Freie Universität Berlin

Internal wave dynamics in the atmosphere

Rupert Klein

Mathematik & Informatik, Freie Universität Berlin

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Scaling Cascades in Complex Systems

ECMWF

Scale-dependent models for atmospheric motions

Background on sound-proof models

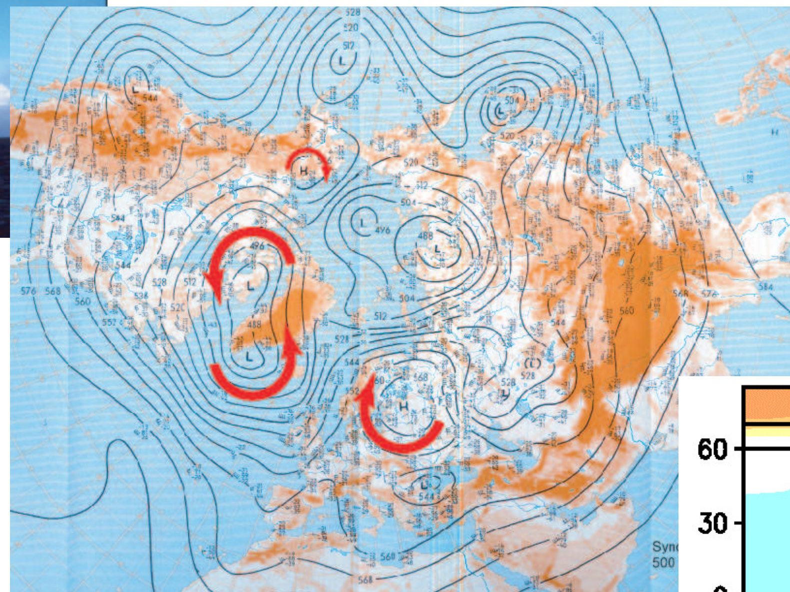
Formal asymptotic regime of validity

Steps towards a rigorous proof

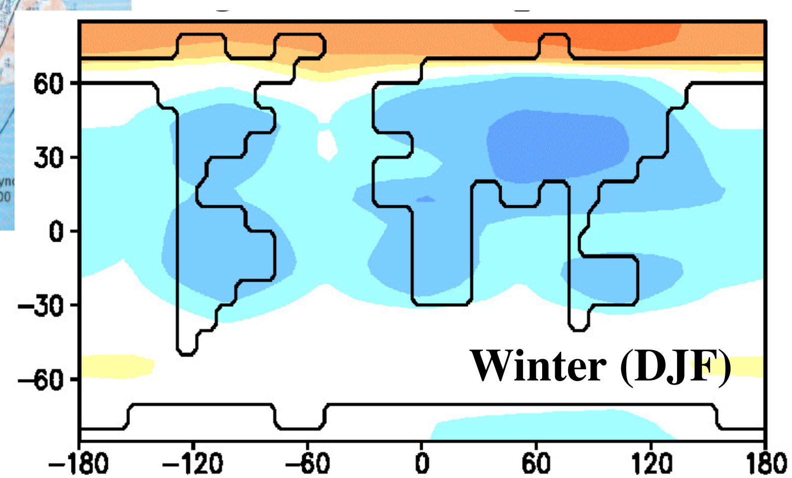
Summary

Scale-Dependent Models

10 km / 20 min



1000 km / 2 days



10000 km / 1 season

Scale-Dependent Models

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\boldsymbol{u}_z + \nabla\pi = \boldsymbol{S}_{\boldsymbol{u}}$$

$$w_t + \boldsymbol{u}\cdot\nabla w + w w_z + \pi_z = -\theta' + S_w$$

$$\theta'_t + \boldsymbol{u}\cdot\nabla\theta' + w\theta'_z = S'_\theta$$

$$\nabla\cdot(\rho_0\boldsymbol{u}) + (\rho_0 w)_z = 0$$

$$\theta = 1 + \varepsilon^4\theta'(\boldsymbol{x}, z, t) + o(\varepsilon^4)$$

Anelastic Boussinesque Model

10 km / 20 min

$$\left(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla\right) q = 0$$

$$q = \zeta^{(0)} + \Omega_0\beta\eta + \frac{\Omega_0}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{d\Theta/dz}\theta^{(3)}\right)$$

$$\zeta^{(0)} = \nabla^2\pi^{(3)}, \qquad \theta^{(3)} = -\frac{\partial\pi^{(3)}}{\partial z}, \qquad \boldsymbol{u}^{(0)} = \frac{1}{\Omega_0}\boldsymbol{k}\times\nabla\pi^{(3)}$$

Quasi-geostrophic theory

1000 km / 2 days

$$\frac{\partial Q_T}{\partial t} + \nabla\cdot\boldsymbol{F}_T = S_T$$

$$\frac{\partial Q_q}{\partial t} + \nabla\cdot\boldsymbol{F}_q = S_q$$

$$Q_\varphi = \int_{z_s}^{H_a}\rho\,\varphi\,dz\,, \qquad \boldsymbol{F}_\varphi = \int_{z_s}^{H_a}\rho\left(\boldsymbol{u}\,\varphi + (\widehat{\boldsymbol{u}'\varphi'}) + \boldsymbol{D}^\varphi\right)dz\,, \qquad \left(\varphi\in\{T,q\}\right)$$

$$T = T_s(t,\boldsymbol{x}) + \Gamma(t,\boldsymbol{x})\left(\min(z,H_T) - z_s\right), \qquad q = q_s(t,\boldsymbol{x})\exp\left(-\frac{z-z_s}{H_q}\right)$$

$$\rho = \rho_*\exp\left(-\frac{z}{h_{sc}}\right), \qquad p = p_*\exp\left(-\frac{\gamma z}{h_{sc}}\right) + p_0(t,\boldsymbol{x}) + g\rho_*\int_0^z\frac{T}{T_*}\,dz'$$

$$\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a\,, \qquad f\rho_*\boldsymbol{k}\times\boldsymbol{u}_g = -\nabla_{\boldsymbol{x}}p \qquad \boldsymbol{u}_\alpha = \alpha\nabla p_0$$

V. Petoukhov et al., *CLIMBER-2 ...*, Climate Dynamics, 16, (2000)

EMIC - equations (CLIMBER-2)

10000 km / 1 season

Scale-Dependent Models

Earth's radius	a	$\sim$	$6 \cdot 10^6$	m
Earth's rotation rate	Ω	$\sim$	10^{-4}	s^{-1}
Acceleration of gravity	g	$\sim$	9.81	ms^{-2}
Sea level pressure	p_{ref}	$\sim$	10^5	$\mathrm{kgm}^{-1}\mathrm{s}^{-2}$
H_2O freezing temperature	T_{ref}	$\sim$	273	K
Latent heat of water vapor	$L_{q_{\mathrm{vs}}}$	$\sim$	$4 \cdot 10^4$	$\mathrm{J\,kg}^{-1}\mathrm{K}^{-1}$
Dry gas constant	R	$\sim$	287	$\mathrm{m}^2\mathrm{s}^{-2}\mathrm{K}^{-1}$
Dry isentropic exponent	γ	$\sim$	1.4	

Distinguished limit:

$$\Pi_1 = \frac{h_{\mathrm{sc}}}{a} \sim 1.6 \cdot 10^{-3} \sim \varepsilon^3$$

$$\Pi_2 = \frac{L_{q_{\mathrm{vs}}}}{c_p T_{\mathrm{ref}}} \sim 1.5 \cdot 10^{-1} \sim \varepsilon$$

$$\Pi_3 = \frac{c_{\mathrm{ref}}}{\Omega a} \sim 4.7 \cdot 10^{-1} \sim \sqrt{\varepsilon}$$

where

$$h_{\mathrm{sc}} = \frac{R T_{\mathrm{ref}}}{g} = \frac{p_{\mathrm{ref}}}{\rho_{\mathrm{ref}} g} \sim 8.5\,\mathrm{km}$$

$$c_{\mathrm{ref}} = \sqrt{R T_{\mathrm{ref}}} = \sqrt{g h_{\mathrm{sc}}} \sim 300\ \mathrm{m/s}$$

$$c_p = \frac{\gamma R}{\gamma - 1}$$

Classical length scales and dimensionless numbers

$$
\begin{aligned}
L_{\text{mes}} &= \varepsilon^{-1}\, h_{\text{sc}} \\
L_{\text{syn}} &= \varepsilon^{-2}\, h_{\text{sc}} \\
L_{\text{Ob}} &= \varepsilon^{-5/2} h_{\text{sc}} \\
L_{\text{p}} &= \varepsilon^{-3}\, h_{\text{sc}}
\end{aligned}
\qquad
\begin{aligned}
\text{Fr}_{\text{int}} &\sim \varepsilon \\
\text{Ro}_{h_{\text{sc}}} &\sim \varepsilon^{-1} \\
\text{Ro}_{L_{\text{Ro}}} &\sim \varepsilon \\
\text{Ma} &\sim \varepsilon^{3/2}
\end{aligned}
$$

Remark:

There aren't *the* low Mach number limit equations.

Asymptotic results depend on the adopted distinguished limit

and scalings of length, time and initial data !

Compressible flow equations with general source terms

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + {}_{\parallel}\boldsymbol{v}\cdot{}_{\parallel}\nabla + w\frac{\partial}{\partial z}\right){}_{\parallel}\boldsymbol{v} \;+\; \varepsilon\;{}_{\parallel}(2\boldsymbol{\Omega}\times\boldsymbol{v}) + \frac{1}{\varepsilon^3\rho}\,\nabla_{\parallel}p &= \boldsymbol{S}_{{}_{\parallel}\boldsymbol{v}}\,,\\
\left(\frac{\partial}{\partial t} + {}_{\parallel}\boldsymbol{v}\cdot{}_{\parallel}\nabla + w\frac{\partial}{\partial z}\right)w \;+\; \varepsilon\,(2\boldsymbol{\Omega}\times\boldsymbol{v})_{\perp} + \frac{1}{\varepsilon^3\rho}\frac{\partial p}{\partial z} &= S_w - \frac{1}{\varepsilon^3}\,,\\
\left(\frac{\partial}{\partial t} + {}_{\parallel}\boldsymbol{v}\cdot{}_{\parallel}\nabla + w\frac{\partial}{\partial z}\right)\rho \;+\; \rho\,\nabla\cdot\boldsymbol{v} &= 0\,,\\
\left(\frac{\partial}{\partial t} + {}_{\parallel}\boldsymbol{v}\cdot{}_{\parallel}\nabla + w\frac{\partial}{\partial z}\right)\Theta &= S_{\Theta}\\
\Theta &= \frac{p^{1/\gamma}}{\rho}
\end{aligned}
$$

Asymptotic single-scale ansatz

$$
\mathrm{U}(t,\boldsymbol{x},z;\varepsilon) = \sum_{i=0}^{m}\phi_i(\varepsilon)\;\mathrm{U}^{(i)}(t,\boldsymbol{x},z;\varepsilon) + o\big(\phi_m(\varepsilon)\big)
$$

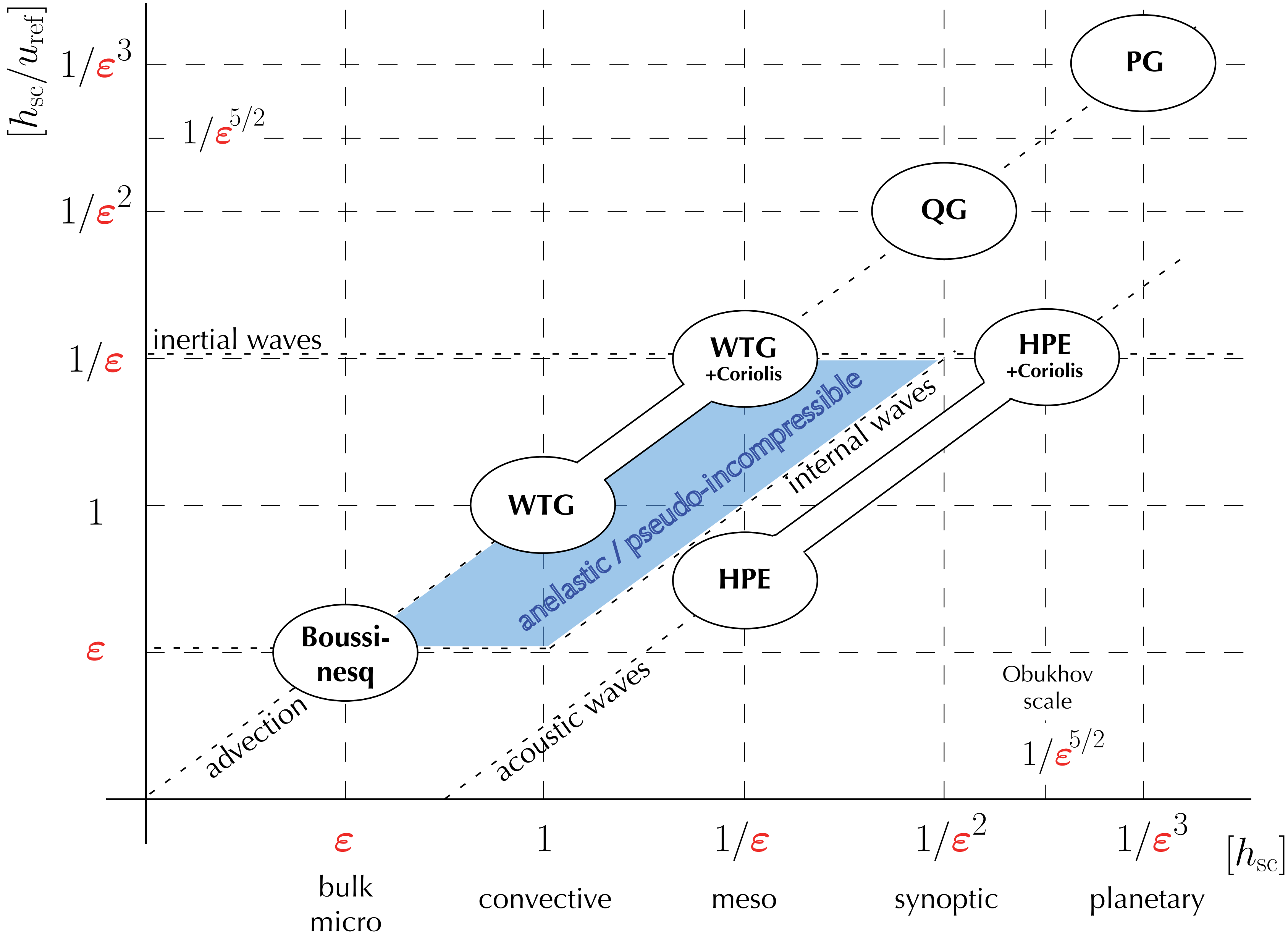
Scale-Dependent Models

Recovered classical single-scale models:

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\frac{t}{\varepsilon}, \boldsymbol{x}, \frac{z}{\varepsilon})$	Linear small scale internal gravity waves
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(t, \boldsymbol{x}, z)$	Anelastic & pseudo-incompressible models
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon t, \varepsilon^2 \boldsymbol{x}, z)$	Linear large scale internal gravity waves
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$	Mid-latitude Quasi-Geostrophic Flow
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$	Equatorial Weak Temperature Gradients
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^{-1} \xi(\varepsilon^2 \boldsymbol{x}), z)$	Semi-geostrophic flow
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\underline{\varepsilon^{3/2}} t, \underline{\varepsilon^{5/2}} x, \underline{\varepsilon^{5/2}} y, z)$	Kelvin, Yanai, Rossby, and gravity Waves

These all share one distinguished limit $\Rightarrow$ Starting point for multiscale analyses!

Scale-Dependent Models



R.K., Ann. Rev. Fluid Mech., **42**, 2010

What about the puzzle?

Compressible flow equations

$$\frac{D_{\shortparallel}\,\boldsymbol{v}}{Dt} + \varepsilon\,(2\boldsymbol{\Omega}\times\boldsymbol{v})_{\shortparallel} + \frac{1}{\varepsilon^3\rho}\,\nabla_{\parallel}p = 0\,,$$

$$\frac{Dw}{Dt} + \varepsilon\,(2\boldsymbol{\Omega}\times\boldsymbol{v})_{\perp} + \frac{1}{\varepsilon^3\rho}\frac{\partial p}{\partial z} = -\frac{1}{\varepsilon^3}\,,$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel}\cdot\nabla_{\shortparallel} + w\frac{\partial}{\partial z}\right)\rho + \rho\,\nabla\cdot\boldsymbol{v} = 0\,,$$

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel}\cdot\nabla_{\shortparallel} + w\frac{\partial}{\partial z}\right)\Theta = 0$$

$$\Theta = \frac{p^{1/\gamma}}{\rho}$$

distinguished limit

$$\mathrm{Fr}_{\mathrm{int}} \sim \varepsilon$$
$$\mathrm{Ro}_{h_{\mathrm{sc}}} \sim \varepsilon^{-1}$$
$$\mathrm{Ro}_{L_{\mathrm{Ro}}} \sim \varepsilon$$
$$\mathrm{Ma} \sim \varepsilon^{3/2}$$

length / time scalings

$$\boldsymbol{x} = \frac{\boldsymbol{x}'}{h_{\mathrm{sc}}}$$

$$z = \frac{z'}{h_{\mathrm{sc}}}$$

$$t = \frac{t'}{h_{\mathrm{sc}}/u_{\mathrm{ref}}}$$

One possible solution

Leading orders

$$\nabla_{\parallel} p = 0 \qquad (1)$$

$$\partial_z p = -\rho \qquad (2)$$

$$\rho_t + \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad (3)$$

$$\frac{D\Theta}{Dt} = 0 \qquad (4)$$

$$\Theta = \frac{p^{1/\gamma}}{\rho}\,. \qquad (5)$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\parallel} \cdot \nabla_{\parallel} + w\frac{\partial}{\partial z}\right)$$

$$(2),\ (5) \ \Rightarrow \qquad \nabla_{\parallel}\rho = \nabla_{\parallel}\Theta = 0 \qquad (6)$$

$$(4)\ \ \&\ \ \Theta = \text{const} \Rightarrow \qquad (4)\checkmark \qquad (7)$$

$$(3) \ \Rightarrow \qquad \nabla \cdot (\rho \boldsymbol{v}) = 0 \qquad (8)$$

$$\Downarrow$$

Anelastic & pseudo-incompressible* models

(key aspect: weak stratification)

*** also called "soundproof models"**

Scale-dependent models for atmospheric motions

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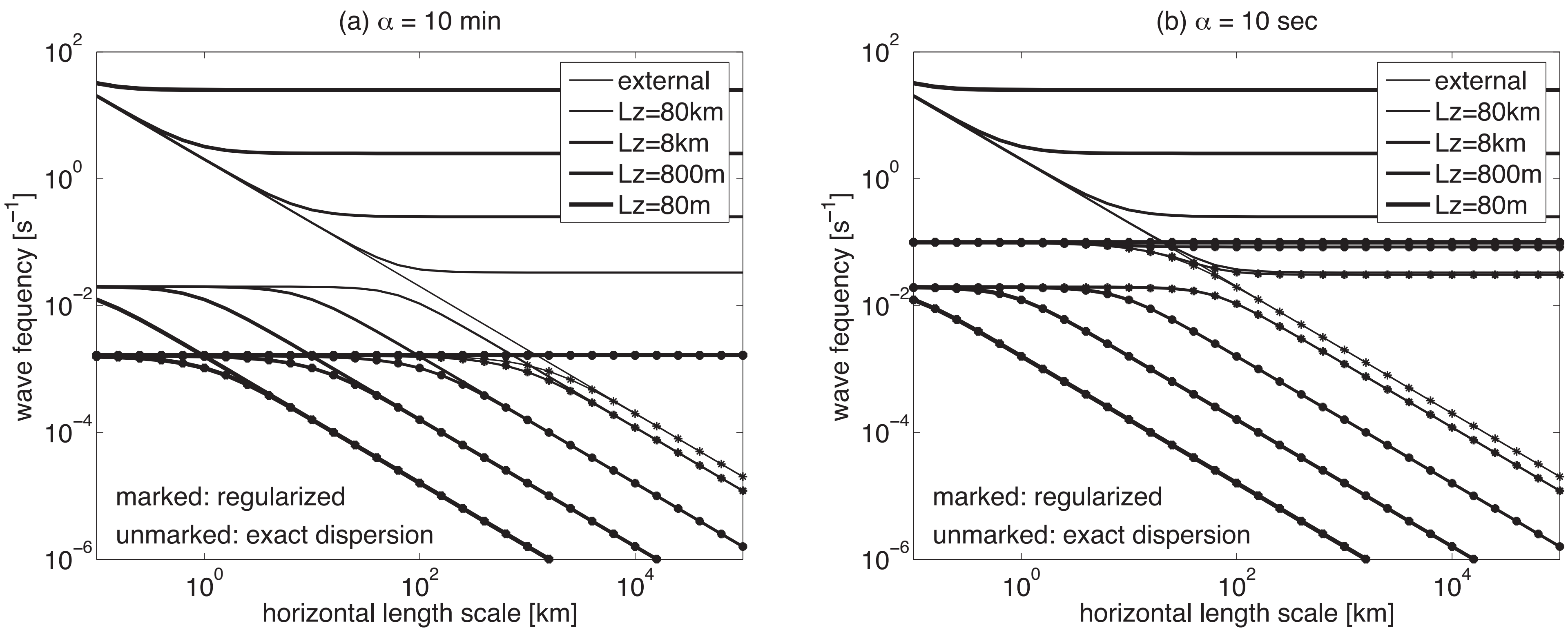
Motivation ... Numerics

Why not simply solve the full compressible flow equations?



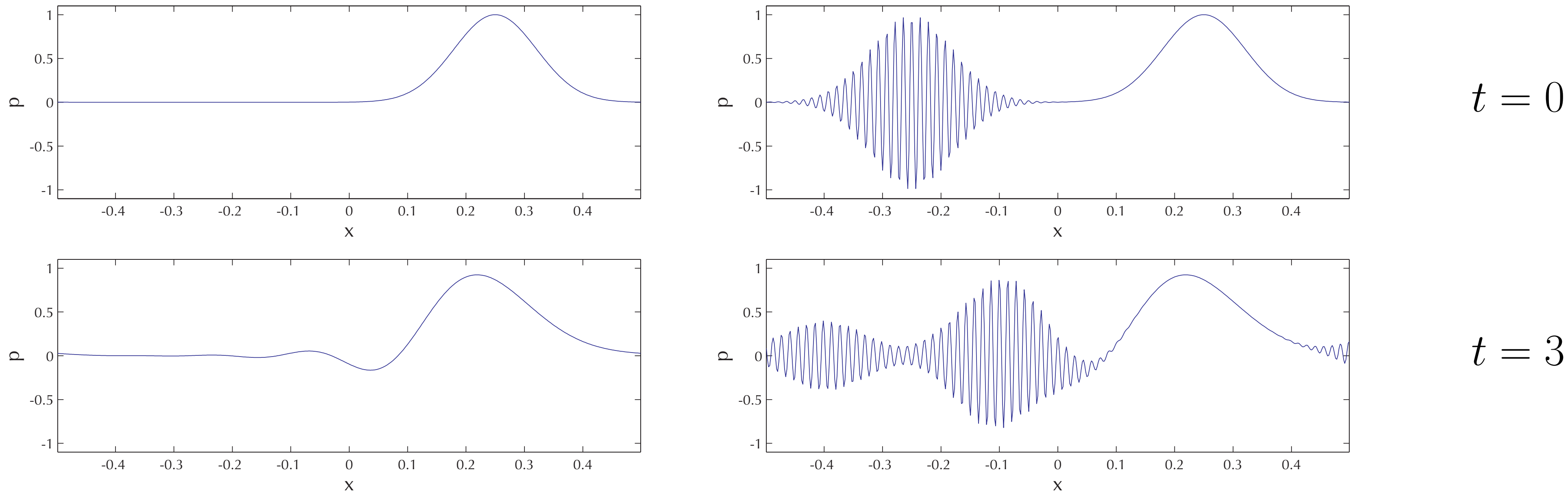
Dispersion relations for acoustic, Lamb, and internal waves

Motivation ... Numerics

Why not simply solve the full compressible equations?

Linear Acoustics, simple wave initial data, periodic domain

(integration: implicit midpoint rule, staggered grid, 512 ***grid pts.,*** $\mathrm{CFL} = 10$***)***



Central questions:

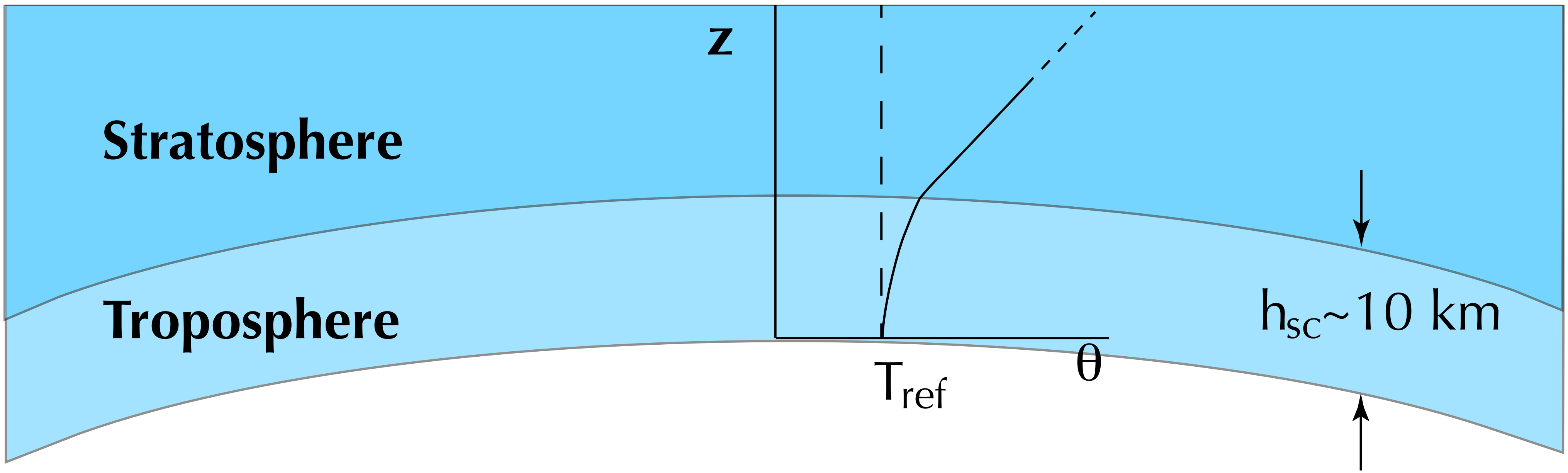
How to characerize a fully compressible flow

at sub-acoustic time scales?

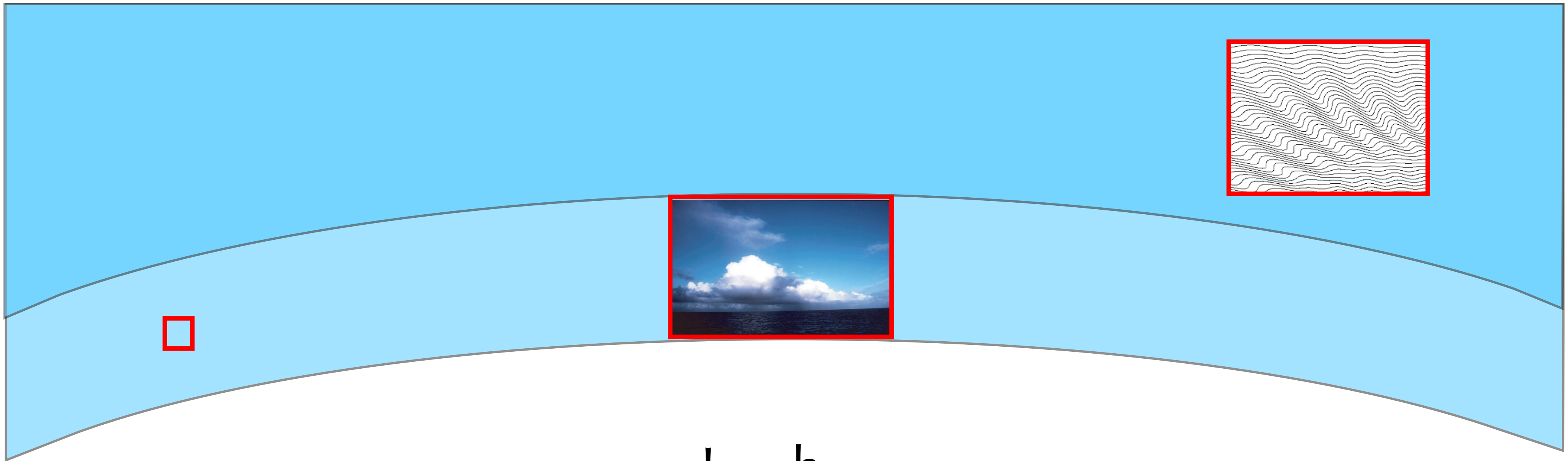
What should be the “required” limit behaviour of a numerical flow solver?

The answers depend on the scaling regimes considered!

Scaling regimes



Scaling regimes



$L \ll h_{sc}$

Boussinesq

$L \sim h_{sc}$

anelastic &
pseudo-incompressible

$l \ll L \sim h_{sc}$

psinc + WKB

R.K., TCFD, 2009; R.K. et al., JAS, 2010; Achatz et al., JFM, 2010

Compressible flow equations $L \sim h_{\rm sc}$

$$
\begin{aligned}
\boldsymbol{\rho_t} + \nabla\cdot(\rho \boldsymbol{v}) &= 0 \\
(\rho \boldsymbol{u})_t + \nabla\cdot(\rho \boldsymbol{v} \circ \boldsymbol{u}) + P \nabla_{\|} \pi &= 0 \\
(\rho w)_t + \nabla\cdot(\rho \boldsymbol{v} w) + P \pi_z &= -\rho g \\
\boldsymbol{P_t} + \nabla\cdot(P \boldsymbol{v}) &= 0
\end{aligned}
$$

drop term for:

anelastic[†] (approx.)

pseudo-incompressible[*]

$$
P = p^{\frac{1}{\gamma}} = \rho\theta\,, \qquad \pi = p/\Gamma P\,, \qquad \Gamma = c_p/R\,, \qquad \boldsymbol{v} = \boldsymbol{u} + w\boldsymbol{k} \quad (\boldsymbol{u}\cdot\boldsymbol{k} \equiv 0)
$$

Parameter range & length and time scales of asymptotic validity ?

[†] e.g. Lipps & Hemler, JAS, **29**, 2192–2210 (1982)

[*] Durran, JAS, **46**, 1453–1461 (1989)

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From here on: ε is the Mach number

Characteristic inverse time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\mathrm{ref}}}{h_{\mathrm{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\mathrm{sc}}}}{u_{\mathrm{ref}}}\sqrt{\dfrac{h_{\mathrm{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon}\sqrt{\dfrac{h_{\mathrm{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}}{h_{\mathrm{sc}}} = \dfrac{\sqrt{g h_{\mathrm{sc}}}}{h_{\mathrm{sc}}}$	$\dfrac{\sqrt{g h_{\mathrm{sc}}}}{u_{\mathrm{ref}}} = \dfrac{1}{\varepsilon}$

Regimes of Validity ... Design Regime

Characteristic inverse time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2)$$

* Ogura & Phillips (1962)

Characteristic inverse time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Ogura & Phillips' regime* with two time scales

$$\overline{\theta} = 1 + \varepsilon^2 \widehat{\theta}(z) + \dots \quad \Rightarrow \quad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^2) \quad \Rightarrow \quad \Delta\overline{\theta}\,\Big|_{z=0}^{h_{\text{sc}}} < 1\text{ K}$$

* Ogura & Phillips (1962)

Regimes of Validity ... Design Regime

Characteristic inverse time scales

		dimensional	dimensionless
advection	:	$\dfrac{u_{\text{ref}}}{h_{\text{sc}}}$	1
internal waves	:	$N = \sqrt{\dfrac{g}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\overline{\theta}}{dz}} = \dfrac{1}{\varepsilon^{\nu}}\sqrt{\dfrac{h_{\text{sc}}}{\overline{\theta}}\dfrac{d\widehat{\theta}}{dz}}$
sound	:	$\dfrac{\sqrt{p_{\text{ref}}/\rho_{\text{ref}}}}{h_{\text{sc}}} = \dfrac{\sqrt{g h_{\text{sc}}}}{h_{\text{sc}}}$	$\dfrac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}} = \dfrac{1}{\varepsilon}$

Realistic regime with three time scales

$$\overline{\theta} = 1 + \varepsilon^{\mu}\widehat{\theta}(z) + \dots \qquad \Rightarrow \qquad \frac{h_{\text{sc}}}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\varepsilon^{\mu}) \qquad (\nu = 1 - \mu/2)$$

Regimes of Validity ... Design Regime

Full compressible flow equations in perturbation variables

$$
\begin{aligned}
\tilde{\theta}_t + \frac{1}{\varepsilon^{\nu}}\,\tilde{w}\,\frac{d\widehat{\theta}}{dz} & = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\theta} \\
\tilde{\boldsymbol{v}}_t - \frac{1}{\varepsilon^{\nu}}\,\frac{\tilde{\theta}}{\overline{\theta}}\,\boldsymbol{k} + \frac{1}{\varepsilon}\,\overline{\theta}\nabla\tilde{\pi} & = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{v}} - \varepsilon^{1-\nu}\tilde{\theta}\nabla\tilde{\pi} \quad . \\
\tilde{\pi}_t + \frac{1}{\varepsilon}\left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) & = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\pi} - \gamma\Gamma\tilde{\pi}\nabla\cdot\tilde{\boldsymbol{v}}
\end{aligned}
$$

Issues to be clarified:

Comparison of the internal wave modes (time scale ε^{ν})

Acoustic-internal wave interactions / resonances

Control of nonlinearities for non-acoustic data

Internal wave scalings for $t = O(\varepsilon^{\nu})$: $\tau = \dfrac{t}{\varepsilon^{\nu}}\,, \qquad \pi^* = \varepsilon^{\nu-1}\tilde{\pi}$

Regimes of Validity ... Design Regime

Fast linear compressible / pseudo-incompressible modes

$$\tilde{\theta}_\tau + \tilde{w}\,\frac{d\widehat{\theta}}{dz} = 0$$

$$\tilde{\boldsymbol{v}}_\tau - \frac{\tilde{\theta}}{\overline{\theta}}\,\boldsymbol{k} + \overline{\theta}\nabla\pi^* = 0$$

$$\varepsilon^{\mu}\,\pi^*_\tau + \left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) = 0$$

Vertical mode expansion (separation of variables)

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\boldsymbol{u}} \\ \tilde{w} \\ \pi^* \end{pmatrix}(\vartheta, \boldsymbol{x}, z) = \begin{pmatrix} \Theta^* \\ \boldsymbol{U}^* \\ W^* \\ \Pi^* \end{pmatrix}(z)\;\exp\left(i\left[\boldsymbol{\omega}\vartheta - \boldsymbol{\lambda}\cdot\boldsymbol{x}\right]\right)$$

Notice the non-constant coefficients involving $\overline{\theta}, \overline{\pi}, \widehat{\theta}$ ...

Regimes of Validity ... Design Regime

Compressible and pseudo-incompressible vertical modes $(W = \overline{P} W^*)$

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^2}}\frac{1}{\overline{\theta}\,\overline{P}}\frac{dW}{dz}\right)+\frac{\lambda^2}{\overline{\theta}\,\overline{P}}W = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}\,\overline{P}}W$$

$\varepsilon^{\mu} = 0$: pseudo-incompressible case

regular Sturm-Liouville problem for internal wave modes ***(rigid lid)***

$\varepsilon^{\mu} > 0$: compressible case

nonlinear Sturm-Liouville problem* ...

$\frac{\omega^2/\lambda^2}{\bar{c}^2} = O(1)$: perturbations of pseudo-incompressible modes & EVals

*** Taylor-Goldstein equation**

Regimes of Validity ... Design Regime

$$-\frac{d}{dz}\left(\frac{1}{1-\varepsilon^{\mu}\frac{\omega^2/\lambda^2}{\bar{c}^2}}\frac{1}{\overline{\theta}\,\overline{P}}\frac{dW}{dz}\right)+\frac{\lambda^2}{\overline{\theta}\,\overline{P}}W = \frac{1}{\omega^2}\frac{\lambda^2 N^2}{\overline{\theta}\,\overline{P}}W$$

Internal wave modes $\left(\frac{\omega^2/\lambda^2}{\bar{c}^2} = O(1)\right)$

- pseudo-incompressible modes/EVals $=$ compressible modes/EVals $+\ O(\varepsilon^{\mu})$ †
- phase errors remain small ***over advection time scales*** for $\mu > \frac{2}{3}$

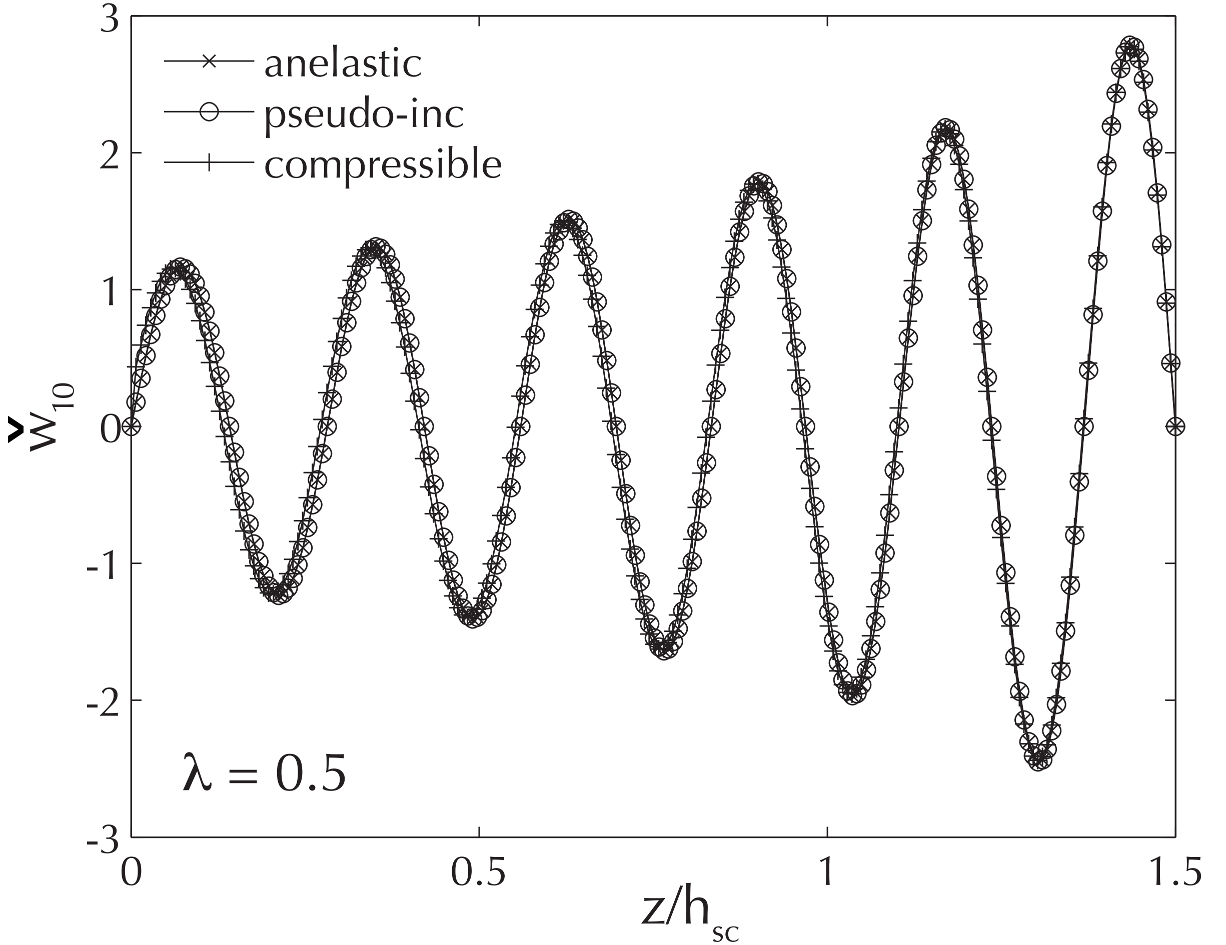
Anelastic and pseudo-incompressible models remain relevant for stratifications

$$\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{dz} < O(\varepsilon^{2/3}) \qquad \Rightarrow \qquad \Delta\theta\big|_0^{h_{sc}} \lesssim 40\ \mathrm{K}$$

not merely up to $O(\varepsilon^2)$ as in Ogura-Phillips (1962)

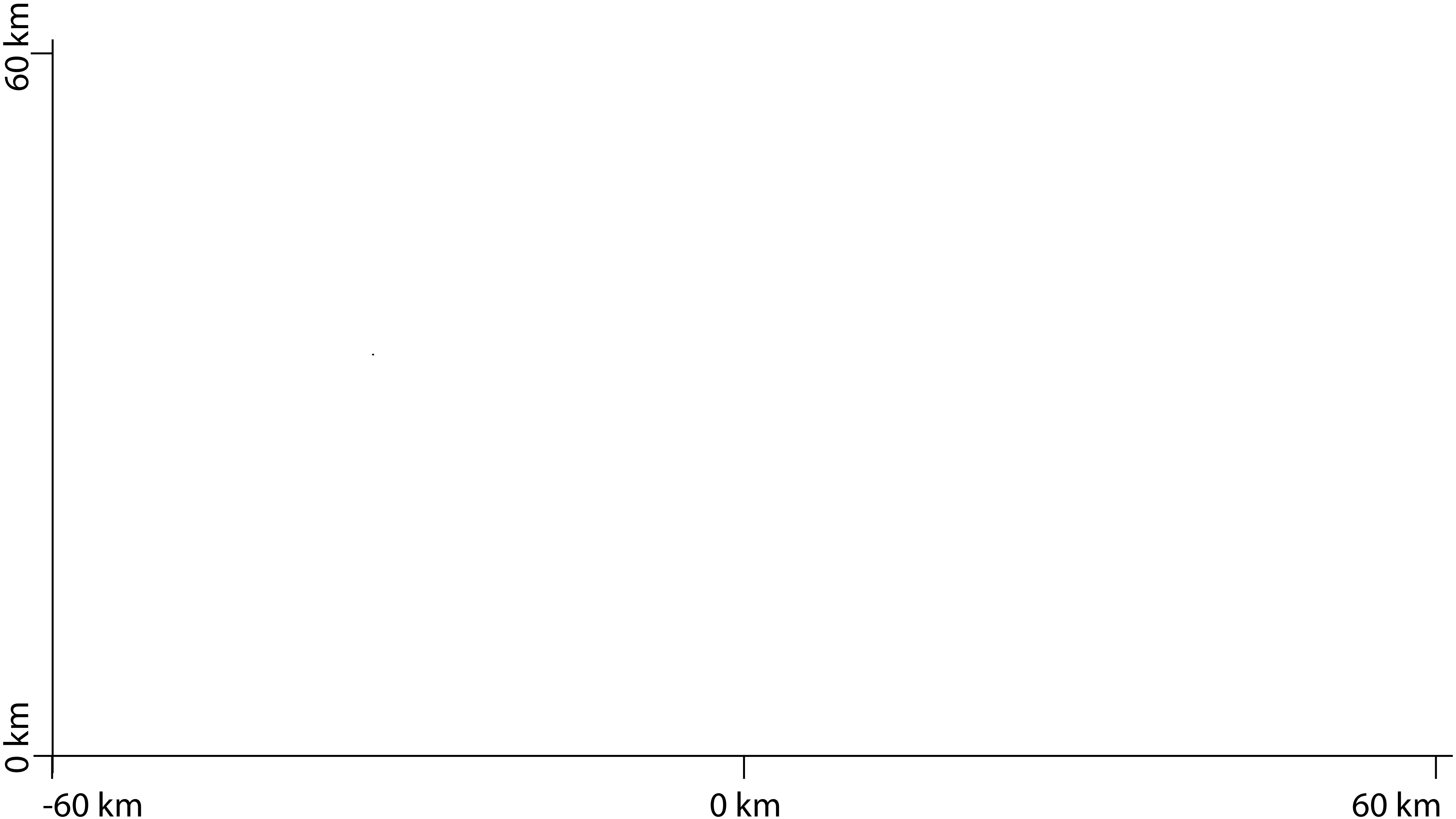
† rigorous proof with D. Bresch

A typical vertical structure function $(L \sim \pi h_{sc} \sim 30 \text{ km}; \quad \varepsilon^{\mu} = \mathbf{0.1})$



thanks to Dr. V. LeDoux, Ghent, for the SL-solver MATSLISE!

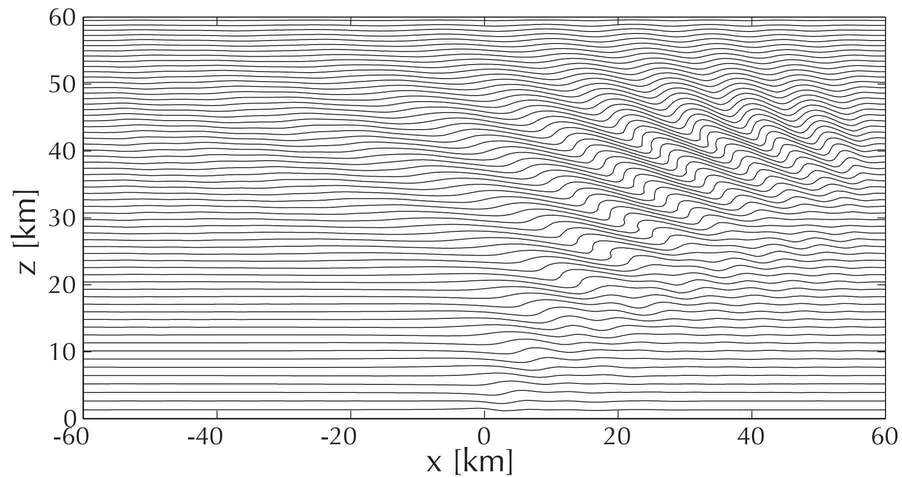
Breaking wave-test for anelastic models (Smolarkiewicz & Margolin (1997))



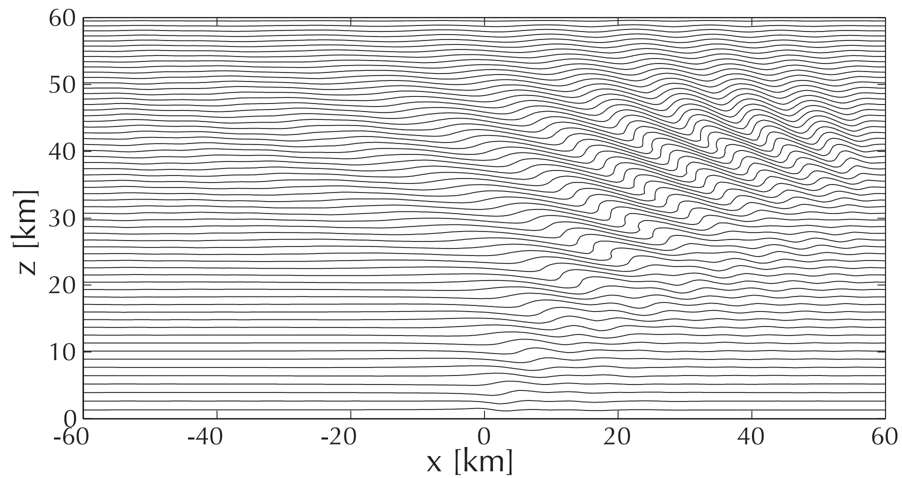
Potential temperature contours

Results at time $t = 2\,h$

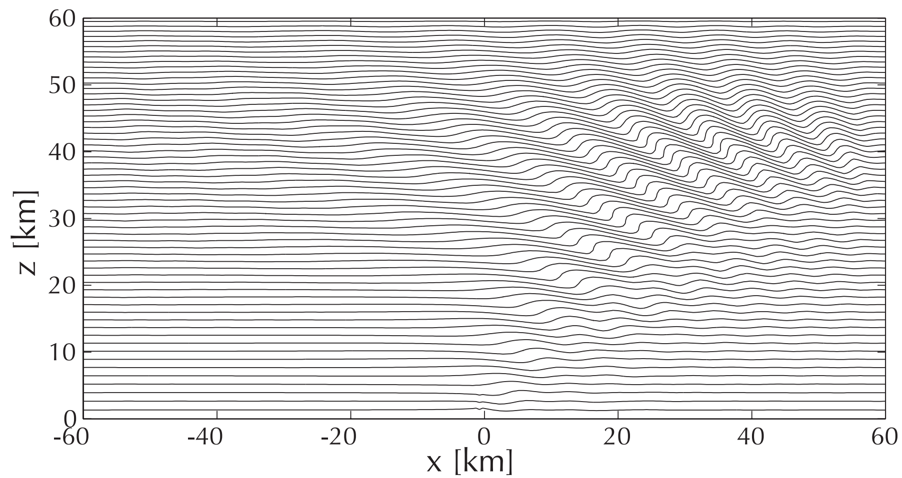
pseudo-incompressible



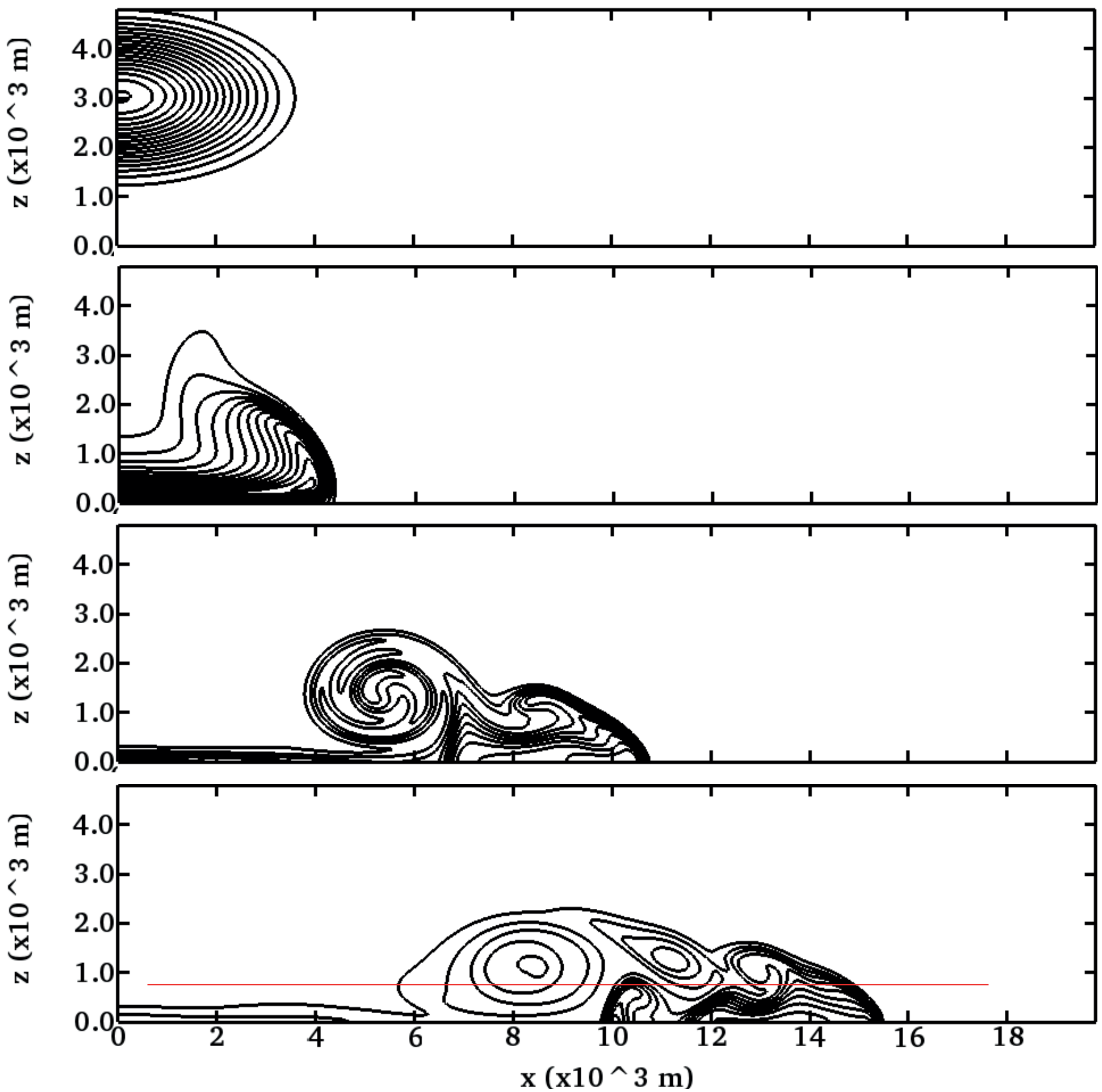
compressible, $\mathrm{CFL}_{\mathrm{adv}} = 1$

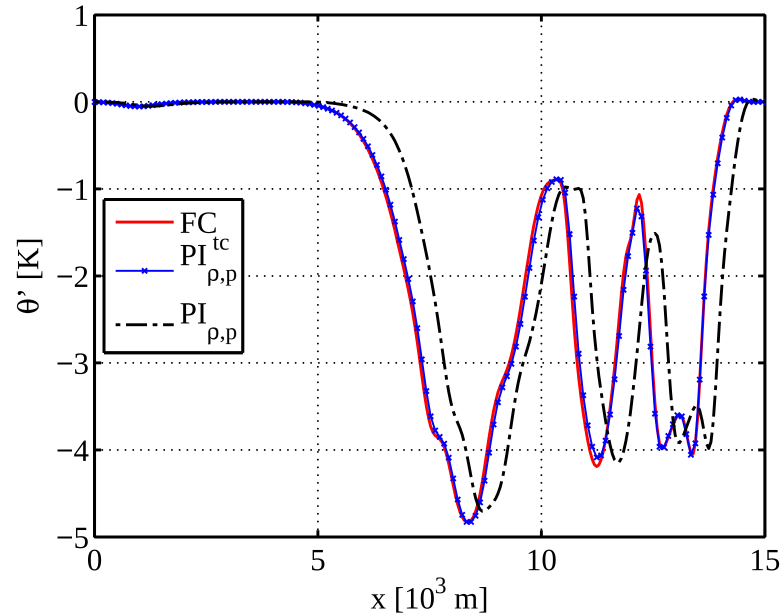


compressible, $\mathrm{CFL}_{\mathrm{ac}} = 2$



Regimes of Validity ... Design Regime





fully compressible

pseudo-incompressible

Benacchio et al., Mon. Wea. Rev., **142**, 4416–4438 (2014)

Steps in the proof

$$
\begin{aligned}
\tilde{\theta}_\tau &+ \frac{1}{\varepsilon^\nu}\,\tilde{w}\,\frac{d\widehat{\theta}}{dz} && = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\theta} \\
\tilde{\boldsymbol{v}}_\tau &+ \frac{1}{\varepsilon^\nu}\frac{\tilde{\theta}}{\overline{\overline{\theta}}}\boldsymbol{k} \quad + \frac{1}{\varepsilon}\,\overline{\theta}\nabla\tilde{\pi} && = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{v}} - \varepsilon^{1-\nu}\tilde{\theta}\nabla\tilde{\pi} \quad \cdot \\
\tilde{\pi}_\tau & \quad + \frac{1}{\varepsilon}\left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) && = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\pi} - \gamma\Gamma\tilde{\pi}\nabla\cdot\tilde{\boldsymbol{v}}
\end{aligned}
$$

Existence & uniqueness of solutions for $t \leq T$ with T independent of ε

1. via energy estimates*
   - **L^2 control of derivatives in the fast linear system**
   - nonlinear terms: Picard iteration exploiting Sobolev embedding
2. via spectral expansions (on bounded domains)*
   - “non-resonance” through non-linear terms or
   - effective eqs. for resonant subsets of modes

*Majda, Metivier, Schochet, Embid, ...

*Babin, Mahalov, Nicolaenco, Dutrifoy ...

Control of derivatives

$$
\begin{aligned}
\tilde{\theta}_t + \frac{1}{\varepsilon^{\nu}}\,\tilde{w}\,\frac{d\widehat{\theta}}{dz} & && = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\theta} \\
\tilde{\boldsymbol{v}}_t + \frac{1}{\varepsilon^{\nu}}\,\frac{\tilde{\theta}}{\overline{\overline{\theta}}}\,\boldsymbol{k} & + \frac{1}{\varepsilon}\,\overline{\theta}\nabla\tilde{\pi} && = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\boldsymbol{v}} - \varepsilon^{1-\nu}\tilde{\theta}\nabla\tilde{\pi} \quad \cdot \\
\tilde{\pi}_t & + \frac{1}{\varepsilon}\left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) && = -\tilde{\boldsymbol{v}}\cdot\nabla\tilde{\pi} - \gamma\Gamma\tilde{\pi}\nabla\cdot\tilde{\boldsymbol{v}}
\end{aligned}
$$

For the linear variable coefficient system:

- ✓ Control of weighted quadratic energy
- ✓ Control of horizontal derivatives
- ✓ Control of time derivatives
- ?? Control of **vertical** derivatives

Control of derivatives

Fast linear compressible / pseudo-incompressible modes

$$\tilde{\theta}_\vartheta + \tilde{w}\,\frac{d\widehat{\theta}}{dz} = 0$$

$$\tilde{\boldsymbol{v}}_\vartheta + \frac{\tilde{\theta}}{\overline{\theta}}\,\boldsymbol{k} + \overline{\theta}\nabla\pi^* = 0$$

$$\varepsilon^{\mu}\,\pi^*_\vartheta + \left(\gamma\Gamma\overline{\pi}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{\pi}}{dz}\right) = 0$$

Vertical mode expansion (separation of variables)

$$\begin{pmatrix} \tilde{\theta} \\ \tilde{\boldsymbol{u}} \\ \tilde{w} \\ \pi^* \end{pmatrix}(\vartheta, \boldsymbol{x}, z) = \begin{pmatrix} \Theta^* \\ \boldsymbol{U}^* \\ W^* \\ \Pi^* \end{pmatrix}(z)\ \exp\left(i\left[\boldsymbol{\omega}\vartheta - \boldsymbol{\lambda}\cdot\boldsymbol{x}\right]\right)$$

Control of derivatives

Pseudo-incompressible case $(W = \overline{P} W^*)$

$$-\frac{d}{dz}\left(\phi\,\frac{dW}{dz}\right) + \lambda^2 \phi\, W \;=\; \frac{\lambda^2}{\boldsymbol{\omega}^2}\,\phi N^2\, W\,, \qquad W(0) = W(H) = 0$$

Orthogonalities for eigenmodes/eigenvalues $(W_k^i; \boldsymbol{\omega}_k^i)$ **for** $\lambda \equiv \lambda^i$

$$\left\langle W_k^i, W_l^i \right\rangle_{L^2,\phi N^2} = \int_H^0 W_k^i W_l^i\, \phi N^2\, dz = \delta_{kl}$$

$$\left\langle W_k^i, W_l^i \right\rangle_{H^1,\phi} = \int_H^0 \left[\frac{dW_k^i}{dz}\frac{dW_l^i}{dz} + (\lambda^i)^2 W_k^i W_l^i\right] \phi\, dz = \left(\frac{\lambda^i}{\omega_k^i}\right)^2\, \delta_{kl}$$

Control of derivatives

Spectral expansion of the (weighted) vertical velocity

$$W(\vartheta, x, z) = \sum_{k,i} w_k^i \, W_k^i(z) \exp\left(\imath[\omega_k^i \vartheta - \lambda^i x]\right)$$

weighted L^2-norm of the vertical velocity:

$$\begin{aligned}
\int_{-L}^{L} \int_{0}^{H} W \overline{W} \phi N^2 \, dz dx &= 2L \sum_{k,i} w_k^i \overline{w_l^i} \left\langle W_k^i, W_l^i \right\rangle_{L^2, \phi N^2} \exp(\imath[\omega_k^i - \omega_l^i]\vartheta) \\
&= 2L \sum_{k,i} |w_k^i|^2 \\
&= \text{const.}
\end{aligned}$$

1st constraint on $|w_k^i|$ for i, k large

Control of derivatives

Spectral expansion of the (weighted) vertical velocity

$$W(\vartheta, x, z) = \sum_{k,i} w_k^i \, W_k^i(z) \exp\left(\imath[\omega_k^i \vartheta - \lambda^i x]\right)$$

weighted H^1-norm of the vertical velocity:

$$\begin{aligned}
\int_{-L}^{L}\int_{0}^{H} \nabla W \cdot \nabla \overline{W} \phi \, dz dx &= 2L \sum_{k,i} w_k^i \overline{w_l^i} \left\langle W_k^i, W_l^i \right\rangle_{H^1,\phi} \exp(\imath[\omega_k^i - \omega_l^i]\vartheta) \\
&= 2L \sum_{k,i} |w_k^i|^2 \frac{\left(\lambda^i\right)^2}{(\omega_k^i)^{2}} \\
&= \text{const.}
\end{aligned}$$

2nd stronger constraint on $|w_k^i|$ for i, k large $(\omega_k^i = O(1/k) \; as \; (k \to \infty))$

Control of derivatives

Higher derivatives via recursion: using Taylor-Goldstein (or SL) eqn.

replace W_{zz} by W and W_z

replace W_{zzz} by W_z and W_{zz} etc.

$\Downarrow$

control of $W_{zz}, W_{zzz}, ...$ in suitable weighted L^2 norms

under increasingly stringent decay conditions for amplitudes w_k^i

Control of derivatives

What about u, θ, π?

Linear system

$$
\begin{aligned}
\tilde{\theta}_\vartheta \;+\; \tilde{w}\,\frac{d\widehat{\theta}}{dz} &= 0\\
\tilde{\boldsymbol{u}}_\vartheta \;+\; \overline{\theta}\nabla\pi^* &= 0\\
\tilde{w}_\vartheta \;-\; \frac{\tilde{\theta}}{\overline{\overline{\theta}}} + \overline{\theta}\frac{\partial\pi^*}{\partial z} &= 0\\
\overline{P}\nabla\cdot\tilde{\boldsymbol{v}} + \tilde{w}\frac{d\overline{P}}{dz} &= 0
\end{aligned}
$$

polarization conditions

$$
\begin{aligned}
\Theta^* &= \frac{\imath}{\omega}\frac{d\widehat{\theta}}{dz}W^*\\
\boldsymbol{U}^* &= -\frac{\boldsymbol{\lambda}}{\omega}\Pi^*\\
\frac{d\Pi^*}{dz} &= \frac{1}{\overline{\theta}^2}\Theta^* - \frac{\imath\omega}{\overline{\theta}}W^*\\
\boldsymbol{\lambda}\cdot\boldsymbol{U}^* &= \frac{\imath}{\overline{\overline{P}}}\frac{d\overline{P}}{dz}W^*
\end{aligned}
$$

Control of derivatives

polarization conditions

$$\Theta^* = \frac{\imath}{\omega}\frac{d\widehat{\theta}}{dz} W^*$$

$$\boldsymbol{U}^* = -\frac{\boldsymbol{\lambda}\overline{\theta}}{\omega}\Pi^*$$

$$\frac{d\Pi^*}{dz} = \frac{1}{\overline{\theta}^2}\Theta^* - \frac{\imath\omega}{\overline{\theta}} W^*$$

$$\boldsymbol{\lambda}\cdot\boldsymbol{U}^* = \frac{\imath}{\overline{P}}\frac{d\overline{P}}{dz} W^*$$

components in terms of W^*

$$\frac{\boldsymbol{\Theta}^*}{d\widehat{\boldsymbol{\theta}}/d\boldsymbol{z}} = \frac{\imath}{\omega} W^*$$

$$\frac{d\boldsymbol{U}^*}{d\boldsymbol{z}} = -\frac{\imath\boldsymbol{\lambda}N^2}{\overline{\theta}\omega^2}\left(1 - \frac{\omega^2}{N^2}\right) W^*$$

$$\frac{d\Pi^*}{dz} = \frac{\imath N^2}{\overline{\theta}\omega}\left(1 - \frac{\omega^2}{N^2}\right) W^*$$

$$\boldsymbol{\lambda}\cdot\boldsymbol{U}^* = \frac{\imath}{\overline{P}}\frac{d\overline{P}}{dz} W^*$$

$$N^2 = \frac{1}{\overline{\theta}}\frac{d\widehat{\theta}}{dz}$$

Control of derivatives

Spectral expansion of the (weighted) potential temperature

$$\frac{\Theta(\vartheta, x, z)}{d\widehat{\theta}/dz} = \imath \sum_{k,i} \frac{w_k^i}{\omega_k^i} W_k^i(z) \exp\left(\imath[\omega_k^i \vartheta - \lambda^i x]\right)$$

new **weighted H^1-norm of the potential temperature:**

$$\int_{-L}^{L} \int_{0}^{H} \left(\nabla \frac{\Theta(\vartheta, x, z)}{d\overline{\theta}/dz}\right)^2 \phi \, dz dx = 2L \sum_{k,i} \frac{w_k^i \overline{w_l^i}}{\omega_k^i \omega_l^i} \left\langle W_k^i, W_l^i \right\rangle_{H^1,\phi} \exp(\imath[\omega_k^i - \omega_l^i]\vartheta)$$

$$= 2L \sum_{k,i} |w_k^i|^2 \frac{(\lambda^i)^2}{(\omega_k^i)^{\mathbf{4}}}$$

$$= \text{const.}$$

3rd strongest constraint on $|w_k^i|$ for i, k large $\quad (\omega_k^i = O(1/k) \; as \; (k \to \infty))$

Control of derivatives

Estimate for $\partial\tilde{\boldsymbol{u}}/\partial z$: **(Idea 1)**

recall

$$\frac{d\boldsymbol{U}^*}{d\boldsymbol{z}} = -\frac{\imath\boldsymbol{\lambda} N^2}{\overline{\theta}\omega^2}\left(1-\frac{\omega^2}{N^2}\right)W^*$$

$$\int_{-L}^{L}\int_{0}^{H}\left(\frac{\overline{\theta}}{N}\frac{\partial\tilde{\boldsymbol{u}}}{\partial z}\right)^2\phi\,dzdx \;\leq\; \sum_i\sum_{k,l} w_k^i\overline{w_l^i}\frac{(\lambda^i)^2}{(\omega_k^i)^2(\omega_l^i)^2}\left(A_{k,l}^i+B_{k,l}^i+C_{k,l}^i\right)$$

where

$$A_k^i \;=\; \left\langle W_k^i, W_l^i\right\rangle_{\phi N^2} = \delta_{kl} \qquad \textbf{OK}$$

$$\boldsymbol{B_k^i} \;=\; \left((\omega_k^i)^2+(\omega_l^i)^2\right)\left\langle W_k^i, W_l^i\right\rangle_\phi \qquad \textbf{double sums !!}$$

$$\boldsymbol{C_k^i} \;=\; (\omega_k^i)^2(\omega_l^i)^2\left\langle W_k^i, W_l^i\right\rangle_{\phi/N^2}$$

The double-sums converge if, for smooth enough positive ψ:

$\left\langle W_k^i, W_l^i\right\rangle_\psi = O\left((k-l)^{-2}\right) \quad \Rightarrow \quad$ **WKB-asymptotics for $\boldsymbol{W_k^i}$ ($\boldsymbol{k}\to\infty$)** ✔

Idea 1 turns out to be crucial in the compressible case!

Control of derivatives

Estimate for $\partial\tilde{\boldsymbol{u}}/\partial z$: **(Idea 2)**

recall

$$\frac{d\boldsymbol{U}^*}{d\boldsymbol{z}} = -\frac{\imath\boldsymbol{\lambda}N^2}{\overline{\theta}\omega^2}\left(1-\frac{\omega^2}{N^2}\right)W^* = \frac{d\boldsymbol{U}_1^*}{d\boldsymbol{z}} + \frac{d\boldsymbol{U}_2^*}{d\boldsymbol{z}}$$

Control

$$\frac{d\boldsymbol{U}_1^*}{d\boldsymbol{z}},\quad \frac{d\boldsymbol{U}_2^*}{d\boldsymbol{z}}$$

in two different weighted norms using same strategy as applied for Θ. Since the weights are bounded, this yields control of standard $L^2, H^1, \ldots$ norms.

No WKB-estimates for near-orthogonality at high wavenumbers needed.

Idea 1 turns out to be crucial in the compressible case!

Control of derivatives

H^s **control for the pseudo-incompressible fast system** ✓

Idea 1 turns out to be crucial in the compressible case!

Towards a rigorous proof

Outlook

- Resonant sets and related evolution equations (pseudo-incompressible)

- Control of derivatives for the compressible system
- Decoupling of acoustic & internal waves
- Resonant sets and related evolution equations (compressible)

- Arakawa & Conor's "Unified Model" for large horizontal scales

- Further "translations" into numerical methods

Idea 1 turns out to be crucial in the compressible case!

Scale-dependent models for atmospheric motions

Background on sound-proof models

Formal asymptotic regime of validity

Steps towards a rigorous proof

Summary