Scales in geophysical flows

Rupert Klein

Mathematik & Informatik, Freie Universität Berlin

Motivation

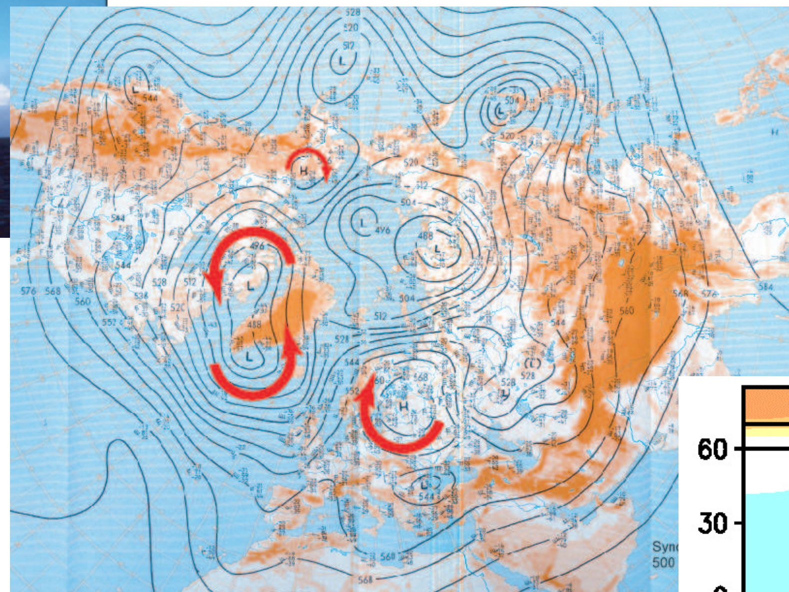
Scale analysis & distinguished limits

Model hierarchy for atmospheric flows

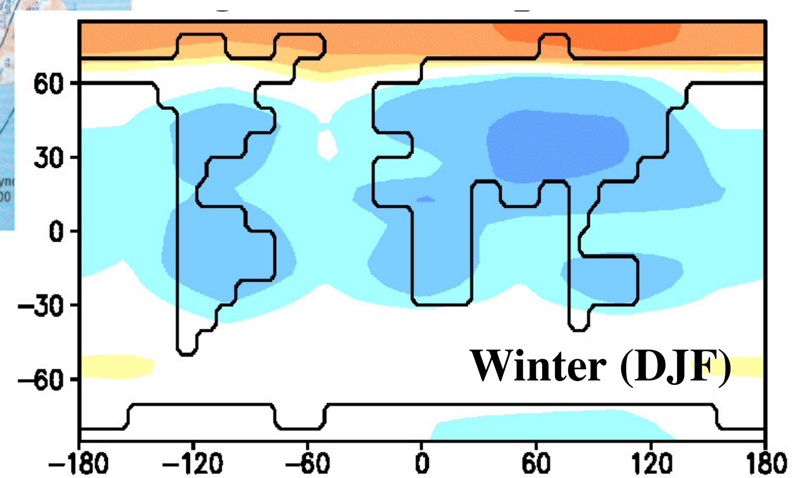
A puzzle

Scale-Dependent Models

10 km / 20 min



1000 km / 2 days



10000 km / 1 season

Thanks to:

P.K. Taylor, Southampton Oceanogr. Inst.; P. Névir, Freie Universität Berlin; S. Rahmstorf, PIK, Potsdam

Scale-Dependent Models

$$
\begin{aligned}
\boldsymbol{u}_t + \boldsymbol{u}\cdot\nabla\boldsymbol{u} + w\boldsymbol{u}_z + \nabla\pi &= \boldsymbol{S}_{\boldsymbol{u}} \\
w_t + \boldsymbol{u}\cdot\nabla w + w w_z + \pi_z &= -\theta' + S_w \\
{\theta'}_t + \boldsymbol{u}\cdot\nabla\theta' + w{\theta'}_z &= S'_\theta \\
\nabla\cdot(\rho_0\boldsymbol{u}) + (\rho_0 w)_z &= 0
\end{aligned}
$$

$$\theta = 1 + \varepsilon^4\theta'(\boldsymbol{x}, z, t) + o(\varepsilon^4)$$

Anelastic Boussinesque Model

10 km / 20 min

$$\left(\partial_\tau + \boldsymbol{u}^{(0)}\cdot\nabla\right) q = 0$$

$$q = \zeta^{(0)} + \Omega_0\beta\eta + \frac{\Omega_0}{\rho^{(0)}}\frac{\partial}{\partial z}\left(\frac{\rho^{(0)}}{d\Theta/dz}\theta^{(3)}\right)$$

$$\zeta^{(0)} = \nabla^2\pi^{(3)}, \qquad \theta^{(3)} = -\frac{\partial\pi^{(3)}}{\partial z}, \qquad \boldsymbol{u}^{(0)} = \frac{1}{\Omega_0}\boldsymbol{k}\times\nabla\pi^{(3)}$$

Quasi-geostrophic theory

1000 km / 2 days

$$
\begin{aligned}
\frac{\partial Q_T}{\partial t} + \nabla\cdot\boldsymbol{F}_T &= S_T \\
\frac{\partial Q_q}{\partial t} + \nabla\cdot\boldsymbol{F}_q &= S_q
\end{aligned}
$$

$$Q_\varphi = \int_{z_s}^{H_a} \rho\,\varphi\,dz\,, \qquad \boldsymbol{F}_\varphi = \int_{z_s}^{H_a} \rho\left(\boldsymbol{u}\,\varphi + (\widehat{\boldsymbol{u}'\varphi'}) + \boldsymbol{D}^\varphi\right) dz\,, \qquad \left(\varphi \in \{T, q\}\right)$$

$$T = T_s(t,\boldsymbol{x}) + \Gamma(t,\boldsymbol{x})\left(\min(z, H_T) - z_s\right), \qquad q = q_s(t,\boldsymbol{x})\exp\left(-\frac{z - z_s}{H_q}\right)$$

$$\rho = \rho_*\exp\left(-\frac{z}{h_{\mathrm{sc}}}\right), \qquad p = p_*\exp\left(-\frac{\gamma z}{h_{\mathrm{sc}}}\right) + p_0(t,\boldsymbol{x}) + g\rho_*\int_0^z \frac{T}{T_*}\,dz'$$

$$\boldsymbol{u} = \boldsymbol{u}_g + \boldsymbol{u}_a\,, \qquad f\rho_*\boldsymbol{k}\times\boldsymbol{u}_g = -\nabla_{\boldsymbol{x}}p \qquad \boldsymbol{u}_\alpha = \alpha\nabla p_0$$

V. Petoukhov et al., *CLIMBER-2 ...*, Climate Dynamics, 16, (2000)

EMIC - equations (CLIMBER-2)

10000 km / 1 season

Compressible flow equations with general source terms

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + {}_{\parallel}\boldsymbol{v}\cdot {}_{\parallel}\nabla + w\frac{\partial}{\partial z}\right) \boldsymbol{v}_{\parallel} &+ {}_{\parallel}(2\boldsymbol{\Omega}\times\boldsymbol{v}) + \frac{1}{\rho}\nabla_{\parallel} p &= \boldsymbol{S}_{\boldsymbol{v}_{\parallel}}\,, \\
\left(\frac{\partial}{\partial t} + {}_{\parallel}\boldsymbol{v}\cdot {}_{\parallel}\nabla + w\frac{\partial}{\partial z}\right) w &+ (2\boldsymbol{\Omega}\times\boldsymbol{v})_{\perp} + \frac{1}{\rho}\frac{\partial p}{\partial z} &= S_w - g\,, \\
\left(\frac{\partial}{\partial t} + {}_{\parallel}\boldsymbol{v}\cdot {}_{\parallel}\nabla + w\frac{\partial}{\partial z}\right) \rho &+ \rho\nabla\cdot\boldsymbol{v} &= 0\,, \\
\left(\frac{\partial}{\partial t} + {}_{\parallel}\boldsymbol{v}\cdot {}_{\parallel}\nabla + w\frac{\partial}{\partial z}\right) \Theta & &= S_{\Theta}\,,
\end{aligned}
$$

$$
\left(\frac{p}{p_{\text{ref}}}\right)^{R/c_p} = \frac{\rho}{\rho_{\text{ref}}}\frac{\Theta}{T_{\text{ref}}}\,.
$$

How do all the simplified models relate to this system?

Motivation

Scale analysis & distinguished limits

Model hierarchy for atmospheric flows

A puzzle

Scale-Dependent Models

Earth's radius	a	$\sim$	$6 \cdot 10^6$	m
Earth's rotation rate	Ω	$\sim$	10^{-4}	s^{-1}
Acceleration of gravity	g	$\sim$	9.81	ms^{-2}
Sea level pressure	p_{ref}	$\sim$	10^5	$\mathrm{kgm}^{-1}\mathrm{s}^{-2}$
H_2O freezing temperature	T_{ref}	$\sim$	273	K
Latent heat of water vapor	$L_{q_{\mathrm{vs}}}$	$\sim$	$4 \cdot 10^4$	$\mathrm{J\,kg}^{-1}\mathrm{K}^{-1}$
Dry gas constant	R	$\sim$	287	$\mathrm{m}^2\mathrm{s}^{-2}\mathrm{K}^{-1}$
Dry isentropic exponent	γ	$\sim$	1.4	

Dimensionless parameters:

$$\Pi_1 = \frac{h_{\mathrm{sc}}}{a} \sim 1.6 \cdot 10^{-3}$$

$$\Pi_2 = \frac{L_{q_{\mathrm{vs}}}}{c_p T_{\mathrm{ref}}} \sim 1.5 \cdot 10^{-1}$$

$$\Pi_3 = \frac{c_{\mathrm{ref}}}{\Omega a} \sim 4.7 \cdot 10^{-1}$$

where

$$h_{\mathrm{sc}} = \frac{R T_{\mathrm{ref}}}{g} = \frac{p_{\mathrm{ref}}}{\rho_{\mathrm{ref}} g} \sim 8.5\,\mathrm{km}$$

$$c_{\mathrm{ref}} = \sqrt{R T_{\mathrm{ref}}} = \sqrt{g h_{\mathrm{sc}}} \sim 300\ \mathrm{m/s}$$

$$c_p = \frac{\gamma R}{\gamma - 1}$$

Scale-Dependent Models

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Distinguished limit:

$$\Pi_1 = \frac{h_{\mathrm{sc}}}{a} \sim 1.6 \cdot 10^{-3} \sim \varepsilon^3$$

$$\Pi_2 = \frac{L_{q_{\mathrm{vs}}}}{c_p T_{\mathrm{ref}}} \sim 1.5 \cdot 10^{-1} \sim \varepsilon$$

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where

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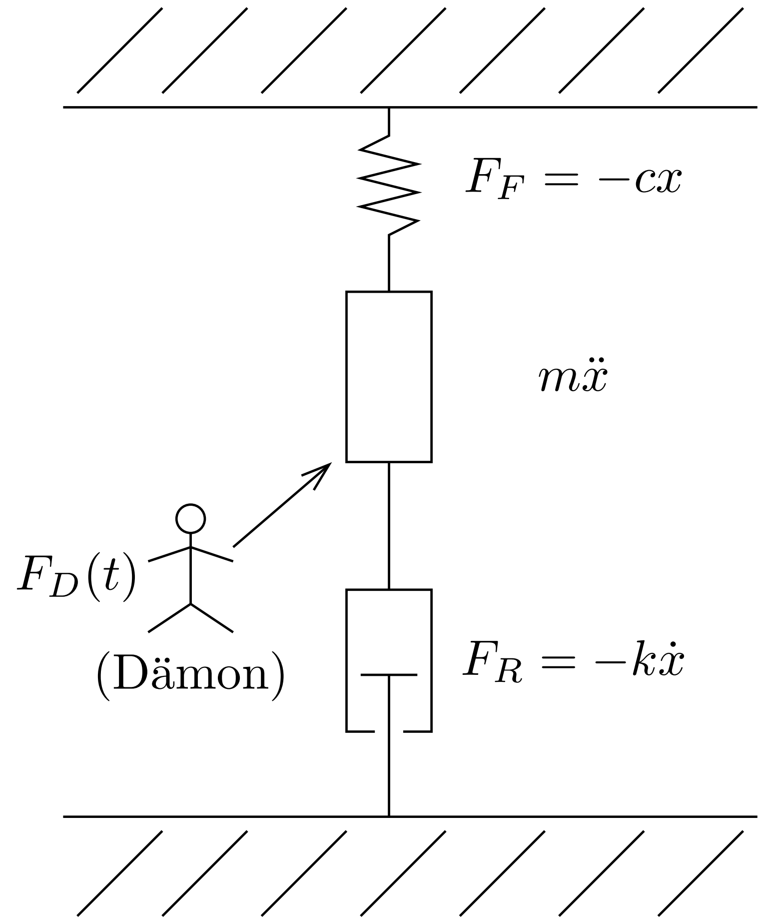
distinguished limits
for the harmonic oscillator



$$m\ddot{x} + k\dot{x} + cx = F_0 \cos(\Omega t)$$

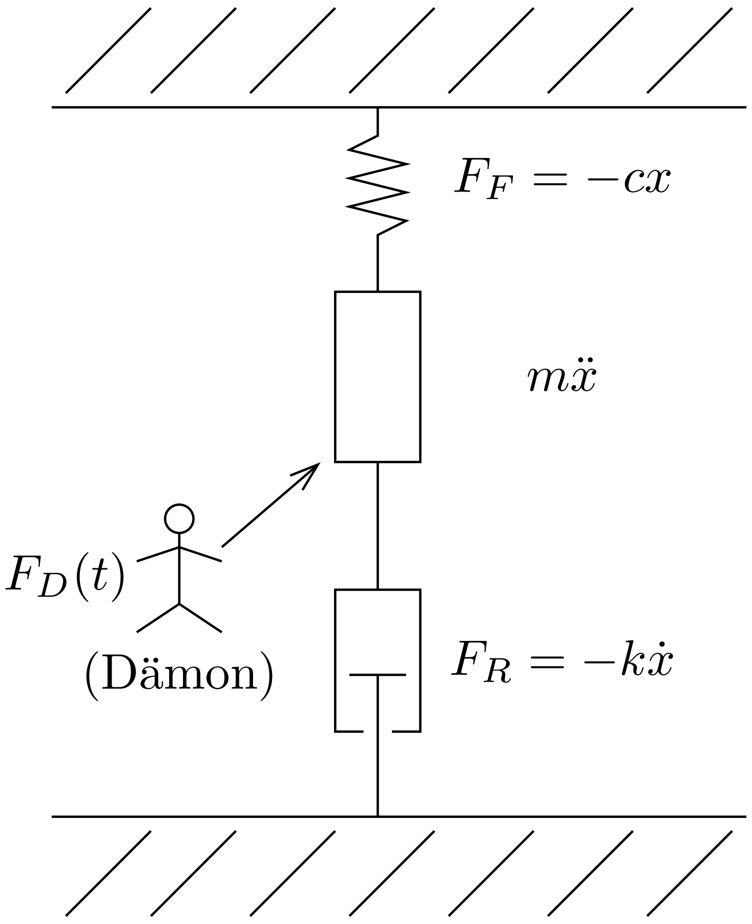
$$x(0) = x_0\,, \qquad \dot{x}(0) = \dot{x}_0$$

$$\boldsymbol{\varepsilon} = \frac{m\Omega^2}{c} \ll 1$$

$$\boldsymbol{\delta} = \frac{k\Omega}{c} \ll 1$$

$$\frac{cx_0}{F_0} = O(1)$$

$$\frac{c\dot{x}_0}{\Omega F_0} = \mathbf{?}$$

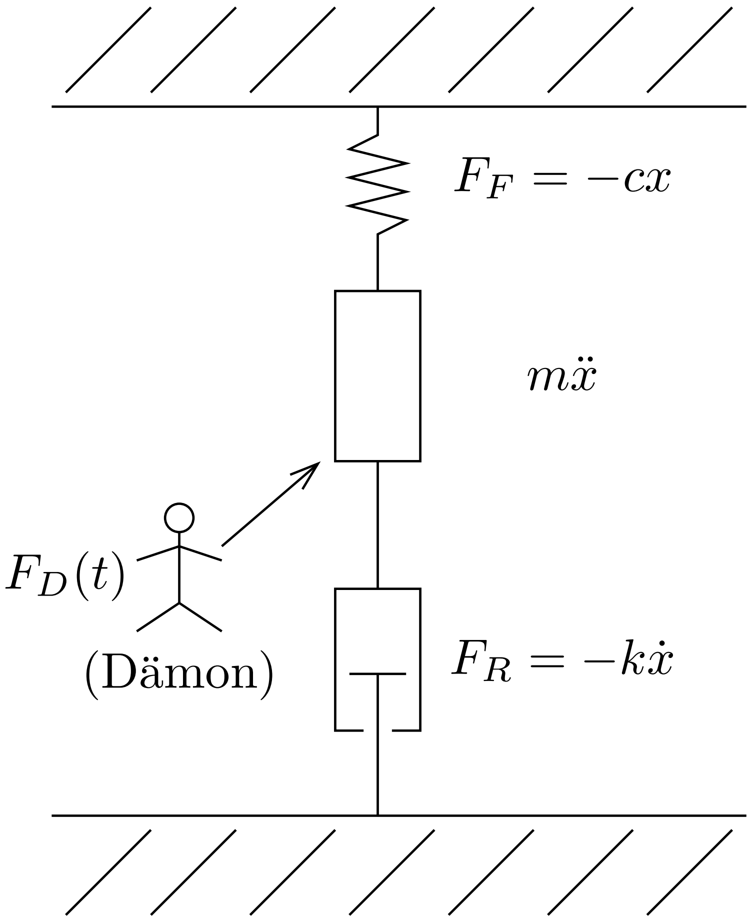


Dimensionless representation

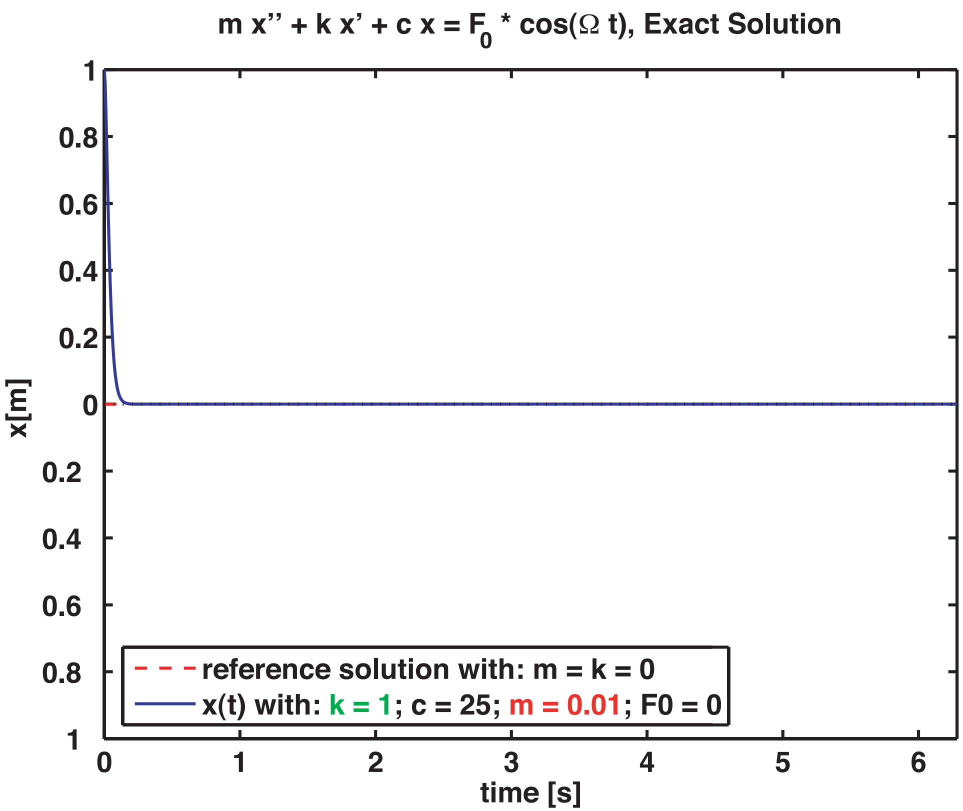
$$x(t) = \frac{F_0}{c}\, y(\tau)\,, \qquad \tau = \Omega t$$

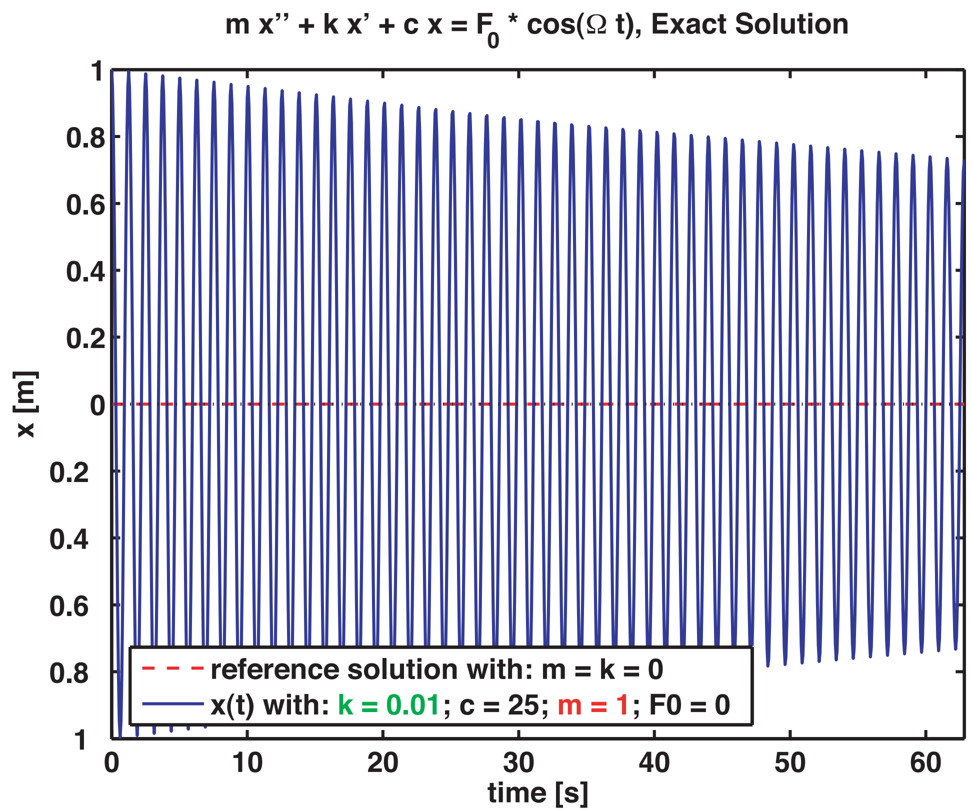
then

$$\boldsymbol{\varepsilon}\, y'' + \boldsymbol{\delta}\, y' + y = \cos(\tau)$$



Is there a unique limit solution for $\boldsymbol{\varepsilon} = \boldsymbol{\delta} = 0$?



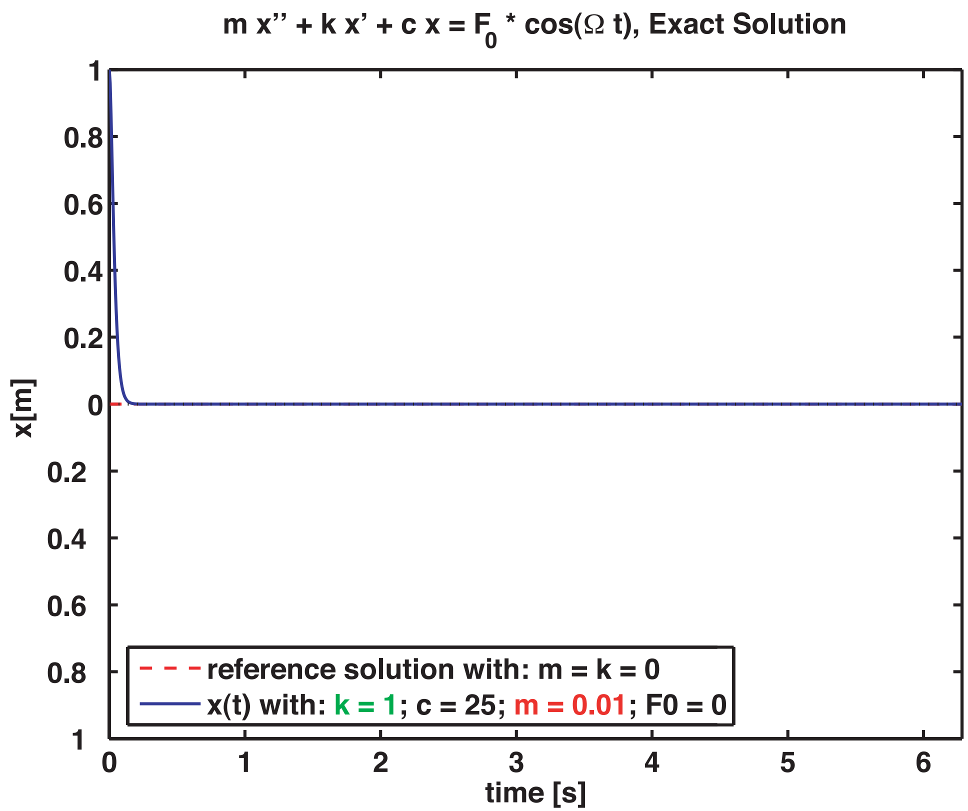


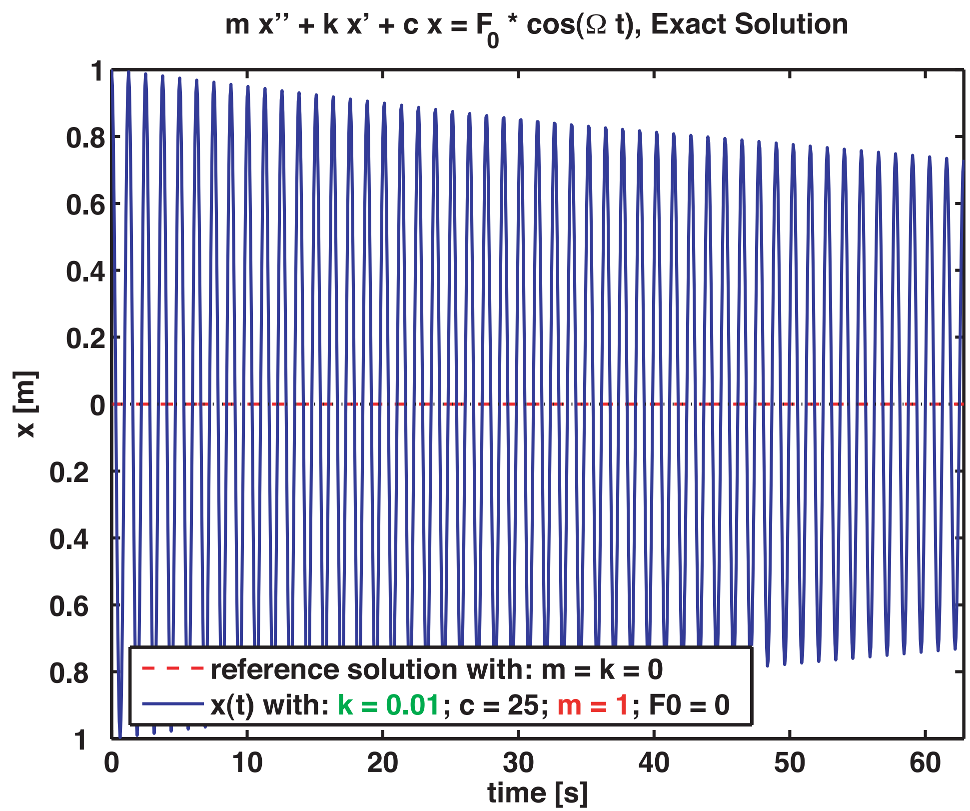
$\varepsilon = 0.0004$

$\delta = 0.04$

$\varepsilon = 0.04$

$\delta = 0.0004$

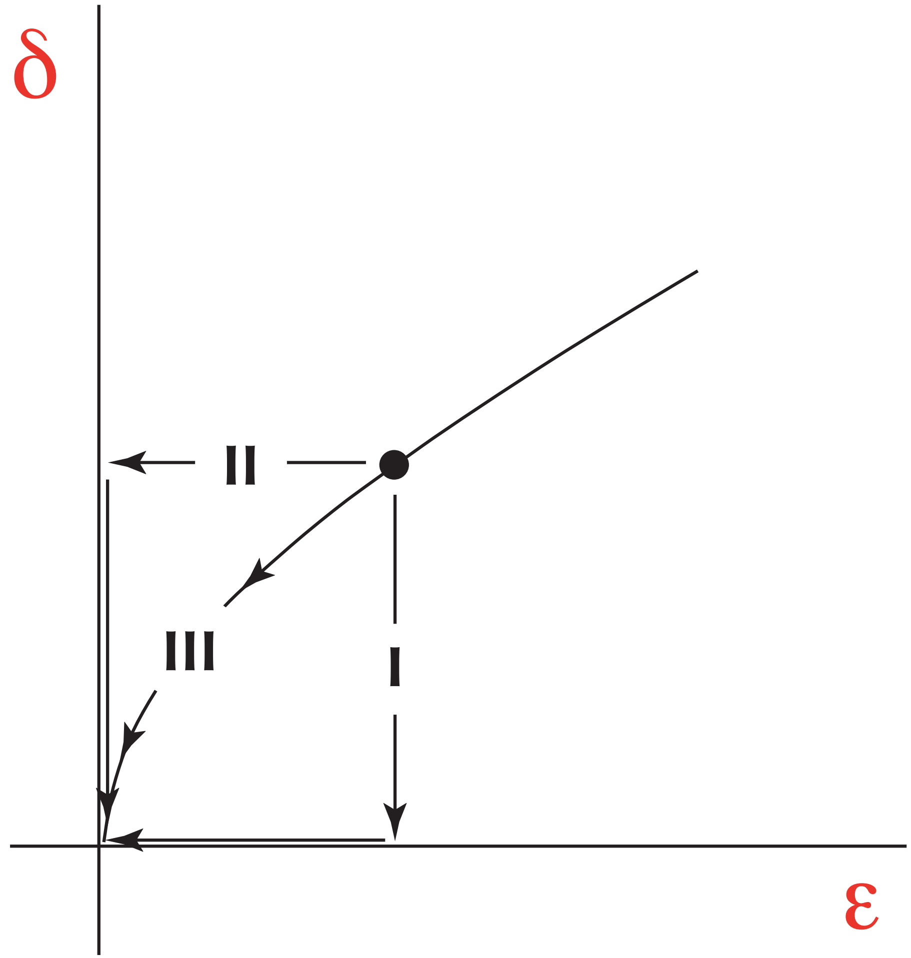




$\varepsilon = 0.0004$ $\qquad$ $\varepsilon = 0.04$

$\delta = 0.04$ $\qquad$ $\delta = 0.0004$

The limit is path-dependent!


δ
II
III
I
ε

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Distinguished limit:

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$$\Pi_3 = \frac{c_{\mathrm{ref}}}{\Omega a} \sim 4.7 \cdot 10^{-1} \sim \sqrt{\varepsilon}$$

where

$$h_{\mathrm{sc}} = \frac{RT_{\mathrm{ref}}}{g} = \frac{p_{\mathrm{ref}}}{\rho_{\mathrm{ref}} g} \sim 8.5\,\mathrm{km}$$

$$c_{\mathrm{ref}} = \sqrt{RT_{\mathrm{ref}}} = \sqrt{g h_{\mathrm{sc}}} \sim 300\ \mathrm{m/s}$$

$$c_p = \frac{\gamma R}{\gamma - 1}$$

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Nondimensionalization

$$(\boldsymbol{x}, z) = \frac{1}{h_{\rm sc}} \, (\boldsymbol{x}', z') \,, \qquad t = \frac{u_{\rm ref}}{h_{\rm sc}} \, t'$$

$$(\boldsymbol{u}, w) = \frac{1}{u_{\rm ref}} \, (\boldsymbol{u}', w') \,, \qquad (p, T, \rho) = \left(\frac{p'}{p_{\rm ref}}, \frac{T'}{T_{\rm ref}}, \frac{\rho' R T_{\rm ref}}{p_{\rm ref}} \right)$$

where

$$u_{\rm ref} = \frac{2}{\pi} \frac{g h_{\rm sc}}{\Omega a} \frac{\Delta\Theta}{T_{\rm ref}} \qquad \text{(thermal wind scaling)}$$

Compressible flow equations with general source terms

$$
\begin{aligned}
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \boldsymbol{v}_{\shortparallel} \; + \; \varepsilon \, (2\boldsymbol{\Omega} \times \boldsymbol{v})_{\shortparallel} + \frac{1}{\varepsilon^3 \rho} \; \nabla_{\|} p \; &= \; \boldsymbol{S}_{\boldsymbol{v}_{\shortparallel}} \, , \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) w \; + \; \varepsilon \, (2\boldsymbol{\Omega} \times \boldsymbol{v})_{\perp} + \frac{1}{\varepsilon^3 \rho} \frac{\partial p}{\partial z} \; &= \; S_w - \frac{1}{\varepsilon^3} \, , \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \rho \; + \; \rho \, \nabla \cdot \boldsymbol{v} \; &= \; 0 \, , \\
\left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel} \cdot \nabla_{\shortparallel} + w \frac{\partial}{\partial z}\right) \Theta \; &= \; S_{\Theta} \, .
\end{aligned}
$$

Classical length scales and dimensionless numbers

$$\begin{aligned} L_{\text{mes}} &= \varepsilon^{-1} \; h_{\text{sc}} \\ L_{\text{syn}} &= \varepsilon^{-2} \; h_{\text{sc}} \\ L_{\text{Ob}} &= \varepsilon^{-5/2} h_{\text{sc}} \\ L_{\text{p}} &= \varepsilon^{-3} \; h_{\text{sc}} \end{aligned}$$

$$\begin{aligned} \text{Fr}_{\text{int}} &\sim \varepsilon \\ \text{Ro}_{h_{\text{sc}}} &\sim \varepsilon^{-1} \\ \text{Ro}_{L_{\text{Ro}}} &\sim \varepsilon \\ \text{Ma} &\sim \varepsilon^{3/2} \end{aligned}$$

Example: the **synoptic scale** *

$$N^2 = \frac{g}{\Theta}\frac{d\Theta}{dz}$$

$$\begin{aligned} L_{\text{syn}} = \frac{N h_{\text{sc}}}{\Omega} \sim \frac{1}{\Omega}\sqrt{\frac{g}{T_{\text{ref}}}\frac{\Delta\Theta}{h_{\text{sc}}}}\, h_{\text{sc}} &= \frac{u_{\text{ref}}}{\Omega h_{\text{sc}}}\frac{\sqrt{g h_{\text{sc}}}}{u_{\text{ref}}}\sqrt{\frac{\Delta\Theta}{T_{\text{ref}}}}\, h_{\text{sc}} \\ &= \text{Ro}_{h_{\text{sc}}}\frac{1}{\text{Ma}}\sqrt{\frac{\Delta\Theta}{T_{\text{ref}}}}\, h_{\text{sc}} = h_{\text{sc}}\varepsilon^{-1-\frac{3}{2}+\frac{1}{2}} = \frac{h_{\text{sc}}}{\varepsilon^2} \end{aligned}$$

* **distance which an internal wave must travel until influenced at leading order by the Coriolis effect**

Single-scale asymptotics

$$\mathbf{U}(t, \boldsymbol{x}, z; \boldsymbol{\varepsilon}) = \sum_{i=0}^{m} \phi_i(\boldsymbol{\varepsilon})\, \mathbf{U}^{(i)}(t, \boldsymbol{x}, z; \boldsymbol{\varepsilon}) + \mathcal{O}\big(\phi_m(\boldsymbol{\varepsilon})\big)$$

Remark

Generally, $m < \infty$, and the series would not converge !

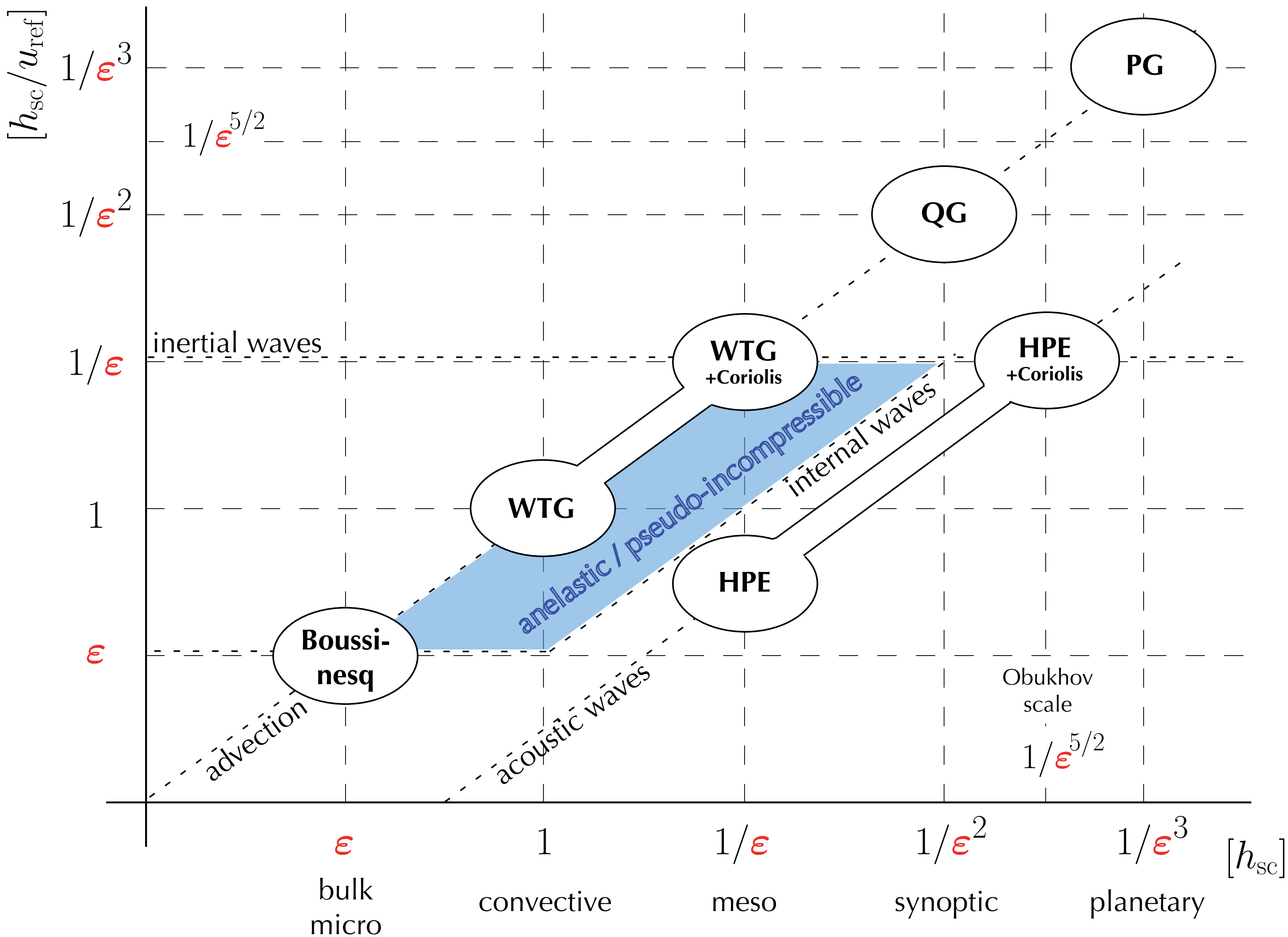
Scale-Dependent Model Hierarchy

Recovered classical single-scale models:

$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\frac{t}{\varepsilon}, \boldsymbol{x}, \frac{z}{\varepsilon})$	Linear small scale internal gravity waves
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(t, \boldsymbol{x}, z)$	Anelastic & pseudo-incompressible models
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon t, \varepsilon^2 \boldsymbol{x}, z)$	Linear large scale internal gravity waves
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$	Mid-latitude Quasi-Geostrophic Flow
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^2 \boldsymbol{x}, z)$	Equatorial Weak Temperature Gradients
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^2 t, \varepsilon^{-1} \xi(\varepsilon^2 \boldsymbol{x}), z)$	Semi-geostrophic flow
$\mathbf{U}^{(i)} = \mathbf{U}^{(i)}(\varepsilon^{3/2} t, \varepsilon^{5/2} x, \varepsilon^{5/2} y, z)$	Kelvin, Yanai, Rossby, and gravity Waves

... and many more

Scale-Dependent Model Hierarchy



R.K., Ann. Rev. Fluid Mech., **42**, 249–274 (2010)

Motivation

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Model hierarchy for atmospheric flows

A puzzle

Compressible flow equations without source terms

$$
\begin{aligned}
\frac{D\boldsymbol{v}_{\shortparallel}}{Dt} &+ \varepsilon\,(2\boldsymbol{\Omega}\times\boldsymbol{v})_{\shortparallel} + \frac{1}{\varepsilon^3\rho}\,\nabla_{\|}p &&= 0\,,\\
\frac{Dw}{Dt} &+ \varepsilon\,(2\boldsymbol{\Omega}\times\boldsymbol{v})_{\perp} + \frac{1}{\varepsilon^3\rho}\frac{\partial p}{\partial z} &&= -\frac{1}{\varepsilon^3}\,,\\
\frac{D\rho}{Dt} &+ \rho\,\nabla\cdot\boldsymbol{v} &&= 0\,,\\
\frac{D\Theta}{Dt} & &&= 0\,.
\end{aligned}
$$

where

$$
\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\shortparallel}\cdot\nabla_{\shortparallel} + w\frac{\partial}{\partial z}\right)
$$

Leading orders

$$\nabla_\parallel p = 0 \qquad (1)$$

$$\partial_z p = -\rho \qquad (2)$$

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\boldsymbol{v} = 0 \qquad (3)$$

$$\frac{D\Theta}{Dt} = 0 \qquad (4)$$

$$\Theta = \frac{p^{1/\gamma}}{\rho}\,. \qquad (5)$$

$$\frac{D}{Dt} = \left(\frac{\partial}{\partial t} + \boldsymbol{v}_{\parallel}\cdot\nabla_{\parallel} + w\frac{\partial}{\partial z}\right)$$

$$(2),\ (5) \ \Rightarrow \ \nabla_\parallel\rho = \nabla_\parallel\Theta = 0 \qquad (6)$$

$$(4) \ \Rightarrow \ \nabla_\parallel w = 0 \qquad (7)$$

$$(3) \ \Rightarrow \ \nabla_\parallel\cdot\boldsymbol{v}_{\parallel} = d(z) \qquad (8)$$

$$\int_{D_\parallel}(8) \ \Rightarrow \ d(z) \equiv 0 \qquad (9)$$

$$(3),\ (4)\,,(5) \ \Rightarrow \ w_{zz} - \frac{\gamma-1}{\gamma}\frac{\rho}{p}w_z = 0 \qquad (10)$$

$$w(0) = w(H) = 0 \ \Rightarrow \ \boldsymbol{w} = \mathbf{0} \qquad (11)$$