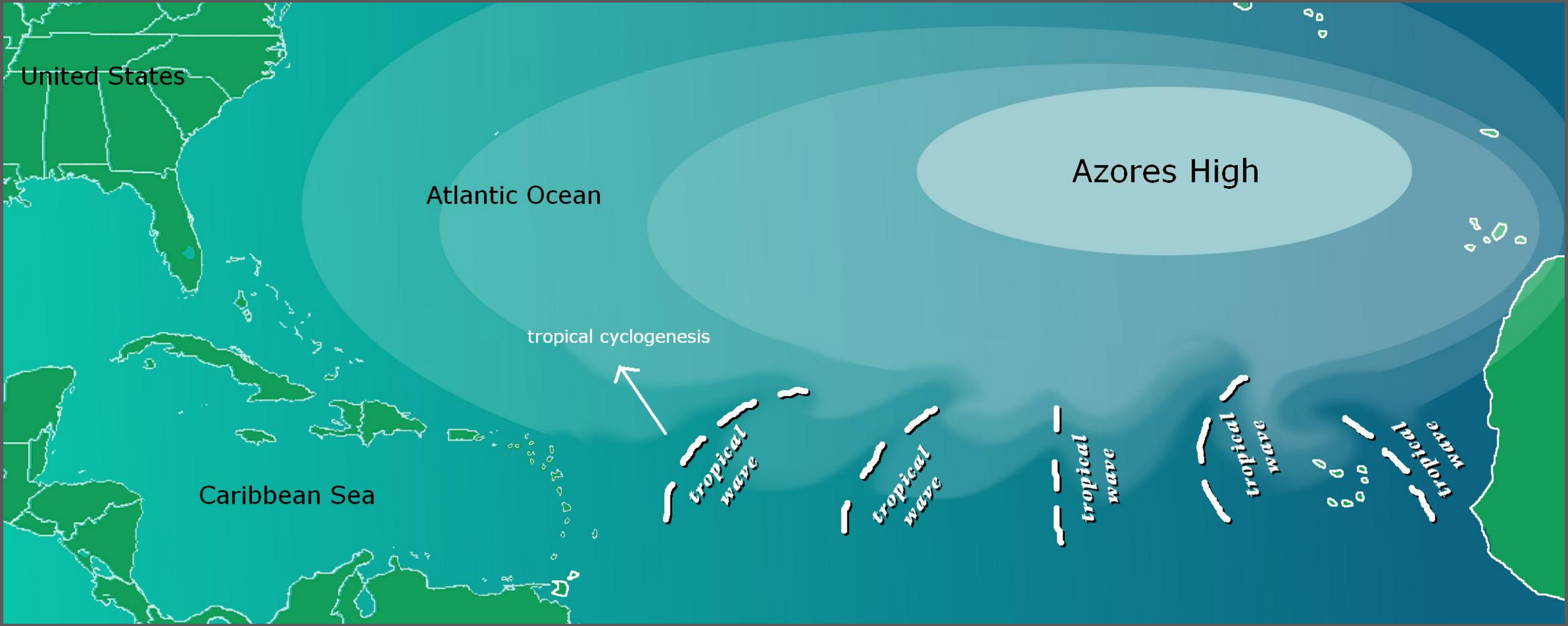
(Päschke et al., JFM, (2012))

Structure of atmospheric vortices II: cascade of scales

(Dörffel et al., arXiv:1708.07674)

Conclusions

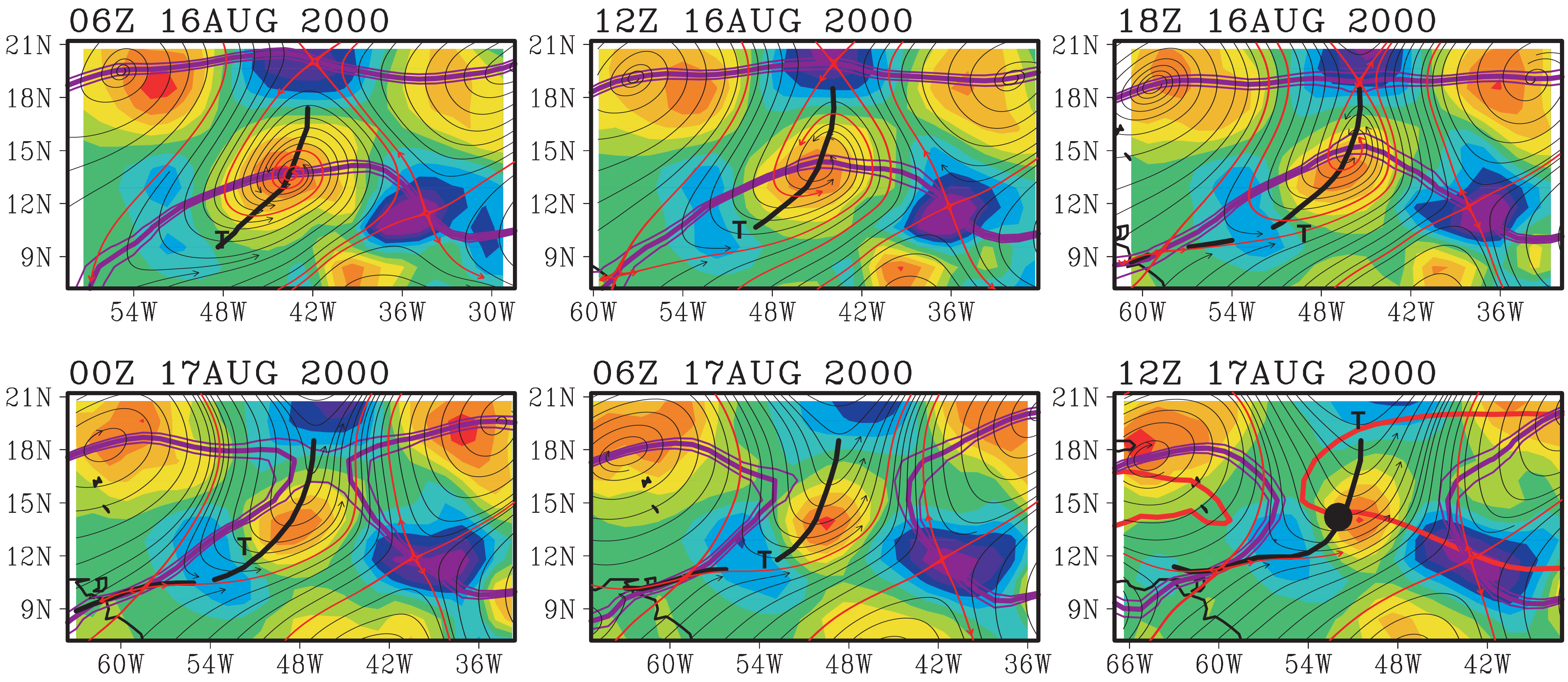
Tropical easterly african waves



http://www.aoml.noaa.gov/hrd/tcfaq/A4.html

Developing tropical storm

(streamlines in co-moving frame and Okubo-Weiss-parameter (color))



$$\mathrm{Ro} = \frac{|\boldsymbol{v}|}{fL} \sim \frac{1}{\mathbf{10}}$$

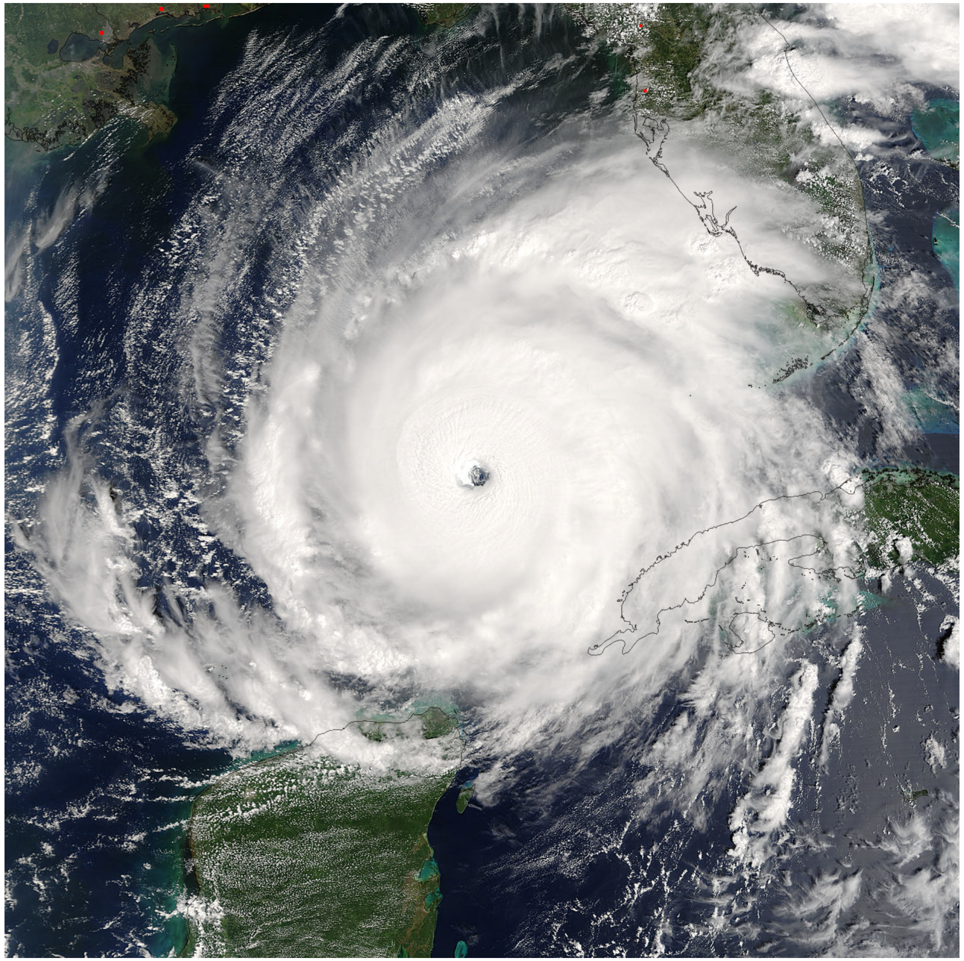
Dunkerton et al., Atmos. Chem. Phys., **9**, 5587–5646 (2009)

Developed hurricane

$$R^*_{\text{mw}} \approx 50\ldots200 \text{ km}$$

$$u_\theta \approx 30\ldots60 \text{ m/s}$$

R_{mw}: radius of max. wind

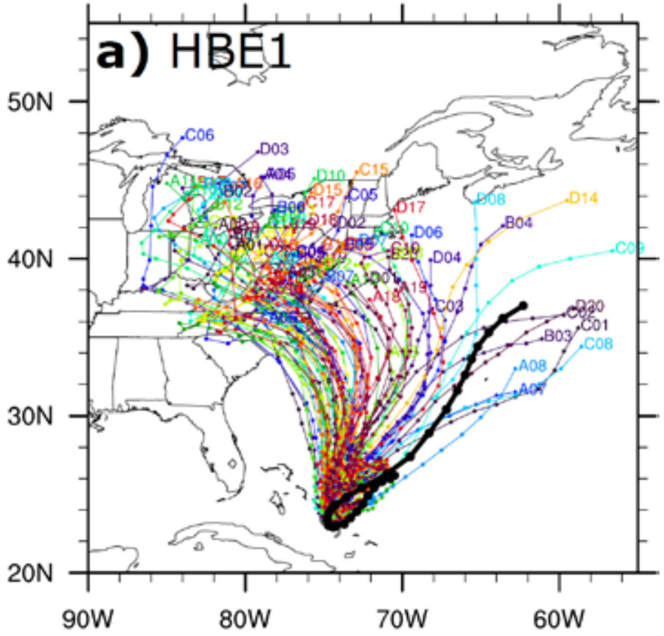


Hurricane ″Rita“

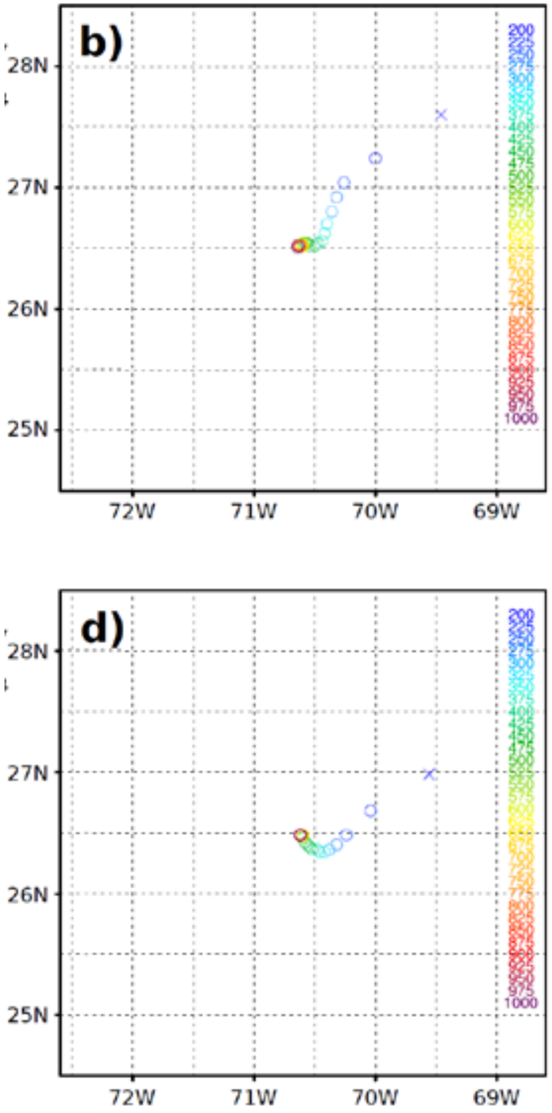
$$\text{Ro} = \frac{u_{\theta,\text{max}}}{f R_{\text{mw}}} \sim \mathbf{10}$$

Photo: Hurricane Rita from https://commons.wikimedia.org/wiki/File:HurricaneRita21Sept05a.jpg

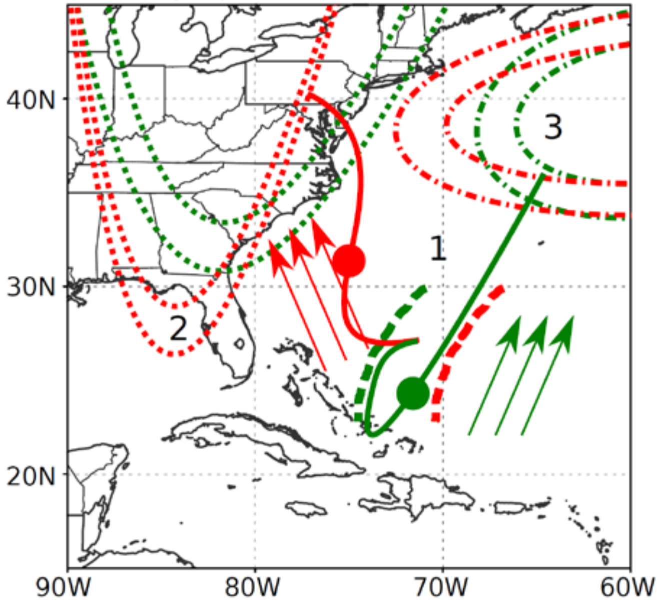
Ensemble of Simulations of “Joaquin”-like Storms



Ensemble Tracks



Vortex Tilts



Storm Evolutions

Gh. Alaka et al. (2019), WAF, submitted

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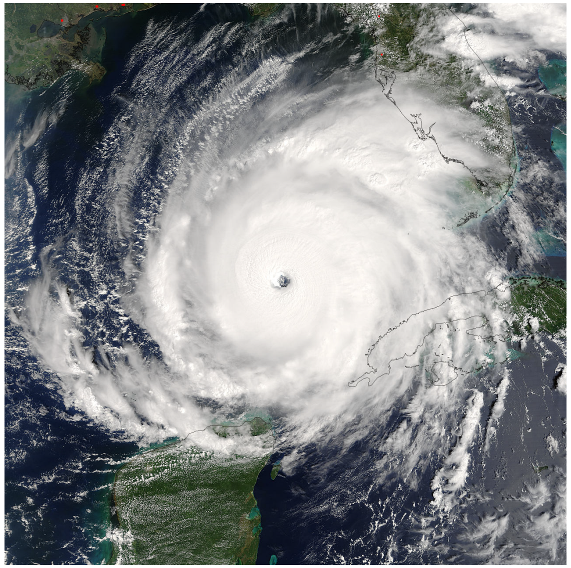
https://commons.wikimedia.org/wiki/
File:HurricaneRita21Sept05a.jpg

Radial momentum balance regimes

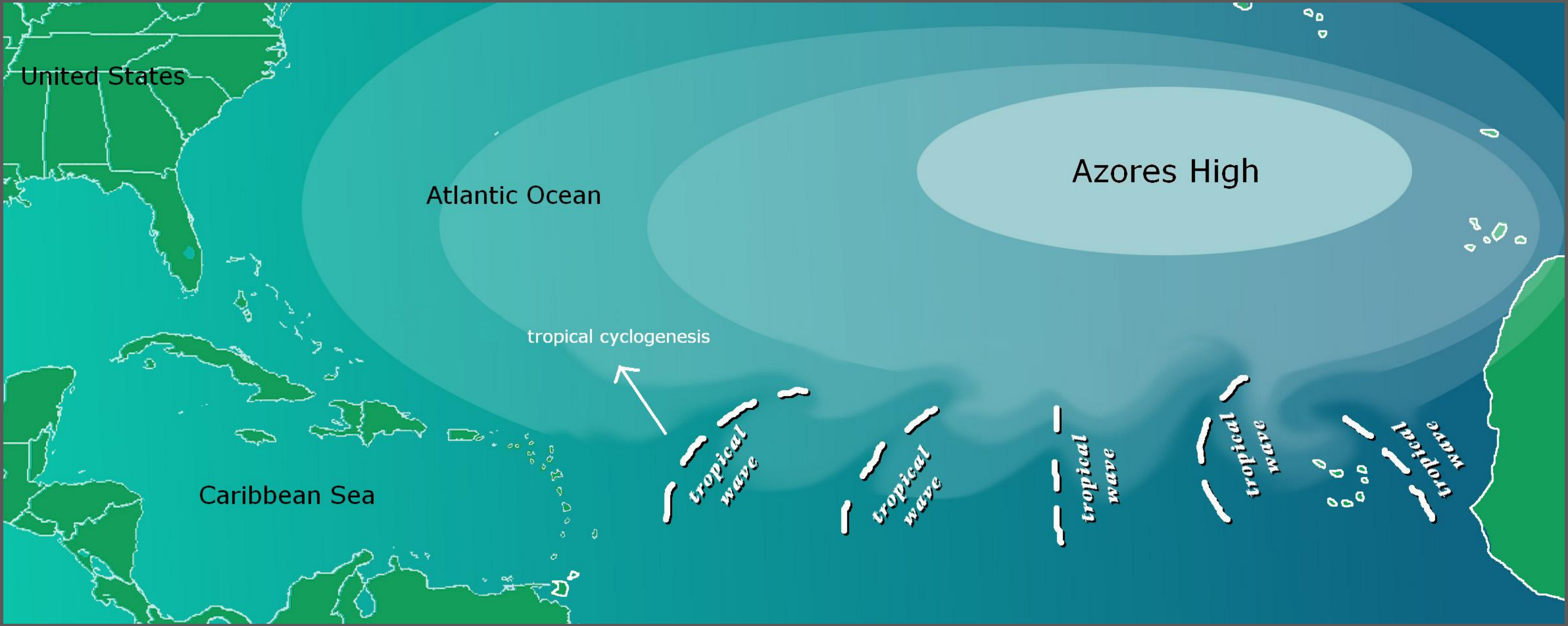


$-\frac{1}{\rho}\frac{\partial p}{\partial r} + f u_\theta = o(1)$	**geostrophic**	$\mathrm{Ro} \ll 1$	**typical "weather"**
$\frac{u_\theta^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} + f u_\theta = o(1)$	**gradient wind**	$\mathrm{Ro} = \mathcal{O}(1)$	**tropical storm** **incipient hurricane**
$\frac{u_\theta^2}{r} - \frac{1}{\rho}\frac{\partial p}{\partial r} = o(1)$	**cyclostrophic**	$\mathrm{Ro} \gg 1$	**hurricane**

Päschke, Marschalik, Owinoh, K., JFM, **701**, 137–170 (2012)

Dörffel et al., preprint, arXiv:1708.07674 (2017)

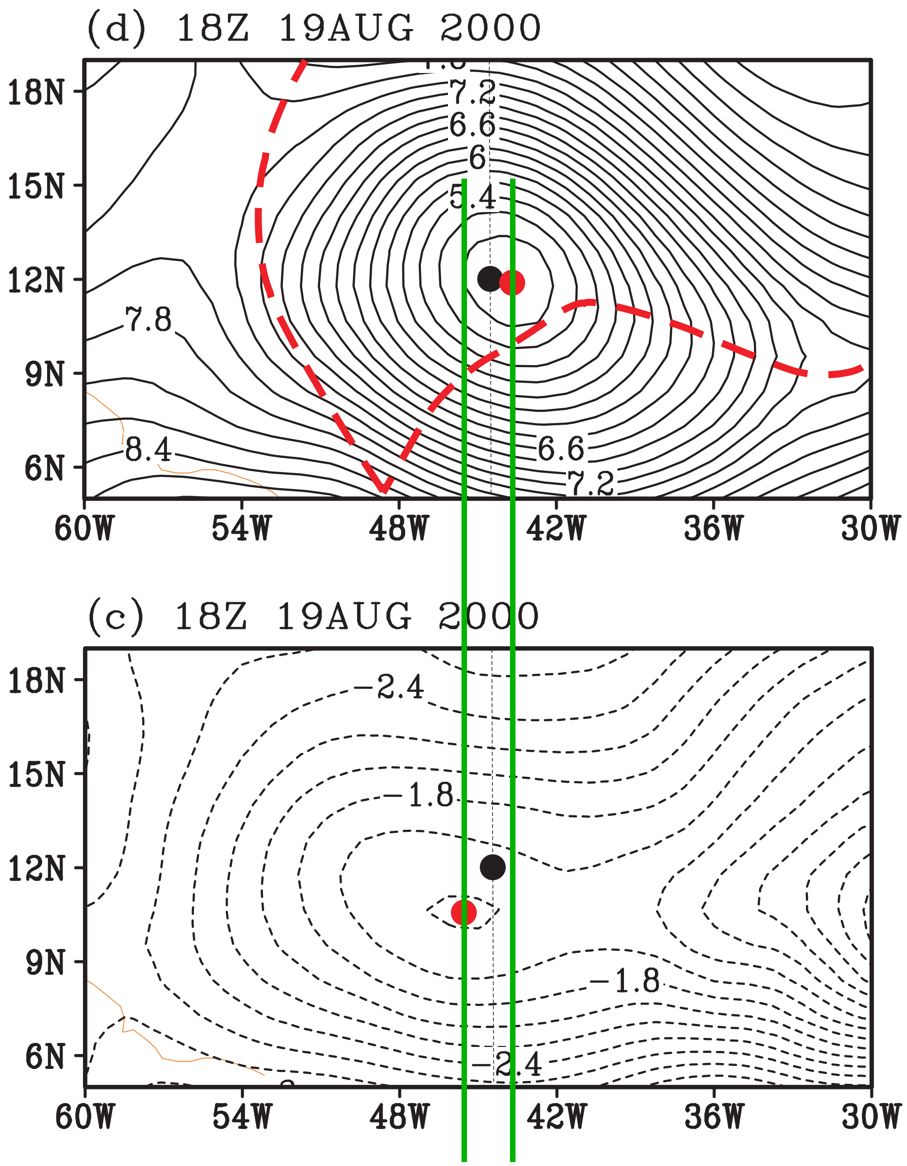
Tropical easterly african waves



http://www.aoml.noaa.gov/hrd/tcfaq/A4.html

Vortex tilt in the incipient hurricane stage

(Velocity potential)



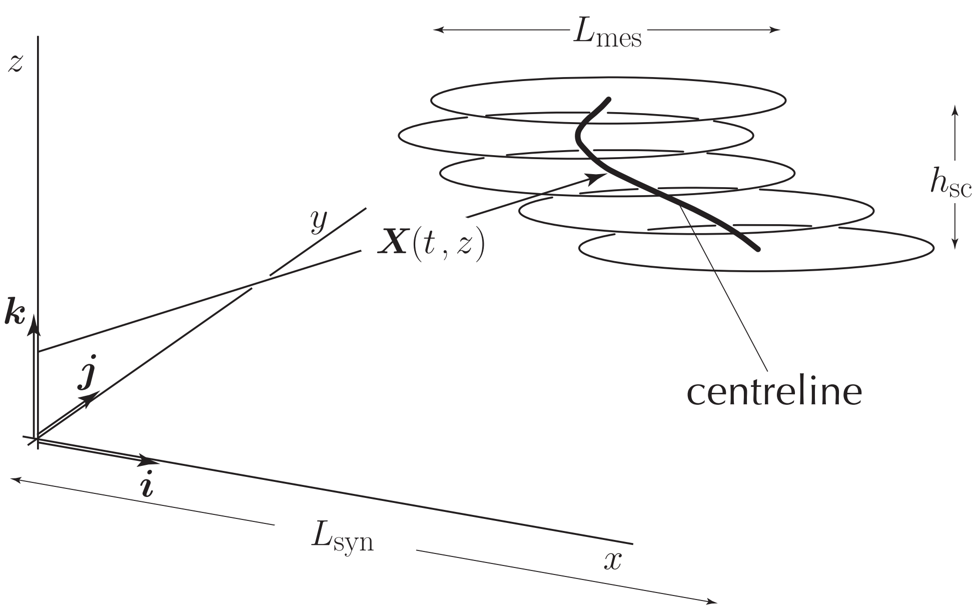
200 hPa
(∼ 12 km)

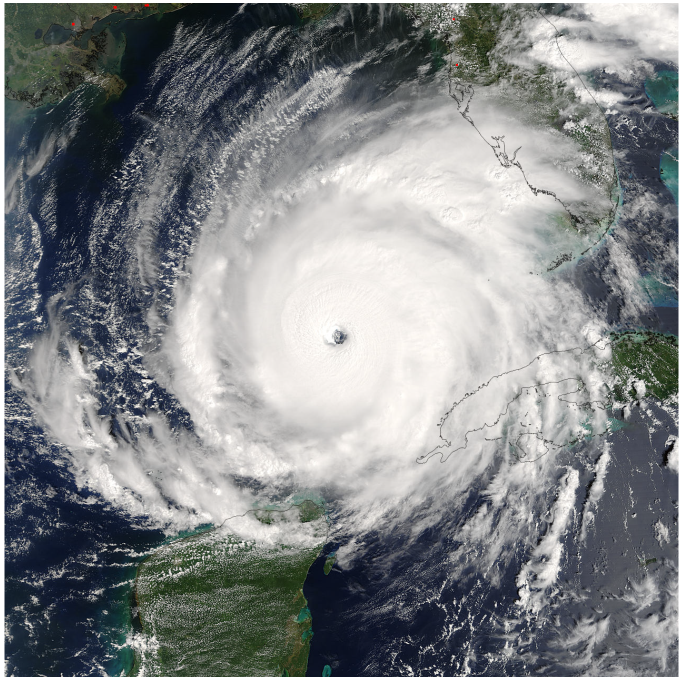
925 hPa
(∼ 0.8 km)

∼ 200 km

Dunkerton et al., Atmos. Chem. Phys., **9**, 5587–5646 (2009)

Scaling regime for matched asymptotic expansions





$$\underbrace{L_{\mathrm{syn}}; \quad |\boldsymbol{v}_{||}| \sim v_{\mathrm{syn}} \quad t_{\mathrm{syn}} \sim L_{\mathrm{syn}}/v_{\mathrm{syn}}}_{\text{farfield: classical QG theory}} \qquad \underbrace{L_{\mathrm{mes}} = \sqrt{\varepsilon}\,L_{\mathrm{syn}}; \quad v_{\mathrm{mes}} = \frac{1}{\sqrt{\varepsilon}}}_{\text{core: gradient wind scaling}}$$

$$\mathcal{C} \sim (vL)_{\mathrm{syn}};\ \mathrm{Ro}_{\mathrm{syn}} \sim \frac{v_{\mathrm{syn}}}{f L_{\mathrm{syn}}} = \mathcal{O}\left(\varepsilon\right) \qquad \mathcal{C} \sim (vL)_{\mathrm{mes}};\ \mathrm{Ro}_{\mathrm{mes}} \sim \frac{v_{\mathrm{mes}}}{f L_{\mathrm{mes}}} = \mathcal{O}\left(1\right)$$

Comparable levels of circulation $\mathcal{C}$!

Photo: Hurricane Rita from https://commons.wikimedia.org/wiki/File:HurricaneRita21Sept05a.jpg

Vortex motion ⇒ precessing quasi-modes*

Centerline evolution

(from the matching condition)

$$\frac{\partial \widehat{\boldsymbol{X}}}{\partial \tau} = \underbrace{\widehat{\boldsymbol{X}} \cdot (\boldsymbol{\nabla}_{\shortparallel} \overline{\boldsymbol{v}}_{\mathrm{QG}}) + \widehat{\boldsymbol{v}}^{*}_{\mathrm{QG}}}_{\text{background advection}} - \underbrace{\left(\ln \frac{1}{\sqrt{\varepsilon}} + \frac{1}{2}\right)(\boldsymbol{k} \times \chi)^{*} + (\boldsymbol{k} \times \boldsymbol{\Psi})}_{\text{self-induced motion}}$$

χ = fct(total circulation, centerline geometry)

$\boldsymbol{\Psi}$ = fct(**core structure**, centerline geometry, diabatic sources)

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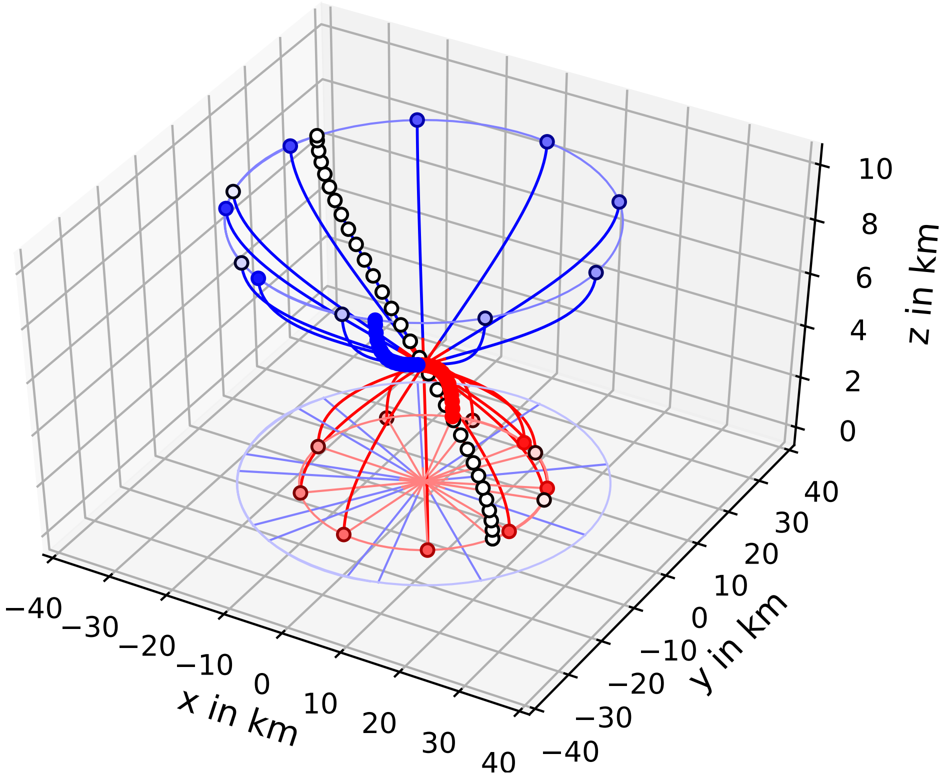
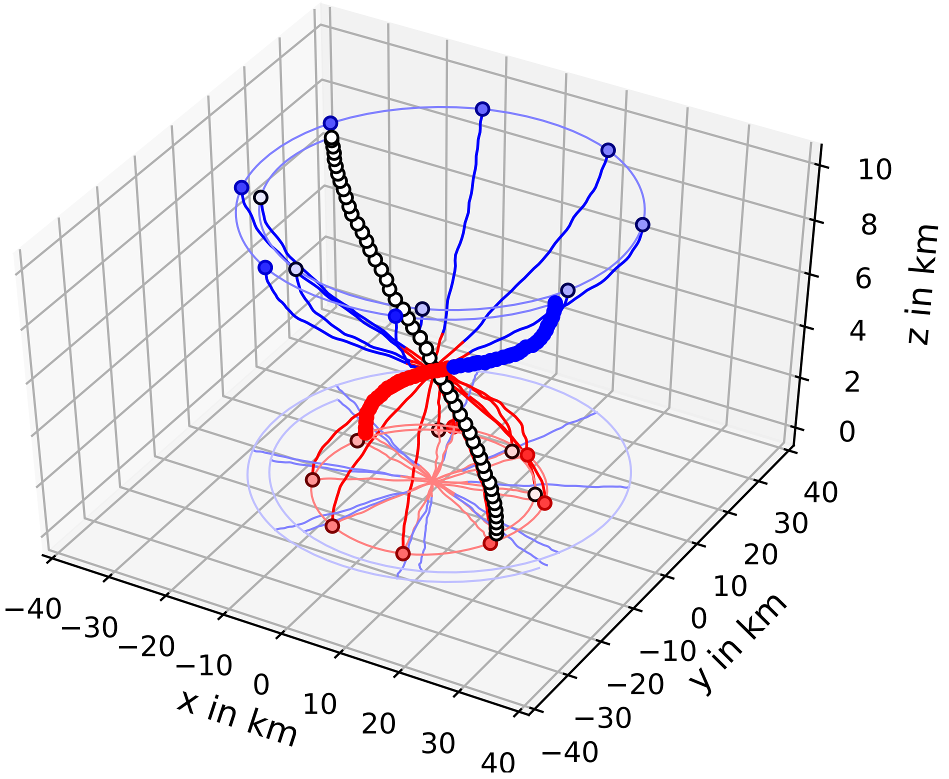
*effect of β-gyres; *akin to local-induction-approximation LIA *Grasso, Kallenbach, Montgomery, Reasor (1997, 2001, 2004)

Vortex motion ⇒ precessing quasi-modes*



asymptotic theory



3D Simulation (EULAG*)

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*

Adiabatic lifting and WTG

(0th & 1st circumferential Fourier modes: $w = w_0 + w_{11}\cos\theta + w_{12}\sin\theta + \ldots$)

gradient wind balance (0th) and hydrostatics (1st) in the tilted vortex

$$\underline{\frac{1}{\overline{\rho}}\frac{\partial p}{\partial r} = \frac{u_\theta^2}{r} + f\,u_\theta}\,, \qquad \underline{\Theta_{1\boldsymbol{k}} = -\frac{1}{\overline{\rho}}\frac{\partial p}{\partial r}\left(\boldsymbol{e}_r \cdot \frac{\partial \widehat{\boldsymbol{X}}}{\partial z}\right)_{1\boldsymbol{k}}}\,,$$

$$\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}} = \widehat{X}\cos\theta + \widehat{Y}\sin\theta\right)$$

potential temperature transport (1st)

$$-(-1)^k \frac{u_\theta}{r}\Theta_{1\boldsymbol{k}^*} + w_{1k}\frac{d\overline{\Theta}}{dz} = Q_{\Theta,1\boldsymbol{k}} \qquad (\boldsymbol{k}^* = 3 - k)$$

1st-mode phase relation: vertical velocity – diabatic sources & vortex tilt

$$\underline{w_{1\boldsymbol{k}}} = \frac{1}{d\overline{\Theta}/dz}\left[\underline{Q_{\Theta,1\boldsymbol{k}}} + \left(\boldsymbol{e}_r \cdot \underline{\frac{\partial \widehat{\boldsymbol{X}}^{\perp}}{\partial z}}\right)_{\boldsymbol{k}} \frac{u_\theta}{r}\left(\frac{u_\theta^2}{r} + f\,u_\theta\right)\right]$$

Spin-up by asynchronous heating

$$\underbrace{\frac{\partial u_{\theta,0}}{\partial \tau} + w_0 \frac{\partial u_{\theta,0}}{\partial z} + u_{r,00}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} + f\right)}_{\text{standard axisymmetric balance}} = -\,\boldsymbol{u_{r,*}}\left(\frac{u_\theta}{r} + f\right)$$

$$\boldsymbol{u_{r,*}} = \left\langle w\,\frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}\right)\right\rangle_\theta$$

$$\begin{aligned} \boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}} \;&=\; \widehat{X}\cos\theta + \widehat{Y}\sin\theta \\ w_{1\boldsymbol{k}} \;&=\; \frac{1}{d\overline{\Theta}/dz}\left[Q_{\Theta,1\boldsymbol{k}} + \frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}^{\perp}\right)_{\boldsymbol{k}} \frac{u_\theta}{r}\left(\frac{u_\theta^2}{r} + f\,u_\theta\right)\right] \end{aligned}$$

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$$\boldsymbol{u_{r,*}} = \left\langle w\,\frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}\right)\right\rangle_\theta = \underline{\frac{1}{d\overline{\Theta}/dz}\left(Q_{\Theta,1\boldsymbol{1}}\frac{\partial \widehat{X}}{\partial z} + Q_{\Theta,1\boldsymbol{2}}\frac{\partial \widehat{Y}}{\partial z}\right)}$$

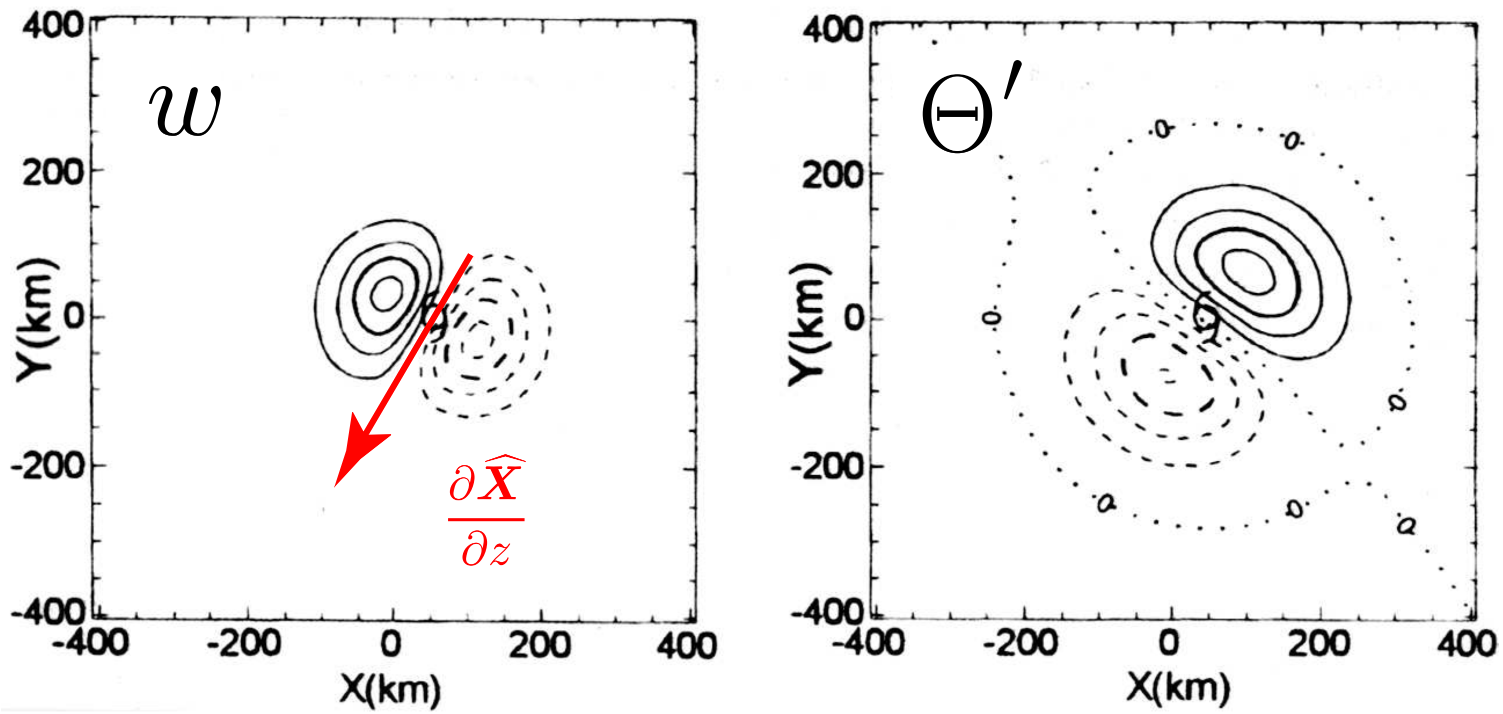
!!

$$\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}} \;=\; \widehat{X}\cos\theta + \widehat{Y}\sin\theta$$

$$w_{1\boldsymbol{k}} \;=\; \frac{1}{d\overline{\Theta}/dz}\left[\underbrace{Q_{\Theta,1\boldsymbol{k}}}_{\text{WTG}} + \underbrace{\frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}^{\perp}\right)_{\boldsymbol{k}}\,\frac{u_\theta}{r}\left(\frac{u_\theta^2}{r} + f\,u_\theta\right)}_{\text{adiabatic lifting}}\right]$$

The adiabatic lifting in a tilted vortex* *

figures adapted from Jones (1995)*



$$w_{1\boldsymbol{k}} = \frac{1}{d\overline{\Theta}/dz}\left[\qquad \frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}^{\perp}\right)_{\boldsymbol{k}} \frac{u_\theta}{r}\left(\frac{u_\theta^2}{r} + f\,u_\theta\right)\right]$$

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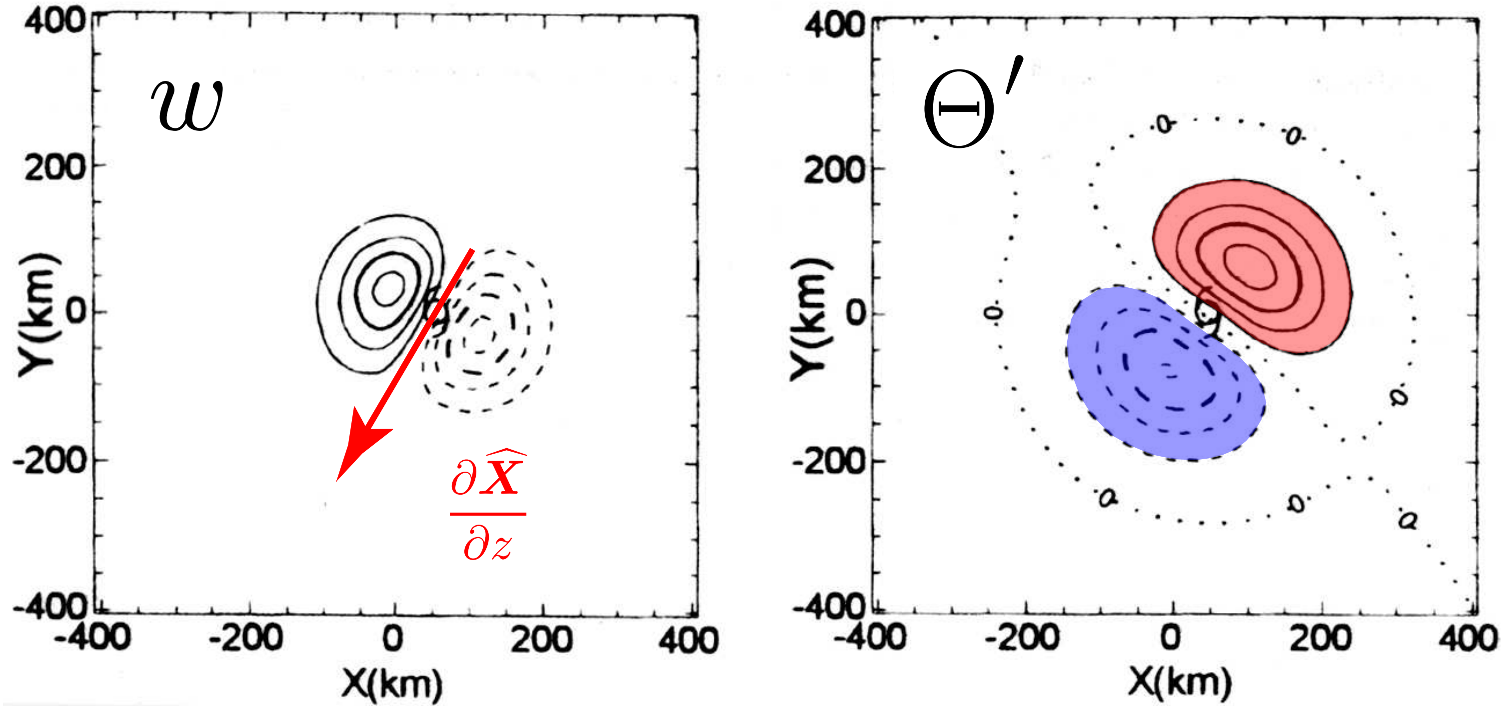
* Jones, Q.J.R. Met. Soc., **121**, 821–851 (1995)

* Frank & Ritchie, Mon. Wea. Rev., **127**, 2044–2061 (1999)

Heating pattern for max intensification (APE-theory)*



$$w_{1\boldsymbol{k}} = \frac{1}{d\overline{\Theta}/dz}\left[Q_{\Theta,1\boldsymbol{k}} + \frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}^{\perp}\right)_{\boldsymbol{k}} \frac{u_\theta}{r}\left(\frac{u_\theta^2}{r} + f\,u_\theta\right)\right]$$

figures adapted from: Jones (1995), Q.J.R. Met. Soc., **121**, 821–851

Lorenz, E. N., Generation of available potential energy and the intensity of the general circulation, *Tech. Rep.*, UCLA, (1955)

Compatibility with Lorenz′ APE theory*

$$\Big(r e_\mathrm{k}\Big)_t + \Big(r u_{r,0}[e_\mathrm{k} + p']\Big)_r + \Big(r w_0[e_\mathrm{k} + p']\Big)_z = \frac{r\overline{\rho}}{N^2\overline{\Theta}^2}\Big(\Theta'_0 Q_{\Theta,0} + \boldsymbol{\Theta}'_1 \cdot \boldsymbol{Q}_{\Theta,1}\Big)$$

$$e_k = \frac{\overline{\rho} u_\theta^2}{2}$$

Symmetric & asymmetric are equally important

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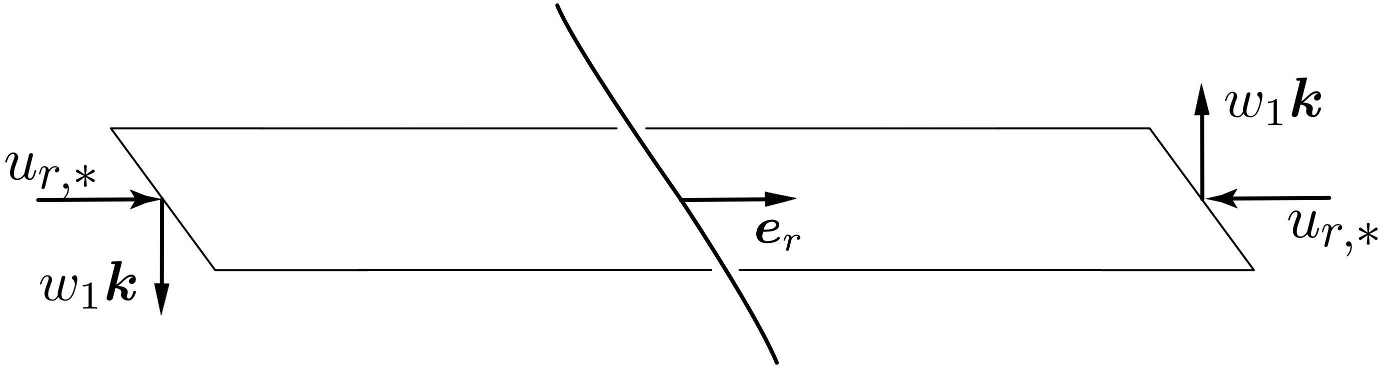
*Thanks to Olivier Pauluis!   “Available Potential Energy”

Dörffel et al., preprint, arXiv:1708.07674 (2017)

Radial transport & tilting by asymmetric heating

Circumferential Fouriermodes of vertical velocity

$$w_{1\boldsymbol{k}} \;=\; \frac{1}{d\overline{\Theta}/dz}\left[Q_{\Theta,1\boldsymbol{k}} + \frac{\partial}{\partial z}\left(\boldsymbol{e}_r\cdot\widehat{\boldsymbol{X}}^{\perp}\right)_{\boldsymbol{k}}\frac{u_\theta}{r}\left(\frac{u_\theta^2}{r} + f\,u_\theta\right)\right]$$

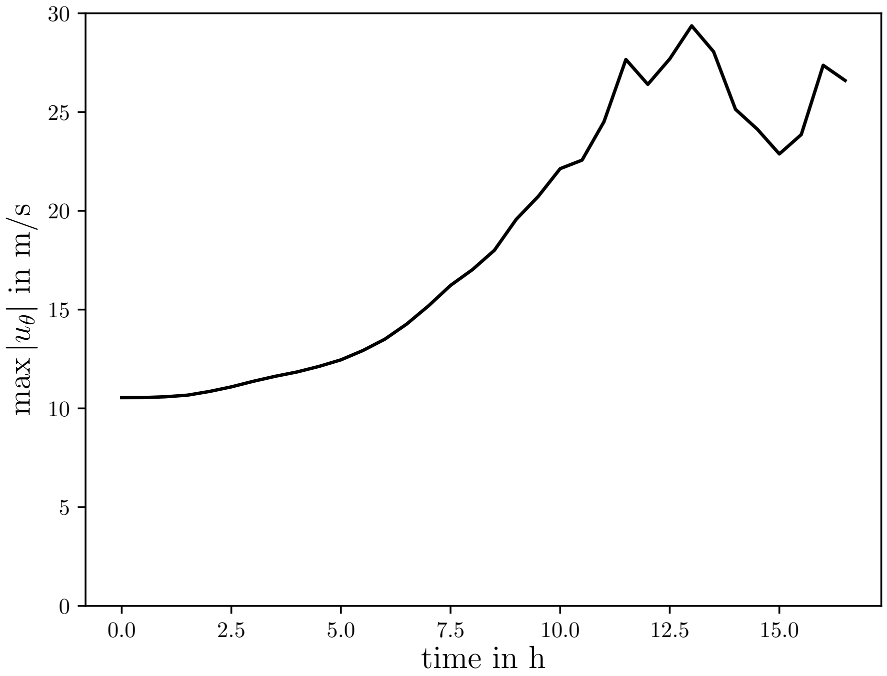


$$\boldsymbol{u_{r,*}} = \left\langle w\,\frac{\partial}{\partial z}\left(\boldsymbol{e}_r\cdot\widehat{\boldsymbol{X}}\right)\right\rangle_\theta = \underline{\frac{1}{d\overline{\Theta}/dz}\left(Q_{\Theta,1\mathbf{1}}\frac{\partial\widehat{X}}{\partial z} + Q_{\Theta,1\mathbf{2}}\frac{\partial\widehat{Y}}{\partial z}\right)}$$

$(w = w_0 + w_{1\mathbf{1}}\cos\theta + w_{1\mathbf{2}}\sin\theta + \ldots)$

Recent results

Qualitative corroboration through 3D-numerics



Artificial heating pattern:

$$w_{1\boldsymbol{k}} = \frac{1}{d\overline{\Theta}/dz}\left[-\frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}\right)_{\boldsymbol{k}} \frac{u_\theta}{r}\left(\frac{{u_\theta}^2}{r} + f\, u_\theta\right) + \frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}^{\perp}\right)_{\boldsymbol{k}} \frac{u_\theta}{r}\left(\frac{u_\theta^2}{r} + f\, u_\theta\right)\right]$$

---

* Ultimately leaves asymptotic regime!

Dörffel et al., preprint, arXiv:1708.07674 (2017)

Motivation

Structure of atmospheric vortices I: two scales

(Päschke et al., JFM, (2012))

Structure of atmospheric vortices II: cascade of scales

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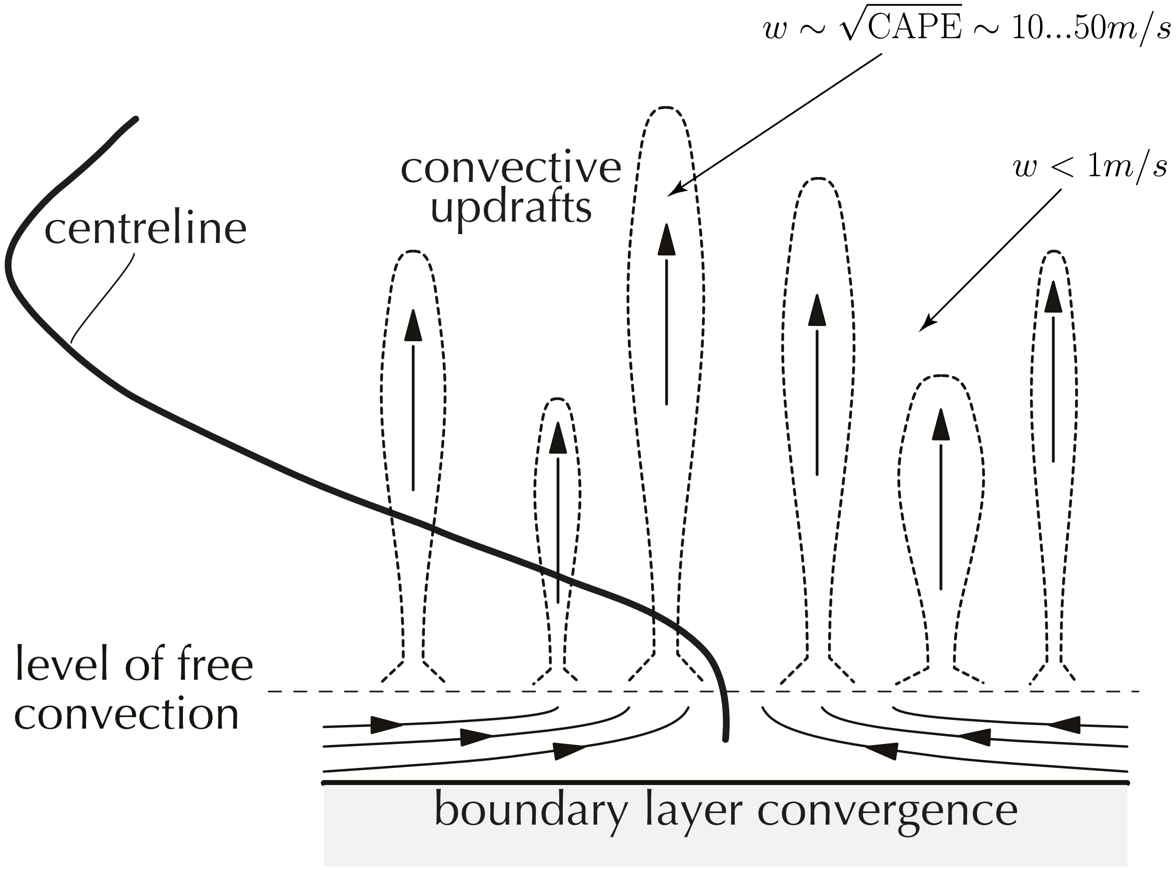
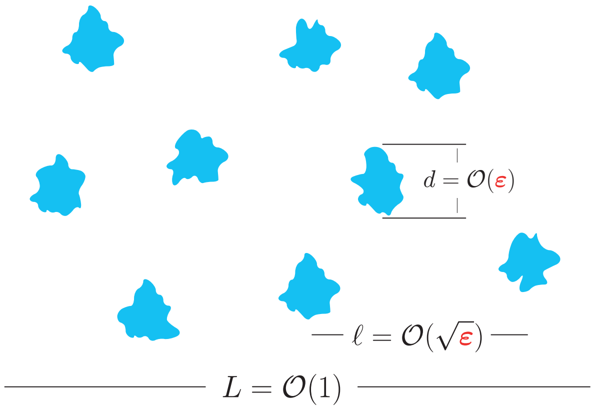
Conclusions

Convective updrafts





Convection concentrates in narrow towers (area fraction $\sigma \ll 1$)

Essentially dry dynamics between towers

Comparable average vertical mass fluxes

Calls for non-standard multiscale analysis

Spin-up by asymmetric convection

$$\underbrace{\frac{\partial u_{\theta,0}}{\partial \tau} + w_0 \frac{\partial u_{\theta,0}}{\partial z} + u_{r,00}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} + f\right)}_{\text{standard axisymmetric balance}} = -\,\boldsymbol{u_{r,*}}\left(\frac{u_\theta}{r} + f\right)$$

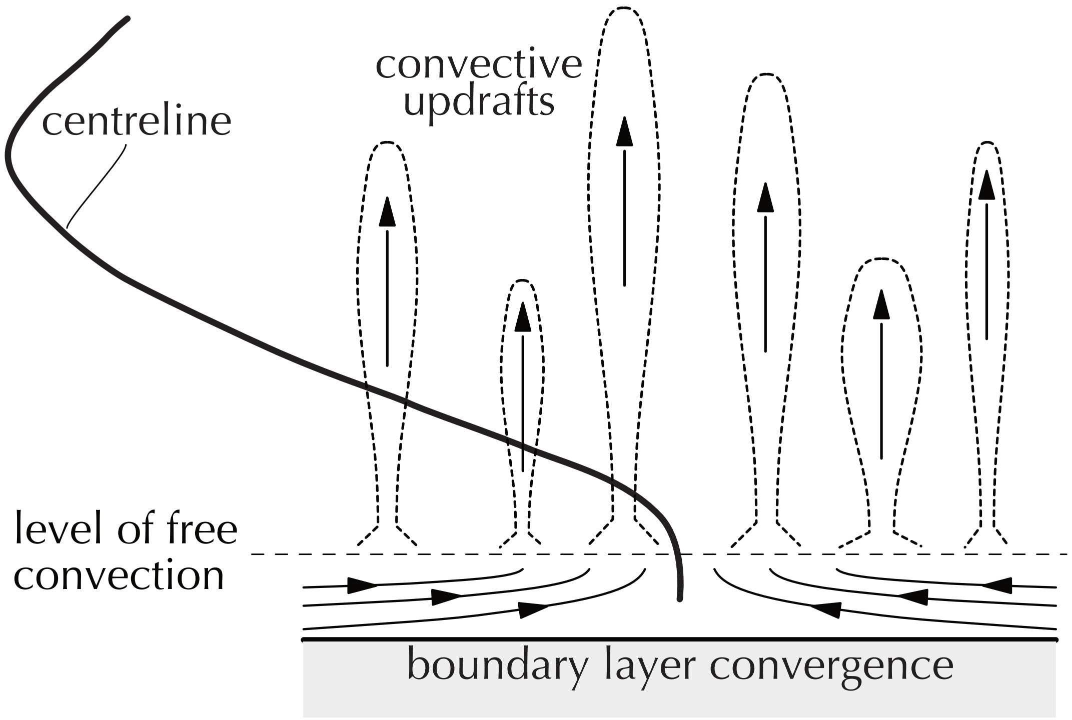
$$\boldsymbol{u_{r,*}} = \left\langle \boldsymbol{w}\,\frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}\right)\right\rangle_\theta = \underline{\overline{\boldsymbol{w}}_{\mathrm{upd},1\mathbf{1}}\frac{\partial \widehat{X}}{\partial z} + \overline{\boldsymbol{w}}_{\mathrm{upd},1\mathbf{2}}\frac{\partial \widehat{Y}}{\partial z}}$$

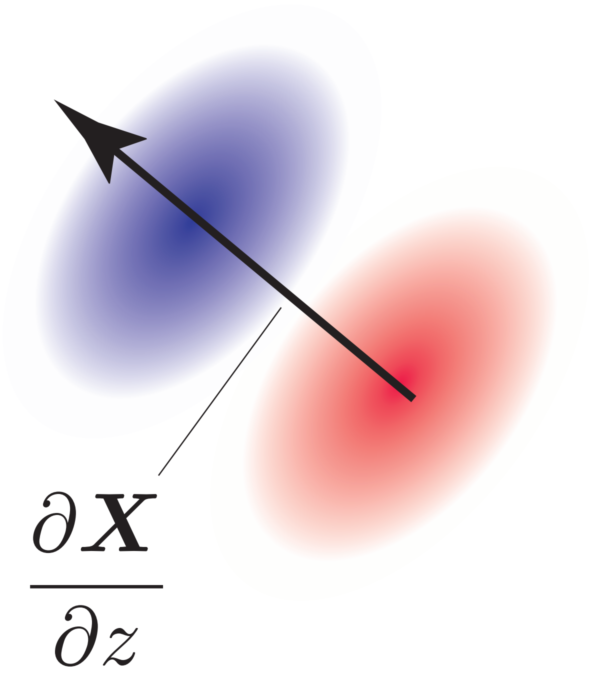
!!

Area averaged updraft fluxes take role of

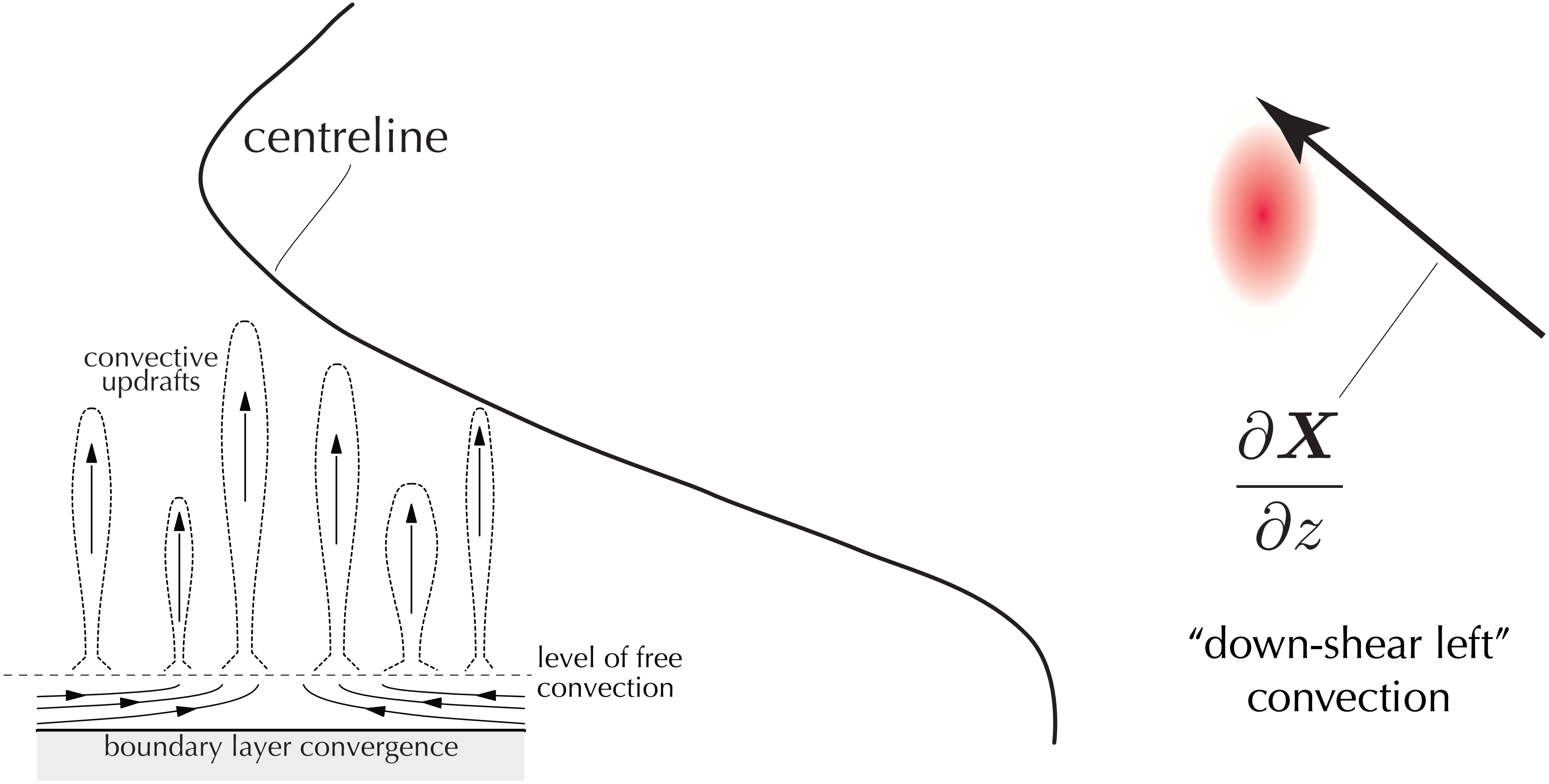
heating-induced vertical velocities

Intensification & tilt destabilization





Attenuation / tilt stabilization



$$\frac{\partial \boldsymbol{X}}{\partial z}$$

“down-shear left”
convection

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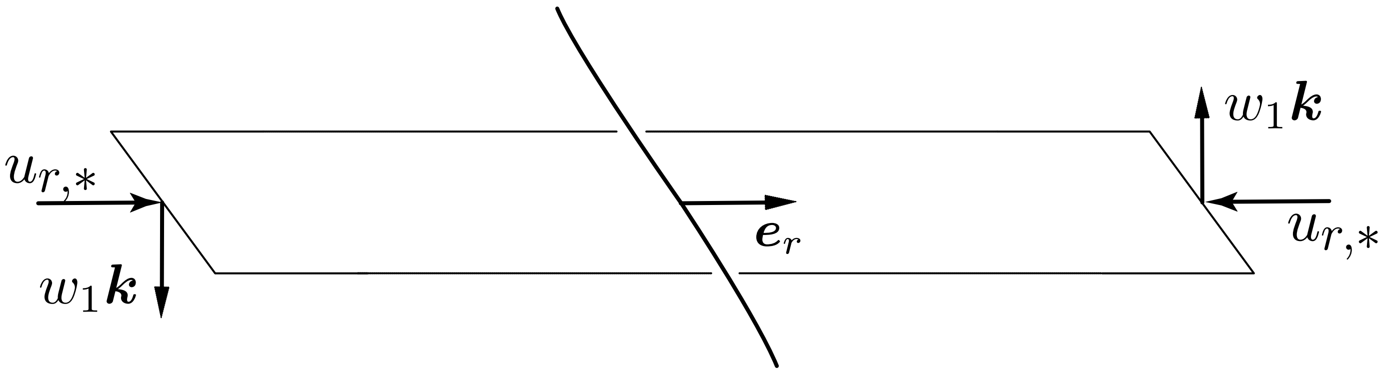
Spin-up by asymmetric heating

$$\underbrace{\frac{\partial u_{\theta,0}}{\partial \tau} + w_0 \frac{\partial u_{\theta,0}}{\partial z} + u_{r,00}\left(\frac{\partial u_\theta}{\partial r} + \frac{u_\theta}{r} + f\right)}_{\text{standard axisymmetric balance}} = -\,\boldsymbol{u_{r,*}}\left(\frac{u_\theta}{r} + f\right)$$

$$\boldsymbol{u_{r,*}} = \left\langle w \frac{\partial}{\partial z}\left(\boldsymbol{e}_r \cdot \widehat{\boldsymbol{X}}\right)\right\rangle_\theta = \underline{\frac{1}{d\overline{\Theta}/dz}\left(Q_{\Theta,1\mathbf{1}} \frac{\partial \widehat{X}}{\partial z} + Q_{\Theta,1\mathbf{2}} \frac{\partial \widehat{Y}}{\partial z}\right)}$$



Radial transport in a tilted vortex induced by asymmetric heating