Finite volume methods for dissipative problems

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Lecture 3 :

Finite volume schemes and long time behavior

Outline

Outline

Some references

- Herbin, 1995
- Coudière, Vila, Villedieu, 1999
- Eymard, Gallouët, Herbin, 1999, 2000, 2010
- Gallouët, Herbin, Vignal, 2000
- Coudière, Gallouët, Herbin, 2001
- Droniou, Gallouët, Herbin, 2003
- Andreianov, Gutnic, Wittbold, 2004
- Filbet, 2006
- Glitzky, Griepentrog, 2010
- Andreianov, Bendahmane, Ruiz Baier, 2011
- Bessemoulin-Chatard, C.-H., Filbet, 2015

Space of approximate solutions and norms

$$X(\mathcal{T}) = \left\{ u_\mathcal{T} = \sum_{K\in\mathcal{T}} u_K \mathbf{1}_K \right\} \subset L^1(\Omega),$$

but $X(\mathcal{T}) \notin H^1(\Omega)$

L^q-norms

- For $1 \leq q < +\infty$,

$$\begin{aligned} \|u_\mathcal{T}\|_{0,q} &= \left(\int_\Omega |u_\mathcal{T}(x)|^q dx \right)^{1/q} \\ &= \left(\sum_{K\in\mathcal{T}} \mathrm{m}(K)|u_K|^q \right)^{1/q}. \end{aligned}$$

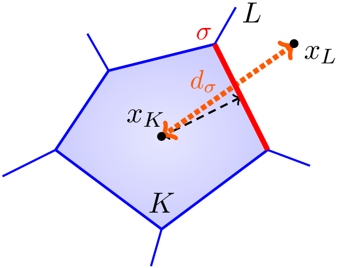
- $\|u_\mathcal{T}\|_{0,\infty} = \max_{K\in\mathcal{T}} |u_K|$.

About the mesh

Regularity of the mesh

- Each control volume K is star-shaped with respect to x_K.
- There exists $\xi > 0$ such that

$$\forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_K,\ \mathrm{d}(x_K, \sigma) \geq \xi \mathrm{d}_\sigma.$$



Remark

- Admissibility assumption not necessary.

Discrete $W^{1,p}$-norms

General framework

- Discrete $W^{1,p}$-semi-norm :

$$|u_\mathcal{T}|_{1,p,\mathcal{T}}^p = \sum_{\sigma=K|L} \mathrm{m}(\sigma)\mathrm{d}_\sigma \frac{|u_L - u_K|^p}{\mathrm{d}_\sigma^p}.$$

- Discrete $W^{1,p}$-norm : $\|u_\mathcal{T}\|_{1,p,\mathcal{T}} = \|u_\mathcal{T}\|_{0,p} + |u_\mathcal{T}|_{1,p,\mathcal{T}}$.

With homogeneous Dirichlet boundary conditions on $\Gamma^0 \subset \Gamma$

$$|u_\mathcal{T}|_{1,p,\Gamma^0,\mathcal{T}}^p = \sum_{\sigma\in\mathcal{E}} \mathrm{m}(\sigma)\mathrm{d}_\sigma \frac{(D_\sigma u)^p}{\mathrm{d}_\sigma^p}$$

$$\text{where } D_\sigma u = \begin{cases} |u_K - u_L| & \text{si } \sigma = K|L, \\ |u_K| & \text{si } \sigma \subset \Gamma^0, \\ 0 & \text{si } \sigma \subset \Gamma \setminus \Gamma^0. \end{cases}$$

Relations between the norms

For $1 \leq s \leq p$, for all $u_{\mathcal{T}} \in X(\mathcal{T})$,

$$\|u_{\mathcal{T}}\|_{0,s} \leq \mathrm{m}(\Omega)^{\frac{p-s}{ps}} \|u_{\mathcal{T}}\|_{0,p},$$

and

$$|u_{\mathcal{T}}|_{1,s,\mathcal{T}} \leq \left(\frac{d\mathrm{m}(\Omega)}{\xi}\right)^{\frac{p-s}{ps}} |u_{\mathcal{T}}|_{1,p,\mathcal{T}}$$

Proof

- Hölder inequality with $p' = \dfrac{p}{s}$ and $q' = \dfrac{p}{p-s}$
- Due to the regularity of the mesh :

$$\sum_{\sigma=K|L} \mathrm{m}(\sigma) \mathrm{d}_\sigma \leq \frac{1}{\xi} \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_K} \mathrm{m}(\sigma) \mathrm{d}(x_K, \sigma) = \frac{d\mathrm{m}(\Omega)}{\xi}.$$

The space $L^1 \cap BV(\Omega)$

Total variation

Let Ω be an open set of $\mathbb{R}^N$ and $u \in L^1(\Omega)$. We define :

$$TV_\Omega(u) = \sup \left\{ \int_\Omega u(x) \mathrm{div} \varphi(x) dx;\ \varphi \in \mathcal{C}_c^1(\Omega, \mathbb{R}^N), \|\varphi\|_\infty \leq 1 \right\}$$

$L^1 \cap BV(\Omega)$

$$L^1 \cap BV(\Omega) = \left\{ u \in L^1(\Omega);\ TV_\Omega(u) < +\infty \right\}.$$

$L^1 \cap BV(\Omega)$ is endowed with the norm :

$$\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + TV_\Omega(u).$$

Relation between $X(\mathcal{T})$ and $L^1 \cap BV(\Omega)$

Total variation of $u_{\mathcal{T}} \in X(\mathcal{T})$

$$TV_{\Omega}(u_{\mathcal{T}}) = \sum_{\sigma=K|L} \mathrm{m}(\sigma)|u_K - u_L| = |u_{\mathcal{T}}|_{1,1,\mathcal{T}}.$$

Inclusion

For all $u_{\mathcal{T}} \in X(\mathcal{T})$, $\|u_{\mathcal{T}}\|_{1,1,\mathcal{T}} < +\infty$ and

$$X(\mathcal{T}) \subset L^1 \cap BV(\Omega).$$

Starting point for the discrete functional inequalities

- Ambrosio, Fusco, Pallara, 2000
- Ziemer, 1989

Theorem

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^N$, $N \geq 2$.

There exists $C > 0$, depending only on Ω such that

$$\left(\int_\Omega |u|^{\frac{N}{N-1}}\right)^{\frac{N-1}{N}} \leq C\|u\|_{BV(\Omega)} \quad \forall u \in L^1 \cap BV(\Omega).$$

$L^1 \cap BV(\Omega) \subset L^{N/(N-1)}(\Omega)$ with continuous embedding.

Discrete Poincaré-Sobolev inequality

Theorem

Let Ω be a polyedral bounded domain of $\mathbb{R}^N$, $N \geq 2$.
Let $(\mathcal{T}, \mathcal{E}, \mathcal{P})$ be a regular mesh of Ω, with regularity ξ.

- If $1 \leq p < N$, let $1 \leq q \leq p^* = \dfrac{pN}{N-p}$.
- If $p \geq N$, let $1 \leq q < +\infty$.

There exists $C > 0$, depending only on p, q, N and Ω such that

$$\|u_{\mathcal{T}}\|_{0,q} \leq \frac{C}{\xi^{(p-1)/p}} \|u_{\mathcal{T}}\|_{1,p,\mathcal{T}} \quad \forall u_{\mathcal{T}} \in X(\mathcal{T}).$$

A crucial lemma

Lemma

Let Ω be an open bounded polyhedral domain of $\mathbb{R}^N$, $N \geq 2$.
Let $(\mathcal{T}, \mathcal{E}, \mathcal{P})$ a regular mesh of Ω, with regularity parameter ξ.

For all $s > 1$, $p > 1$, we have :

$$\|u_{\mathcal{T}}\|^s_{0,sN/(N-1)} \leq \frac{C}{\xi^{(p-1)/p}} \|u_{\mathcal{T}}\|^{(s-1)}_{0,(s-1)p/(p-1)} \|u_{\mathcal{T}}\|_{1,p,\mathcal{T}} \qquad \forall u_{\mathcal{T}} \in X(\mathcal{T}).$$

Proof

- Application of the Theorem on $L^1 \cap BV$ to $v_{\mathcal{T}} = |u_{\mathcal{T}}|^s$.
- $lhs \leq C \left(\frac{1}{\xi^{(p-1)/p}} |u_{\mathcal{T}}|_{1,p,\mathcal{T}} \|u_{\mathcal{T}}\|^{(s-1)}_{0,(s-1)p/(p-1)} + \|u_{\mathcal{T}}\|^s_{0,s} \right)$
- Interpolation : $\|u_{\mathcal{T}}\|_{0,s} \leq \|u_{\mathcal{T}}\|^{1/s}_{0,p} \|u_{\mathcal{T}}\|^{(s-1)/s}_{0,(s-1)p/(p-1)}$.

The key points of the proof of (PSdis)

$p = 1$

- Direct consequence of the embedding Theorem :
$$\|u_{\mathcal{T}}\|_{0,N/(N-1)} \leq C\|u_{\mathcal{T}}\|_{1,1,\mathcal{T}}.$$
- $p^* = \dfrac{N}{N-1} \Longrightarrow$ result sill holds $\forall 1 \leq q \leq p^*$.

$1 < p < N$

- Let $s = \dfrac{(N-1)p}{N-p}$. Then,
$$s > 1,\ \frac{(s-1)p}{p-1} = \frac{sN}{N-1} \text{ and } \frac{sN}{N-1} = \frac{Np}{N-p}.$$
- Application of the lemma :
$$\|u_{\mathcal{T}}\|_{0,pN/(N-p)} \leq \frac{C}{\xi^{(p-1)/p}}\|u_{\mathcal{T}}\|_{1,p,\mathcal{T}}.$$
- Result $\forall 1 \leq q \leq p^* = \frac{pN}{N-p}$.

The key points of the proof of (PSdis)

$p = N$

- Application of the lemma with $p = N$:

$$\|u_{\mathcal{T}}\|_{0,sN/(N-1)}^{s} \leq \frac{C}{\xi^{(N-1)/N}} \|u_{\mathcal{T}}\|_{0,(s-1)N/(N-1)}^{(s-1)} \|u_{\mathcal{T}}\|_{1,N,\mathcal{T}}.$$

- But $L^{sN/(N-1)}(\Omega) \subset L^{(s-1)N/(N-1)}(\Omega)$, so that

$$\|u_{\mathcal{T}}\|_{0,(s-1)N/(N-1)} \leq \frac{C}{\xi^{(N-1)/N}} \|u_{\mathcal{T}}\|_{1,N,\mathcal{T}}$$

- $s = 1 + (N-1)q/N$.

$p > N$

- We have : $\|u_{\mathcal{T}}\|_{1,N,\mathcal{T}} \leq \dfrac{C}{\xi^{(p-N)/(pN)}} \|u_{\mathcal{T}}\|_{1,p,\mathcal{T}}$.
- We apply the result for $p = N$.

Discrete Poincaré-Sobolev inequality, Dirichlet case

Theorem

Let Ω be a polyedral bounded domain of $\mathbb{R}^N$, $N \geq 2$.
Let $\Gamma^0 \subset \Gamma$, $\mathrm{m}(\Gamma^0) > 0$.
Let $(\mathcal{T}, \mathcal{E}, \mathcal{P})$ be a regular mesh of Ω, with regularity ξ.

- If $1 \leq p < N$, let $1 \leq q \leq p^* = \dfrac{pN}{N-p}$.
- If $p \geq N$, let $1 \leq q < +\infty$.

There exists $C > 0$, depending only on p, q, N, Γ^0 and Ω such that

$$\|u_{\mathcal{T}}\|_{0,q} \leq \frac{C}{\xi^{(p-1)/p}} \, |u_{\mathcal{T}}|_{1,p,\Gamma^0,\mathcal{T}} \quad \forall u_{\mathcal{T}} \in X(\mathcal{T}).$$

Discrete Poincaré-Wirtinger inequality

Theorem

Let Ω be a polyedral bounded domain of $\mathbb{R}^N$, $N \geq 2$.
Let $(\mathcal{T}, \mathcal{E}, \mathcal{P})$ be a regular mesh of Ω, with regularity ξ.

For all $1 \leq p < +\infty$, there exists $C > 0$, depending only on p, N and Ω such that

$$\|u_{\mathcal{T}} - \bar{u}_{\mathcal{T}}\|_{0,p} \leq \frac{C}{\xi^{(p-1)/p}} \, |u_{\mathcal{T}}|_{1,p,\mathcal{T}} \quad \forall u_{\mathcal{T}} \in X(\mathcal{T}),$$

where

$$\bar{u}_{\mathcal{T}} = \frac{1}{\mathrm{m}(\Omega)} \int_{\Omega} u_{\mathcal{T}}.$$

Outline

FV scheme for the evolutive equation

$$\begin{cases} \partial_t f = \Delta f^\beta, \text{ in } \Omega \times \mathbb{R}_+ \\ f = f^D \text{ on } \Gamma^D \times \mathbb{R}_+, \quad \nabla f \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+ \\ f(\cdot, 0) = f_0 > 0. \end{cases}$$

The scheme

$$\begin{cases} \mathrm{m}(K)\dfrac{f_K^{n+1} - f_K^n}{\Delta t} - \displaystyle\sum_{\sigma \in \mathcal{E}_K} \tau_\sigma D_{K,\sigma}(f^{n+1})^\beta = 0 \quad \forall K \in \mathcal{T} \\ f_\sigma^D = \dfrac{1}{\mathrm{m}(\sigma)} \displaystyle\int_\sigma f^D, \quad f_K^0 = \dfrac{1}{\mathrm{m}(K)} \displaystyle\int_K f_0 \end{cases}$$

with the notation : $D_{K,\sigma} u = \begin{cases} u_L - u_K & \text{if } \sigma = K|L \\ u_\sigma^D - u_K & \text{if } \sigma \subset \Gamma^D \\ 0 & \text{if } \sigma \subset \Gamma^N \end{cases}$

Hypotheses and first result

Hypotheses

- Admissibility and regularity of the mesh
- $\mathcal{E}_{ext}^D \neq \emptyset$
- $f_K^0 \geq 0 \quad \forall K \in \mathcal{T}$
- $\exists m^D$ and M^D such that

$$0 < m^D \leq f_\sigma^D \leq M^D \quad \forall \sigma \in \mathcal{E}_{ext}^D.$$

Proposition

The scheme has a unique nonnegative solution $(f_K^n)_{K\in\mathcal{T},n\geq 0}$.

❑ Eymard, Gallouët, Hilhorst, Naït Slimane, 1998

Scheme for the steady state

$$\begin{cases} \Delta f^{\beta} = 0, \text{ in } \Omega \times \mathbb{R}_+ \\ f = f^D \text{ on } \Gamma^D \times \mathbb{R}_+, \quad \nabla f \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+ \end{cases}$$

The scheme

$$\sum_{\sigma \in \mathcal{E}_K} \tau_\sigma D_{K,\sigma} (f^\infty)^\beta \; = \; 0 \, , \quad \forall K \in \mathcal{T} \, .$$

Proposition

The scheme has a unique nonnegative solution $(f_K^\infty)_{K\in\mathcal{T}}$, which satisfies :

$$m^D \leq f_K^\infty \leq M^D \quad \forall K \in \mathcal{T}.$$

Outline

At the continuous level

$$E(t) = \int_\Omega \frac{f^{\beta+1} - (f^\infty)^{\beta+1}}{\beta+1} - (f^\infty)^\beta (f - f^\infty)$$
$$D(t) = \int_\Omega |\nabla \left(f^\beta - (f^\infty)^\beta \right)|^2$$

- Relation between entropy and dissipation :

$$D(t) \geq \frac{(m^D)^{\beta-1}}{C_P} E(t).$$

- Exponential decay of the entropy :

$$E(t) \leq E(0)e^{-\lambda t}, \text{ with } \lambda = \frac{(m^D)^{\beta-1}}{C_P}.$$

At the discrete level

Discrete relative entropy

$$E^n = \sum_{K\in\mathcal{T}} \mathrm{m}(K) \left(\frac{(f_K^n)^{\beta+1} - (f_K^\infty)^{\beta+1}}{\beta+1} - (f_K^\infty)^\beta (f_K^n - f_K^\infty) \,.\right)$$

Discrete dissipation

$$\mathcal{D}^n = \sum_{\sigma\in\mathcal{E}} \tau_\sigma \left(D_{K,\sigma}((f^{n+1})^\beta - (f^\infty)^\beta) \right)^2$$

Discrete entropy-entropy dissipation property

$$\frac{E^{n+1} - E^n}{\Delta t} + \mathcal{D}^{n+1} \leq 0 \quad \forall n \geq 0.$$

Exponential decay towards the steady-state

Discrete Poincaré inequality

$$\sum_{K\in\mathcal{T}} \mathrm{m}(K)\Big((f_K^{n+1})^\beta - (f_K^\infty)^\beta\Big)^2 \le \frac{C_P}{\xi}\mathcal{D}^{n+1}.$$

Elementary inequality

$$(x^\beta - y^\beta)^2 \ge y^{\beta-1}\left(\frac{x^{\beta+1}-y^{\beta+1}}{\beta+1} - y^\beta(x-y)\right) \quad \forall x,y \ge 0.$$

Consequences

$$E^{n+1} \le \frac{C_P}{\xi\,(m^D)^{\beta-1}}\mathcal{D}^{n+1}$$

$$E^{n+1} \le \left(1 + \Delta t\,\frac{\xi\,(m^D)^{\beta-1}}{C_P}\right)^{-1} E^n$$

Exponential decay towards the steady-state

Theorem

$$E^n \leq e^{-\lambda t^n} E^0 \quad \forall n \geq 0$$

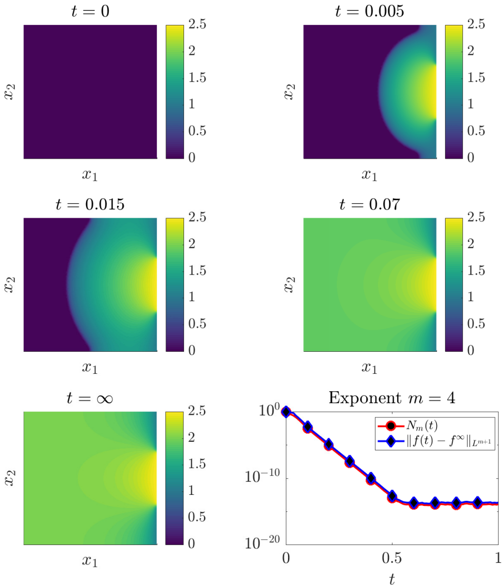
and $$\sum_{K \in \mathcal{T}} \mathrm{m}(K) |f_K^n - f_K^\infty|^{\beta+1} \; \leq \; (\beta+1)\, e^{-\lambda t^n} E^0$$

Another elementary inequality

$$|x-y|^{\beta+1} \geq x^{\beta+1} - y^{\beta+1} - (\beta+1) y^\beta (x-y) \quad \forall x, y \geq 0.$$

❑ C.-H., Herda, 2019

Numerical results ($\beta = 4$)



Outline

General case

$$\begin{cases} \partial_t f + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = -\nabla f + \mathbf{U} f, \text{ in } \Omega \times \mathbb{R}_+ \\ f = f^D \text{ on } \Gamma^D \times \mathbb{R}_+ \text{ and } \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+ \end{cases}$$

Steady-state

$$\begin{cases} \nabla \cdot \mathbf{J}^\infty = 0, \quad \mathbf{J}^\infty = -\nabla f^\infty + \mathbf{U} f^\infty, \text{ in } \Omega \times \mathbb{R}_+ \\ f^\infty = f^D \text{ on } \Gamma^D \times \mathbb{R}_+ \text{ and } \mathbf{J}^\infty \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+. \end{cases}$$

$$f = f^\infty h \quad \Longrightarrow \quad \mathbf{J} = \mathbf{J}^\infty h - f^\infty \nabla h$$

Exponential decay towards the steady-state

- Entropy/dissipation, with $\Phi_2(x) = (x-1)^2$,

$$H_2(t) = \int_\Omega f^\infty \Phi_2(h) \text{ and } D_2(t) = \int_\Omega f^\infty \Phi_2''(h) |\nabla h|^2$$

- Poincaré inequality + bounds on f^∞

Adaptation to the discrete level ?

❑ FILBET, HERDA, '17

Strategy

- Forward/backward Euler in time + finite volume in space
- Numerical scheme for the steady-state f^∞

  $\Longrightarrow$ approximation of the steady flux $\mathbf{J}^\infty$
- Approximation of the flux $\mathbf{J}$ as $\mathbf{J} = \mathbf{J}^\infty h - f^\infty \nabla h$

Main result

$$\|f_\delta(t^n) - f_\delta^\infty\|_1^2 \leq C e^{-\kappa t^n}$$

"Drawback"

Pre-computation of the steady-state needed for the definition of the scheme

Outline

Schemes for the evolutive drift-diffusion equation

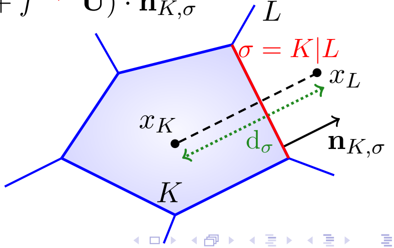
From the equation...

$$\begin{cases} \partial_t f + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = -\nabla f + \mathbf{U} f, \\ f(\cdot, 0) = f_0 \geq 0 \ \ + \text{boundary conditions} \end{cases}$$

... to the scheme

$$\begin{cases} \mathsf{m}(K) \dfrac{f_K^{n+1} - f_K^n}{\Delta t} + \displaystyle\sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{K,\sigma}^{n+1} = 0 \\ \mathcal{F}_{K,\sigma}^{n+1} \approx \displaystyle\int_\sigma (-\nabla f^{n+1} + f^{n+1} \mathbf{U}) \cdot \mathbf{n}_{K,\sigma} \end{cases}$$

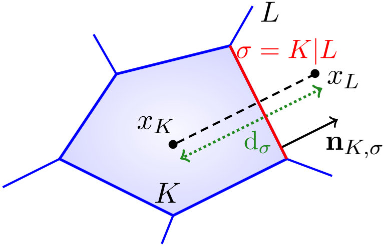
- $\mathcal{T}$: control volumes, $K \in \mathcal{T}$
- $\mathcal{E}$: edges, $\sigma \in \mathcal{E}$
- Δt : time step



Numerical fluxes

$$\mathcal{F}_{K,\sigma} \approx \int_\sigma (-\nabla f + f\mathbf{U}) \cdot \mathbf{n}_{K,\sigma}$$

$$U_{K,\sigma} \approx \frac{1}{\mathrm{m}(\sigma)} \int_\sigma \mathbf{U} \cdot \mathbf{n}_{K,\sigma}$$



Generic form

$$\mathcal{F}_{K,\sigma} = \tau_\sigma \Big(B(-U_{K,\sigma}\mathrm{d}_\sigma) f_K - B(U_{K,\sigma}\mathrm{d}_\sigma) f_L \Big), \ \tau_\sigma = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}$$

with $B(0) = 1, \ B(x) > 0$ and $B(x) - B(-x) = -x \ \forall x \in \mathbb{R}$

Classical examples

$$B_{up}(s) = 1 + s^-, \quad B_{ce}(s) = 1 - \frac{s}{2}$$

❑ C.-H., Droniou, '05

Scharfetter-Gummel fluxes

Generic form

$$\mathcal{F}_{K,\sigma} = \tau_\sigma \Big(B(-U_{K,\sigma}\mathrm{d}_\sigma) f_K - B(U_{K,\sigma}\mathrm{d}_\sigma) f_L \Big), \; \tau_\sigma = \frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}$$

with $B(0) = 1, \; B(x) > 0$ and $B(x) - B(-x) = -x \; \forall x \in \mathbb{R}$

Preservation of a thermal equilibrium $\mathbf{U} = -\nabla \Psi$

$$f = \lambda e^{-\Psi} \Longrightarrow -\nabla f - f \nabla \Psi = 0$$

At the discrete level $U_{K,\sigma}\mathrm{d}_\sigma = (\Psi_K - \Psi_L)$

$$(f_K = \lambda e^{-\Psi_K} \Longrightarrow \mathcal{F}_{K,\sigma} = 0) \Longleftrightarrow B(x) = \frac{x}{e^x - 1}$$

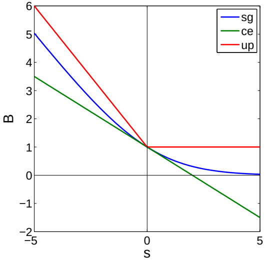
- Scharfetter, Gummel, 1969

Family of B-schemes for the Fokker-Planck equation

$$
\begin{cases}
\mathsf{m}(K)\dfrac{f_K^{n+1}-f_K^n}{\Delta t}+\displaystyle\sum_{\sigma\in\mathcal{E}_K}\mathcal{F}_{K,\sigma}^{n+1}=0\\[2ex]
\mathcal{F}_{K,\sigma}^{n+1}=\begin{cases}
\tau_\sigma\Big(B(-U_{K,\sigma}\mathrm{d}_\sigma)f_K^{n+1}-B(U_{K,\sigma}\mathrm{d}_\sigma)f_L^{n+1}\Big), & \sigma=K|L,\\
\tau_\sigma\Big(B(-U_{K,\sigma}\mathrm{d}_\sigma)f_K^{n+1}-B(U_{K,\sigma}\mathrm{d}_\sigma)f_\sigma^{D}\Big), & \sigma\in\mathcal{E}_{ext}^D,\\
0, & \sigma\in\mathcal{E}_{ext}^N.
\end{cases}
\end{cases}
$$

Hypotheses on B

- $B(0)=1$,
- $B(x)>0 \quad \forall x\in\mathbb{R}$,
- $B(x)-B(-x)=-x$.



Additional hypotheses

- Admissibility and regularity of the mesh
- $\mathcal{E}_{ext}^{D} \neq \emptyset$
- $f_K^0 \geq 0 \quad \forall K \in \mathcal{T}$
- $\exists m^D$ and M^D such that

$$0 < m^D \leq f_\sigma^D \leq M^D \quad \forall \sigma \in \mathcal{E}_{ext}^D.$$

- $\exists V \geq 0$ such that

$$\max_{K \in \mathcal{T}} \max_{\sigma \in \mathcal{E}_K} |U_{K,\sigma}| \leq V.$$

Proposition

The scheme has a unique nonnegative solution $(f_K^n)_{K \in \mathcal{T}, n \geq 0}$.

Associated steady-state

$$\begin{cases} \displaystyle\sum_{\sigma\in\mathcal{E}_K} \mathcal{F}_{K,\sigma}^{\infty} = 0 \\ \mathcal{F}_{K,\sigma}^{\infty} = \begin{cases} \tau_\sigma\Big(B(-U_{K,\sigma}\mathrm{d}_\sigma)f_K^\infty - B(U_{K,\sigma}\mathrm{d}_\sigma)f_L^\infty\Big), & \sigma = K|L \\ \tau_\sigma\Big(B(-U_{K,\sigma}\mathrm{d}_\sigma)f_K^\infty - B(U_{K,\sigma}\mathrm{d}_\sigma)f_\sigma^D\Big), & \sigma \in \mathcal{E}_{ext}^D \\ 0, & \sigma \in \mathcal{E}_{ext}^N \end{cases} \end{cases}$$

Proposition

- Existence and uniqueness of a solution to the scheme $(f_K^\infty)_{K\in\mathcal{T}}$.
- $\exists m^\infty, M^\infty$ such that

$$0 < m^\infty \le f_K^\infty \le M^\infty \quad \forall K \in \mathcal{T}.$$

Outline

How to rewrite the numerical fluxes ?

$$f = f^{\infty} h \Longrightarrow \mathbf{J} = \mathbf{J}^{\infty} h - f^{\infty} \nabla h$$

$$\begin{aligned}
\mathcal{F}_{K,\sigma} &= \tau_\sigma \Big(B(-U_{K,\sigma}\mathrm{d}_\sigma) f_K - B(U_{K,\sigma}\mathrm{d}_\sigma) f_L \Big), \\
&= \tau_\sigma \Big(B(-U_{K,\sigma}\mathrm{d}_\sigma) h_K f_K^{\infty} - B(U_{K,\sigma}\mathrm{d}_\sigma) h_L f_L^{\infty} \Big), \\
&= \mathcal{F}^{\infty}_{K,\sigma} h_K + \tau_\sigma B(U_{K,\sigma}\mathrm{d}_\sigma) f_L^{\infty} (h_K - h_L), \\
&= \mathcal{F}^{\infty}_{K,\sigma} h_L + \tau_\sigma B(-U_{K,\sigma}\mathrm{d}_\sigma) f_K^{\infty} (h_K - h_L)
\end{aligned}$$

Reformulation of the fluxes

$$\mathcal{F}_{K,\sigma} = \mathcal{F}^{upw}_{K,\sigma} + \tau_\sigma f^{\infty}_{B,\sigma} (h_K - h_L)$$

with $\mathcal{F}^{upw}_{K,\sigma} = (\mathcal{F}^{\infty}_{K,\sigma})^+ h_K - (\mathcal{F}^{\infty}_{K,\sigma})^- h_L$

and $f^{\infty}_{B,\sigma} = \min\Big(B(-U_{K,\sigma}\mathrm{d}_\sigma) f_K^{\infty}, B(U_{K,\sigma}\mathrm{d}_\sigma) f_L^{\infty} \Big)$

Entropy-entropy dissipation property

$$\Phi'' > 0,\ \Phi(1) = 0,\ \Phi'(1) = 0$$

Discrete relative Φ-entropy

$$H_\Phi^n = \sum_{K \in \mathcal{T}} \mathrm{m}(K) \Phi(h_K^n) f_K^\infty$$

Discrete dissipation

$$D_\Phi^n = \sum_{\sigma \in \mathcal{E}} \tau_\sigma f_{B,\sigma}^\infty (h_K^n - h_L^n)(\Phi'(h_K^n) - \Phi'(h_L^n)).$$

Discrete entropy-entropy dissipation property

$$\frac{H_\Phi^{n+1} - H_\Phi^n}{\Delta t} + D_\Phi^{n+1} \leq 0 \quad \forall n \geq 0.$$

Main results

Uniform bounds

$$m^{\infty}\min(1,\min_{K\in\mathcal{T}}\frac{f_K^0}{f_K^{\infty}})\le f_K^n\le M^{\infty}\max(1,\max_{K\in\mathcal{T}}\frac{f_K^0}{f_K^{\infty}})$$

Proof

- $\Phi_+(s)=(s-M)^+,\quad M=\max(1,\max h_K^0)$
- $\Phi_-(s)=(s-m)^-,\quad m=\min(1,\min h_K^0)$

Exponential decay

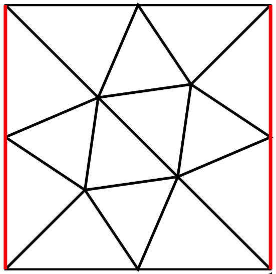
$$\Phi_2(s)=(s-1)^2,$$

$$H_{\Phi_2}^n\le H_{\Phi_2}^0 e^{-\kappa t^n},$$

$$\left(\sum_{K\in\mathcal{T}}\mathrm{m}(K)|f_K^n-f_K^{\infty}|\right)^2\le H_{\Phi_2}^0\left(\sum_{K\in\mathcal{T}}\mathrm{m}(K)f_K^{\infty}\right)e^{-\kappa t^n}.$$

Test case

$$\begin{cases} \partial_t f + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = -\nabla f + \mathbf{U} f \\ \mathbf{U} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$



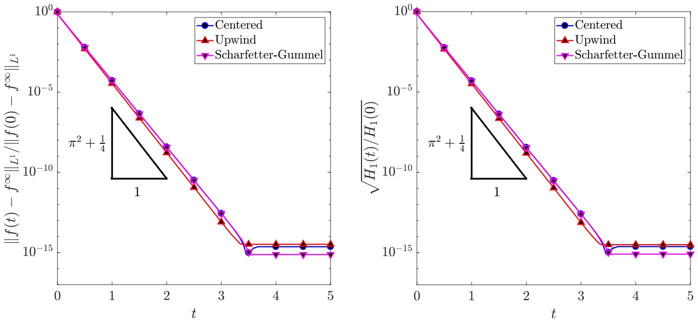
Solution and steady-state

$$f(x_1, x_2, t) = \exp(x_1) + \exp\left(\frac{x_1}{2} - \left(\pi^2 + \frac{1}{4}\right) t\right) \sin(\pi x_1)$$

$$f^\infty(x_1, x_2) = \exp(x_1)$$

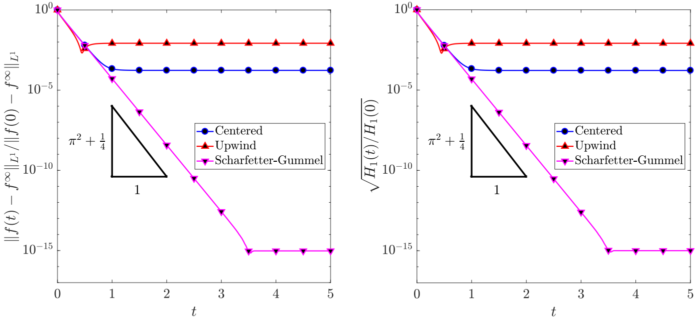
Long time behavior

Decay to the steady-state associated to the scheme



Long time behavior

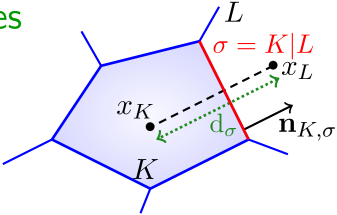
Decay to the real steady-state



Outline of the talk

1. Discrete functional inequalities
2. Results for the porous media equations
   - Presentation of the schemes
   - Long time behavior
3. Results for the Fokker-Planck equations
   - Presentation of the B-schemes and first results
   - Long time behavior
   - About nonlinear schemes

Design of nonlinear TPFA schemes



Numerical fluxes

$$\mathbf{J} = -\nabla f - f\nabla\Psi = -f\nabla(\log f + \Psi)$$

$$\mathcal{F}_{K,\sigma} \approx \int_\sigma -f\nabla(\log f + \Psi) \cdot \mathbf{n}_{K,\sigma}$$

$$\mathcal{F}_{K,\sigma} = \tau_\sigma \; r(f_K, f_L)\Big(\log f_K + \Psi_K - \log f_L - \Psi_L\Big)$$

Examples of r functions

$$r(x,y) = \frac{x+y}{2}, \quad r(x,y) = \frac{x-y}{\log x - \log y}, \ldots$$

Design of nonlinear TPFA schemes

$$\begin{cases} \partial_t f + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = -\nabla f - \nabla \Psi f \text{ in } \Omega \times \mathbb{R}_+, \\ \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma \times \mathbb{R}_+, \\ f(.,0) = f_0 \geq 0. \end{cases}$$

The nonlinear schemes

$$\begin{cases} \mathrm{m}(K) \dfrac{f_K^{n+1} - f_K^n}{\Delta t} + \displaystyle\sum_{\sigma \in \mathcal{E}_K^{int}} \mathcal{F}_{K,\sigma}^{n+1} = 0, \\ \mathcal{F}_{K,\sigma} = \tau_\sigma \; r(f_K, f_L) \Big(\log f_K + \Psi_K - \log f_L - \Psi_L \Big). \end{cases}$$

Preservation of the thermal equilibrium

- $f_K^\infty = \lambda e^{-\Psi_K}$ is a steady-state,
- λ is fixed by the conservation of mass.

Dissipativity of the schemes

$$\begin{cases} \mathsf{m}(K)\dfrac{f_K^{n+1}-f_K^n}{\Delta t} + \displaystyle\sum_{\sigma\in\mathcal{E}_K^{int}} \mathcal{F}_{K,\sigma}^{n+1} = 0, \\ \mathcal{F}_{K,\sigma} = \tau_\sigma r(f_K, f_L)\Big(\log\dfrac{f_K}{f_K^\infty} - \log\dfrac{f_L}{f_L^\infty}\Big). \end{cases}$$

Dissipation of the discrete entropies

Discrete relative entropy : $H_\Phi^n = \sum_{K\in\mathcal{T}} f_K^\infty \Phi(\frac{f_K^n}{f_K^\infty})$

$$\frac{H_\Phi^{n+1}-H_\Phi^n}{\Delta t} + D_\Phi^{n+1} \le 0$$

with

$$D_\Phi = \sum_{\sigma\in\mathcal{E}_{int}} \tau_\sigma r(f_K, f_L)\big(\log\frac{f_K}{f_K^\infty} - \log\frac{f_L}{f_L^\infty}\big)\big(\Phi'(\frac{f_K}{f_K^\infty}) - \Phi'(\frac{f_L}{f_L^\infty})\big)$$

Main results for the nonlinear TPFA schemes

A priori estimates

- Uniform bounds obtained with

$$\Phi(s) = (s-M)^+ \text{ and } \Phi(s) = (s-m)^-$$

for $M = \max(1, \max \frac{f_K^0}{f_K^\infty})$, $m = \min(1, \min \frac{f_K^0}{f_K^\infty})$

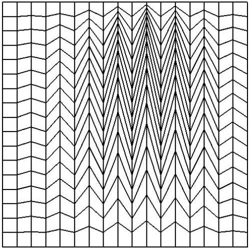
Existence of a solution to the scheme

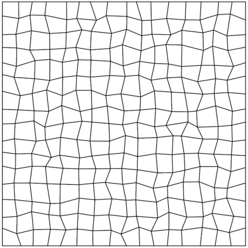
- via a topological degree argument

Exponential decay of H_1^n

- based on a discrete Log-Sobolev inequality

On general meshes ?





- The nonlinear strategy is applicable to other kinds of finite volume schemes.
- DDFV schemes, for instance.

❑ Cancès, Guichard, 2016

❑ Cancès, C.-H., Krell, 2018

Convergence with respect to the grid

On Kershaw meshes

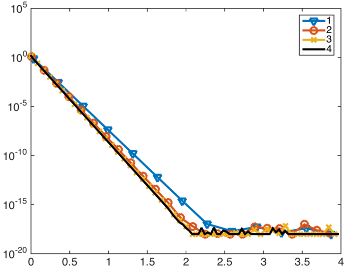
M	dt	errf	ordf	N_{max}	N_{mean}	Min f^n
1	2.0E-03	7.2E-03	—	9	2.15	1.010E-01
2	5.0E-04	1.7E-03	2.09	8	2.02	2.582E-02
3	1.2E-04	7.2E-04	2.20	7	1.49	6.488E-03
4	3.1E-05	4.0E-04	2.11	7	1.07	1.628E-03
5	3.1E-05	2.6E-04	1.98	7	1.04	1.628E-03

On quadrangle meshes

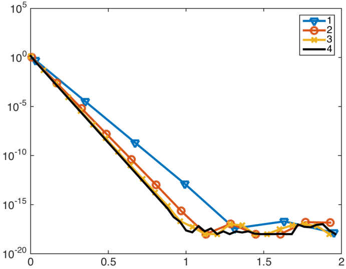
M	dt	errf	ordf	N_{max}	N_{mean}	Min f^n
1	4.0E-03	2.1E-02	—	9	2.26	1.803E-01
2	1.0E-03	5.1E-03	2.08	9	2.04	5.079E-02
3	2.5E-04	1.3E-03	2.06	8	1.96	1.352E-02
4	6.3E-05	3.3E-04	2.09	8	1.22	3.349E-03
5	1.2E-05	7.7E-05	1.70	7	1.01	8.695E-04

Long time behavior

Exponential decay of the discrete relative entropy



Kershaw



quadrangles

Conclusion

- FV schemes are well adapted for the discretization of conservation laws/system of conservation laws.
- They are able to preserve physical properties like positivity, conservation of mass, entropy dissipation,...
- For dissipative problems, they satisfy discrete entropy - dissipation properties.
  - → bounds on the solution, leading to compactness properties
  - → knowledge of the long time behavior
- Classical results like compactness properties or functional inequalities may be adapted to the discrete setting.