Finite volume methods for dissipative problems

Claire Chainais-Hillairet

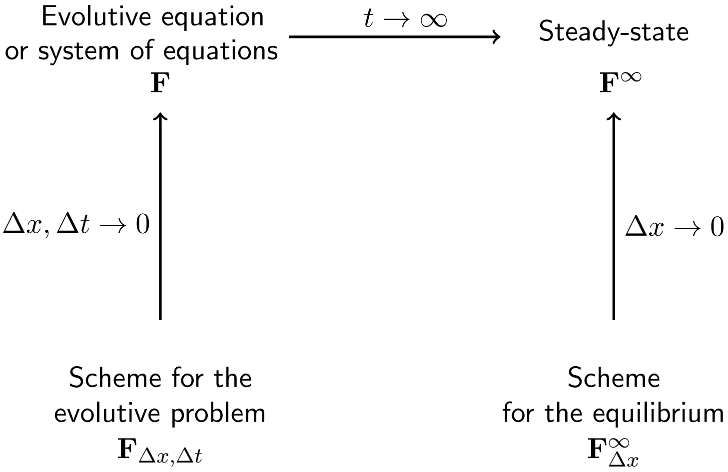
CEMRACS summer school

Marseille, July 2019, 15-20

Lecture 2 :

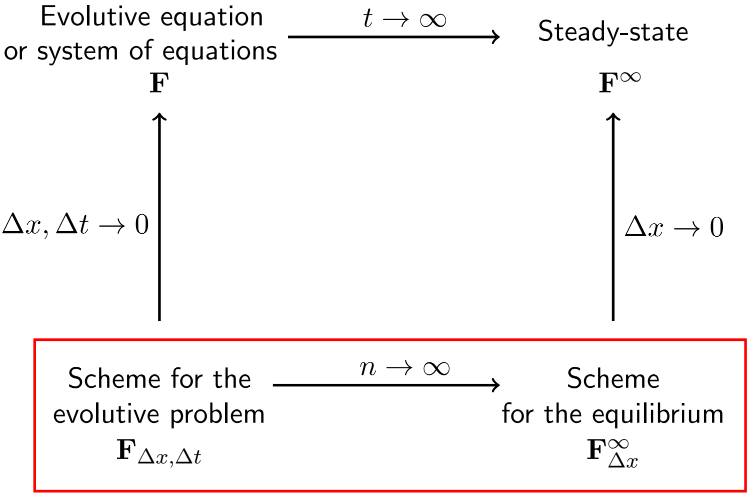
Dissipative problems and long time behavior

Overview



Evolutive equation
or system of equations
$\mathbf{F}$
$t \to \infty$
Steady-state
$\mathbf{F}^{\infty}$
$\Delta x, \Delta t \to 0$
$\Delta x \to 0$
Scheme for the
evolutive problem
$\mathbf{F}_{\Delta x, \Delta t}$
Scheme
for the equilibrium
$\mathbf{F}^{\infty}_{\Delta x}$

Overview


Evolutive equation
or system of equations
$\mathbf{F}$
$t \to \infty$
Steady-state
$\mathbf{F}^{\infty}$
$\Delta x, \Delta t \to 0$
$\Delta x \to 0$
Scheme for the
evolutive problem
$\mathbf{F}_{\Delta x, \Delta t}$
$n \to \infty$
Scheme
for the equilibrium
$\mathbf{F}^{\infty}_{\Delta x}$

Framework : dissipative problems

$$(\mathbf{F}) \quad \begin{cases} \partial_t u + Au = 0, \ t \geq 0, \\ u(0) = u_0. \end{cases} \qquad (\mathbf{F}^\infty) \quad \begin{cases} u_\infty, \\ Au_\infty = 0. \end{cases}$$

Main features

- Existence of a Lyapunov convex function E satisfying

$$\frac{d}{dt}E(u) = -\langle Au, E'(u)\rangle \leq 0.$$

- E is general given by the physics : it is a physical energy or entropy, which is dissipated along time.
- $D(u) = \langle Au, E'(u)\rangle$ is the dissipation of energy/entropy.
- The steady-state is a minimizer for E.

Dissipativity and long time behavior

$$(\mathbf{F}) \quad \begin{cases} \partial_t u + Au = 0, \ t \geq 0, \\ u(0) = u_0. \end{cases} \qquad (\mathbf{F}^\infty) \quad \begin{cases} u_\infty, \\ Au_\infty = 0. \end{cases}$$

Main features

$$\frac{d}{dt}E(u) = -D(u)$$

- $D(u) \geq \lambda E(u) \Longrightarrow$ exponential decay :

$$E(u) \leq E(u_0)e^{-\lambda t}.$$

- $D(u) \geq KE(u)^{1+\gamma} \Longrightarrow$ polynomial decay $\sim t^{-1/\gamma}$

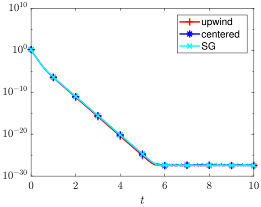
Overview

Evolutive equation
or system of equations
$\mathbf{F}$

$t \to \infty$

Steady-state
$\mathbf{F}^{\infty}$

$\Delta x, \Delta t \to 0$



$\Delta x \to 0$

+ exponential convergence ?

Scheme for the
evolutive problem
$\mathbf{F}_{\Delta x, \Delta t}$

$n \to \infty$

Scheme
for the equilibrium
$\mathbf{F}^{\infty}_{\Delta x}$

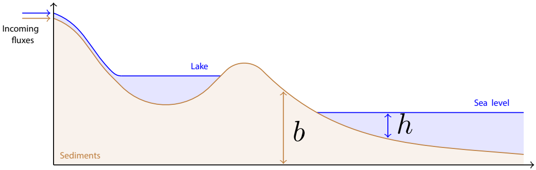
Outline of the lecture

1. Some example of dissipative problems

2. Long time behavior of the Fokker-Planck equation

3. Long time behavior of the porous media equation

Shallow water equations with viscous terms



h : water height

b : ground topography

u : velocity

Shallow water equations with friction

$$\begin{cases} \partial_t h + \operatorname{div}(h\mathbf{u}) = 0 \\ \partial_t h\mathbf{u} + \operatorname{div}(h\mathbf{u} \otimes \mathbf{u}) + gh\nabla(h+b) = -C|\mathbf{u}|\mathbf{u} \end{cases}$$

Energy/dissipation $$E(h,u) = \frac{1}{2}\int_{\mathbb{T}^2} \left(h|\mathbf{u}|^2 + g(h+b)^2 \right)$$

$$D(h,u) = \int_{\mathbb{T}^2} C|\mathbf{u}|^3$$

Study over large time scales

❑ Peton, '18

$$\begin{cases} \partial_t h + \mathrm{div}(h\mathbf{u}) = 0 \\ gh\nabla(h+b) = -C|\mathbf{u}|\mathbf{u} \end{cases}$$

$$\Longrightarrow \mathbf{u} = -(\frac{g}{C})^{1/2} h^{1/2} |\nabla(h+b)|^{-1/2} \nabla(h+b)$$

Conservation law for the water height

$$\partial_t h + \mathrm{div}(-K h^{3/2} |\nabla(h+b)|^{-1/2} \nabla(h+b)) = 0$$

- degenerate 3/2-laplacian equation,
- with drift.

Dissipative behavior

$$\begin{cases} \partial_t h + \operatorname{div}(-K h^{3/2} |\nabla(h+b)|^{-1/2} \nabla(h+b)) = 0 \\ + \text{ no-flux boundary condition} \end{cases}$$

Energy/dissipation $E(h) = \frac{1}{2} \int_\Omega (h+b)^2$

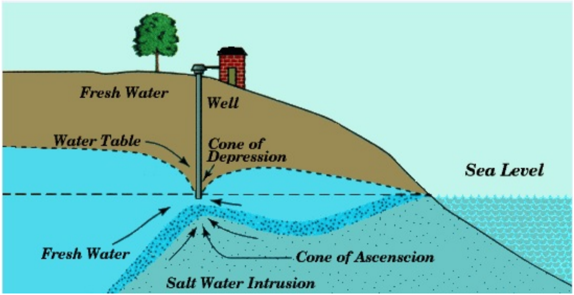
$$\begin{aligned} \frac{d}{dt} E(h) &= \int_\Omega \partial_t h (h+b) \\ &= -\int_\Omega \operatorname{div}\big(-K h^{3/2} |\nabla(h+b)|^{-1/2} \nabla(h+b)\big)(h+b) \\ &= -\int_\Omega K h^{3/2} |\nabla(h+b)|^{3/2} \end{aligned}$$

Steady states

$h = 0$ or $h + b = constant.$

Saltwater intrusion model

Management of fresh water resources in costal regions



Quantities of interest

- height of the freshwater
- height of the interface with the saltwater

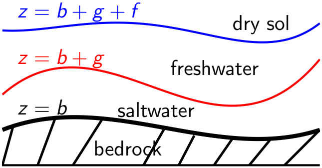
Saltwater intrusion model

- Escher, Laurençot, Matioc, '12,
- Laurençot, Matioc, '14, '17
- Jazar, Monneau, '14

Assumptions

- Thin layers, sharp interfaces
- Large time scales
- Incompressible immiscible phases
- Mainly horizontal displacements

Notations



ρ : ratio of the densities, $\rho = \dfrac{\rho_f}{\rho_s} < 1$,

ν : ratio of the kinematic viscosities, $\nu = \dfrac{\nu_s}{\nu_f} \in (0, +\infty)$

Saltwater intrusion model

$$\begin{cases} \partial_t f + \operatorname{div}(-\nu f \nabla(f+g+b)) = 0, \\ \partial_t g + \operatorname{div}(-g\nabla(\rho f + g + b)) = 0, \\ \ + \text{ no-flux boundary conditions} \end{cases}$$

Dissipative behavior

$$E(f,g) = \int_\Omega \frac{\rho}{2}(f+g)^2 + \frac{1-\rho}{2}g^2 + b(\rho f + g).$$

$$\begin{aligned} \frac{d}{dt}E(f,g) &= \int_\Omega \partial_t f(\rho(f+g+b)) + \partial_t g(\rho f + g + b) \\ &= -\int_\Omega \rho\nu f\left|\nabla(\rho(f+g+b))\right|^2 + g\left|\nabla(\rho f + g + b)\right|^2 \\ &= -\int_\Omega \rho\nu f\left|\nabla\Phi_f\right|^2 + g\left|\nabla\Phi_g\right|^2 \end{aligned}$$

About porous media equation : $\Omega = \mathbb{R}^d$, $b = 0$, $f = 0$

$$\partial_t g + \mathrm{div}(-g\nabla g) = 0$$

Self-similar solutions

- Passage to self-similar variables :

$$\tau = \log(1+t),\ \xi = \frac{x}{(1+t)^{1/(d+2)}},\ g(t,x) = e^{-\tau\frac{d}{d+2}} u(\tau,\xi)$$

- Nonlinear Fokker-Planck equation on u :

$$\partial_t u + \mathrm{div}(-u\nabla(u + \frac{|x|^2}{2(d+2)})) = 0$$

- Self-similar solution in $g \Longleftrightarrow$ steady-state in u
- Exponential decay in u and polynomial decay in g

❑ Barenblatt, '1952

❑ Carrillo, Toscani, '00

Salt water intrusion model : self-similar solutions

Initial system with $b = 0$, $\Omega = \mathbb{R}^2$

$$\begin{cases} \partial_t f + \operatorname{div}(-\nu f \nabla(f+g)) = 0, \\ \partial_t g + \operatorname{div}(-g \nabla(\rho f + g)) = 0, \end{cases}$$

Self-similar variables and new system

$$(f,g)(t,x) = \frac{1}{(1+t)^{1/2}} (\tilde{f}, \tilde{g}) \left(\log(1+t), \frac{x}{(1+t)^{1/4}} \right)$$

$$\begin{cases} \partial_t f + \operatorname{div}(-\nu f \nabla(f + g + \frac{b}{\nu})) = 0, \\ \partial_t g + \operatorname{div}(-g \nabla(\rho f + g + b)) = 0, \\ b(x) = \dfrac{|x|^2}{8} \end{cases}$$

Salt water intrusion model : self-similar solutions

$$E(f,g) = \int_\Omega \frac{\rho}{2}(f+g)^2 + \frac{1-\rho}{2}g^2 + b(\frac{\rho}{\nu}f + g),$$

$$D(f,g) = \int_\Omega \rho\nu f\left|\nabla\Phi_f\right|^2 + g\left|\nabla\Phi_g\right|^2$$

with $\Phi_f = f + g + \frac{b}{\nu},\ \Phi_g = \rho f + g + b$

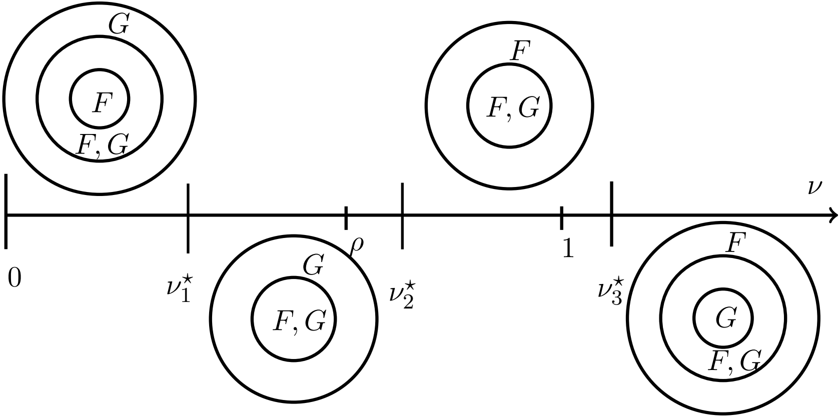
Characterization of the steady-states (self-similar solutions)

- Stationary solutions have vanishing fluxes :
  $$F\nabla\Phi_F = 0,\ G\nabla\Phi_G = 0.$$
- The minimizer of the energy is a stationary solution.
- There exists a unique minimizer of E which is radially symmetric.

❑ Ait Hammou Oulhaj, Cancès, C.-H., Laurençot, '19

❑ Laurençot, Matioc, '14

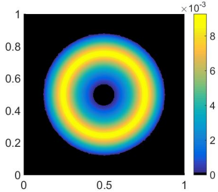
Self-similar profiles



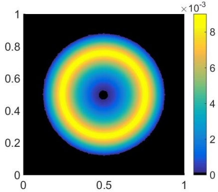
$$\nu_1^\star = \frac{\rho^2 \frac{M_f}{M_g}}{1 + \rho\left(\frac{M_f}{M_g} - 1\right)}, \quad \nu_2^\star = \frac{\rho \frac{M_f}{M_g} + 1}{\frac{M_f}{M_g} + 1}, \quad \nu_3^\star = 1 + (1 - \rho)\frac{M_f}{M_g}.$$

Numerical experiments

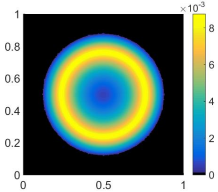
Topological change of E_G near the critical value $\nu_1^\star = 0.81$



$\nu = 0.80$



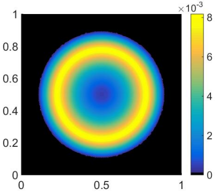
$\nu = \nu_1^\star = 0.81$



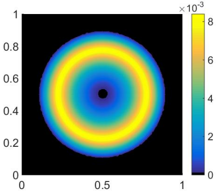
$\nu = 0.82$

Numerical experiments

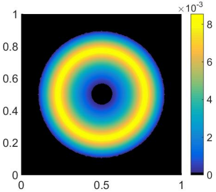
Topological change of E_F near the critical value $\nu_3^\star = 1.1$



$\nu = 1.09$



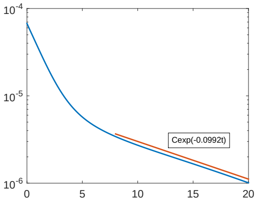
$\nu = \nu_3^\star = 1.10$



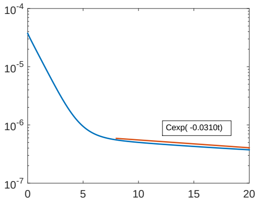
$\nu = 1.11$

Convergence towards the steady-state

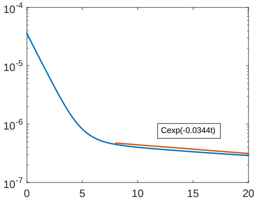
$$(f(t), g(t)) \to (F, G) \text{ in } L^2(\mathbb{R}^2; \mathbb{R}^2) \text{ as } t \to \infty$$



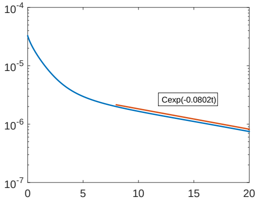
$\nu = 0.4$



$\nu = 0.9$



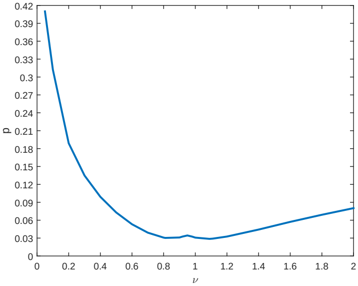
$\nu = 0.95$



$\nu = 2$

Exponential convergence towards the steady-state ?

Rate of convergence with respect to ν



Outline of the lecture

Focus on Fokker-Planck equations

$$\begin{cases} \partial_t f + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = -\nabla f + \mathbf{U} f, \text{ in } \Omega \times \mathbb{R}_+ \\ \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+ \\ f = f^D \text{ on } \Gamma^D \times \mathbb{R}_+ \\ f(\cdot, 0) = f_0 > 0. \end{cases}$$

Some references

- Carrillo, Toscani, '98
- Arnold, Markowich, Toscani, Unterreiter, '01
- Carrillo et al., '01
- Bodineau, Lebowitz, Mouhot, Villani, '14
- Gajewski, Gröger, '86, '89
- Jüngel, '95

Thermal equilibrium, when $\mathbf{U} = -\nabla\Psi$

$$\begin{cases} \partial_t f + \nabla\cdot\mathbf{J} = 0, \quad \mathbf{J} = -\nabla f - \nabla\Psi f, \text{ in } \Omega\times\mathbb{R}_+ \\ \mathbf{J}\cdot\mathbf{n} = 0 \text{ on } \Gamma^N\times\mathbb{R}_+ \\ f = f^D \text{ on } \Gamma^D\times\mathbb{R}_+ \\ f(\cdot,0) = f_0 > 0. \end{cases}$$

$$f = \lambda e^{-\Psi} \Longrightarrow \mathbf{J} = 0$$

Existence of a thermal equilibrium $\quad f^\infty = \lambda e^{-\Psi}$

- if $\Gamma^D = \emptyset$, with $\lambda = \int_\Omega f_0 \,/\, \int_\Omega e^{-\Psi}$,
- if $\log f^D + \Psi^D = \alpha$, with $\lambda = e^\alpha$.

$$\Longrightarrow \quad \mathbf{J} = -f\nabla(\log f + \Psi) = -f\nabla\log\frac{f}{f^\infty}$$

Entropy-dissipation property

$$\partial_t f + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = -f \nabla \log \frac{f}{f^\infty}$$

Relative entropy

$$\Phi_1(x) = x \log x - x + 1$$
$$H_1(t) = \int_\Omega f^\infty \Phi_1(\frac{f}{f^\infty})$$

Dissipation of the entropy

$$\frac{d}{dt} H_1(t) = -D_1(t),$$

with $D_1(t) = \int_\Omega f \left| \nabla \log \frac{f}{f^\infty} \right|^2 \geq 0$

Exponential decay towards thermal equilibrium

No-flux boundary conditions

- conservation of mass : $\int_\Omega f = \int_\Omega f_0 = \int_\Omega f^\infty$
- $H_1(t) = \int_\Omega f \log(f/f^\infty)$
- $D_1(t) = \int_\Omega f |\nabla \log(f/f^\infty)|^2 = 4 \int_\Omega f^\infty \left|\nabla \sqrt{f/f^\infty}\right|^2$
- thanks to Logarithmic Sobolev inequality :

$$0 \leq H_1(t) \leq H_1(0) e^{-\kappa t}$$

- and with Csiszar-Kullback inequality :

$$\|f(t) - f^\infty\|_1^2 \leq 2 H_1(0) e^{-\kappa t}$$

Exponential decay towards thermal equilibrium

Dirichlet boundary conditions

- Upper and lower bounds on f and f^∞
- $H_1(t) = \int_\Omega \Phi_1(f) - \Phi_1(f^\infty) - (f - f^\infty)\Phi_1'(f^\infty)$

$$c\|f(t) - f^\infty\|_2^2 \leq H_1(t) \leq C\|f(t) - f^\infty\|_2^2$$

- $D_1(t) = \int_\Omega f\,|\nabla(\log f - \log f^\infty)|^2$
- with Poincaré inequality :

$$D_1(t) \geq \mathcal{C}\|f(t) - f^\infty\|_2^2$$

- Conclusion :

$$c\|f(t) - f^\infty\|_2^2 \leq H_1(t) \leq H_1(0)e^{-\kappa t}$$

General case

$$\begin{cases} \partial_t f + \nabla \cdot \mathbf{J} = 0, \quad \mathbf{J} = -\nabla f + \mathbf{U} f, \text{ in } \Omega \times \mathbb{R}_+ \\ f = f^D \text{ on } \Gamma^D \times \mathbb{R}_+ \text{ and } \mathbf{J} \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+ \end{cases}$$

Steady-state

$$\begin{cases} \nabla \cdot \mathbf{J}^\infty = 0, \quad \mathbf{J}^\infty = -\nabla f^\infty + \mathbf{U} f^\infty, \text{ in } \Omega \times \mathbb{R}_+ \\ f^\infty = f^D \text{ on } \Gamma^D \times \mathbb{R}_+ \text{ and } \mathbf{J}^\infty \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+. \end{cases}$$

$$f = f^\infty h \quad \Longrightarrow \quad \mathbf{J} = \mathbf{J}^\infty h - f^\infty \nabla h$$

Exponential decay towards the steady-state

- Entropy/dissipation, with $\Phi_2(x) = (x-1)^2$,

$$H_2(t) = \int_\Omega f^\infty \Phi_2(h) \text{ and } D_2(t) = \int_\Omega f^\infty \Phi_2''(h) |\nabla h|^2$$

- Poincaré inequality + bounds on f^∞

Outline of the lecture

Focus on the porous media equations

$\beta \geq 1$

$$\begin{cases} \partial_t f = \Delta f^\beta, \text{ in } \Omega \times \mathbb{R}_+ \\ f = f^D \text{ on } \Gamma^D \times \mathbb{R}_+ \\ \nabla f \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \times \mathbb{R}_+ \\ f(\cdot, 0) = f_0 > 0. \end{cases}$$

Existence and uniqueness of a weak solution

- Vázquez

Steady-state

$$\begin{cases} \Delta (f^\infty)^\beta = 0, \text{ in } \Omega \\ f^\infty = f^D \text{ on } \Gamma^D, \quad \nabla f^\infty \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \end{cases}$$

Existence and uniqueness of the steady state ?

Dissipative behavior, $\Gamma^D \neq \emptyset$

Existence and uniqueness of the steady state

$$\begin{cases} \Delta (f^\infty)^\beta = 0, \text{ in } \Omega \\ f^\infty = f^D \text{ on } \Gamma^D, \quad \nabla f^\infty \cdot \mathbf{n} = 0 \text{ on } \Gamma^N \end{cases}$$

Dissipation of the relative entropy

$$E(t) = \int_\Omega \frac{f^{\beta+1} - (f^\infty)^{\beta+1}}{\beta+1} - (f^\infty)^\beta (f - f^\infty)$$

$$\begin{aligned} \frac{d}{dt} E(t) &= \int_\Omega (f^\beta - (f^\infty)^\beta) \partial_t f \\ &= - \int_\Omega \left| \nabla \left(f^\beta - (f^\infty)^\beta \right) \right|^2 \end{aligned}$$

- Bodineau, Lebowitz, Mouhot, Villani, 2014

Relation between entropy and dissipation, $\Gamma^D \neq \emptyset$

$$E(t) = \int_\Omega \frac{f^{\beta+1} - (f^\infty)^{\beta+1}}{\beta+1} - (f^\infty)^\beta (f - f^\infty)$$

$$D(t) = \int_\Omega \left| \nabla \left(f^\beta - (f^\infty)^\beta \right) \right|^2$$

Poincaré inequality $\quad f^\beta - (f^\infty)^\beta = 0$ on Γ^D

$$\int_\Omega \left(f^\beta - (f^\infty)^\beta \right)^2 \leq C_P \int_\Omega \left| \nabla \left(f^\beta - (f^\infty)^\beta \right) \right|^2$$

Functional inequalities

$$(z^\beta - 1)^2 \geq \frac{1}{\beta+1} \left(z^{\beta+1} - (\beta+1)z + \beta \right) \quad \forall z \geq 0. \quad (\beta \geq 1)$$

$$(x^\beta - y^\beta)^2 \geq y^{\beta-1} \left(\frac{x^{\beta+1} - y^{\beta+1}}{\beta+1} - y^\beta (x - y) \right) \quad \forall x, y \geq 0$$

Exponential decay towards the steady-state, $\Gamma^D \neq \emptyset$

$$E(t) = \int_\Omega \frac{f^{\beta+1} - (f^\infty)^{\beta+1}}{\beta+1} - (f^\infty)^\beta (f - f^\infty)$$

$$D(t) = \int_\Omega |\nabla \Big(f^\beta - (f^\infty)^\beta \Big)|^2$$

- Relation between entropy and dissipation :

$$D(t) \geq \frac{(m^D)^{\beta-1}}{C_P} E(t).$$

- Exponential decay of the entropy :

$$E(t) \leq E(0) e^{-\lambda t}, \text{ with } \lambda = \frac{(m^D)^{\beta-1}}{C_P}.$$

What happens when $\Gamma^D = \emptyset$?

$$\begin{cases} \partial_t f = \Delta f^{\beta}, \text{ in } \Omega \times \mathbb{R}_+ \\ \nabla f \cdot \mathbf{n} = 0 \text{ on } \Gamma \times \mathbb{R}_+ \\ f(\cdot, 0) = f_0 \geq 0, \text{with } \displaystyle\int_\Omega f_0 > 0. \end{cases}$$

with $\Omega = [0,1]^d$, such that $m(\Omega) = 1$.

Steady-state

- the associate steady-state is a constant function,
- the constant is fixed by the conservation of mass :

$$f^\infty = \frac{1}{m(\Omega)} \int_\Omega f_0 = \int_\Omega f_0$$

Towards the long time behavior

References

- Alikakos, Rostamian, '1981
- Bonforte, Grillo, '05
- Grillo, Muratori, '13, '14
- C.-H., Jüngel, Schuchnigg,

Families of entropies

- Zeroth order entropies :

$$E_\alpha(f) = \frac{1}{\alpha+1}\left(\int_\Omega f^{\alpha+1}dx - \left(\int_\Omega f\,dx\right)^{\alpha+1}\right)$$

- First order entropies :

$$F_\alpha[f] = \frac{1}{2}\int_\Omega |\nabla f^{\alpha/2}|^2 dx$$

Dissipation of the order 0 entropies

$$\begin{cases} \partial_t f - \Delta(f^\beta) = 0, \text{ in } [0,1]^d \times \mathbb{R}_+, \\ \nabla f \cdot n = 0 \text{ on } \Gamma \times \mathbb{R}_+, \quad f(0) = f_0 \\ f_\infty = \int_\Omega f_0. \end{cases}$$

- $E_\alpha(f) = \frac{1}{\alpha+1}\left(\int_\Omega f^{\alpha+1}dx - \left(\int_\Omega f dx\right)^{\alpha+1}\right)$

$$\frac{d}{dt}E_\alpha(f) = \int_\Omega f^\alpha \partial_t f = \int_\Omega f^\alpha \Delta(f^\beta)$$
$$= -\int_\Omega \nabla f^\alpha \cdot \nabla f^\beta$$
$$= -\frac{4\alpha\beta}{(\alpha+\beta)^2}\int_\Omega |\nabla f^{(\alpha+\beta)/2}|^2.$$

- $D_\alpha(f) = \frac{4\alpha\beta}{(\alpha+\beta)^2}\int_\Omega |\nabla f^{(\alpha+\beta)/2}|^2.$

Use of Poincaré-Wirtinger inequality ?

Poincaré-Wirtinger inequality $u \in H^1(\Omega)$ $(\mathrm{m}(\Omega) = 1)$

$$\int_\Omega \left(u - \frac{1}{\mathrm{m}(\Omega)} \int_\Omega u \right)^2 \leq C_P^2 \|\nabla u\|^2_{L^2(\Omega)}$$

$$\|u\|^2_{L^2(\Omega)} - \|u\|^2_{L^1(\Omega)} \leq C_P^2 \|\nabla u\|^2_{L^2(\Omega)}$$

Application $u = f^{(\alpha+\beta)/2}$

$$\begin{aligned} D_\alpha(f) &= \frac{4\alpha\beta}{(\alpha+\beta)^2} \int_\Omega |\nabla f^{(\alpha+\beta)/2}|^2 \\ &\geq \frac{4\alpha\beta}{C_P^2(\alpha+\beta)^2} \left(\int_\Omega |f^{(\alpha+\beta)/2}|^2 - \left(\int_\Omega |f^{(\alpha+\beta)/2}| \right)^2 \right) \\ &\geq ???K\left(E_\alpha(f)\right)^\kappa \end{aligned}$$

Beckner inequalities

Original Beckner inequality

- Beckner 1989

$$\int_\Omega |u|^2 - \left(\int_\Omega |u|^{2/r}\right)^r \le C_B^0(r)\|\nabla u\|^2_{L^2(\Omega)}, \quad 1 \le r \le 2$$

Generalizations

$$\int_\Omega |u|^q - \left(\int_\Omega |u|^{1/p}\right)^{pq} \le C_B(p,q)\|\nabla u\|^q_{L^2(\Omega)},$$

$$0 \le q < 2,\ pq \ge 1.$$

$$\|u\|^{2-q}_{L^q(\Omega)}\left(\int_\Omega |u|^q - \left(\int_\Omega |u|^{1/p}\right)^{pq}\right) \le C'_B(p,q)\|\nabla u\|^2_{L^2(\Omega)},$$

$$0 \le q < 2, pq \ge 1$$

Entropy/dissipation relation

$$\int_\Omega |u|^q - \left(\int_\Omega |u|^{1/p}\right)^{pq} \leq C_B(p,q)\|\nabla u\|^q_{L^2(\Omega)}, \quad 0 \leq q < 2,\ pq \geq 1$$

Application : $u = f^{\frac{\alpha+\beta}{2}},\ p = \dfrac{\alpha+\beta}{2},\ q = \dfrac{2(\alpha+1)}{\alpha+\beta}$

$$\int_\Omega f^{\alpha+1} - \left(\int_\Omega f\right)^{\alpha+1} \leq C_B(p,q)\left\|\nabla f^{(\alpha+\beta)/2}\right\|^{\frac{2(\alpha+1)}{\alpha+\beta}}_{L^2(\Omega)}$$

for $\alpha > 0,\ \beta > 1$.

Consequence

$$\begin{aligned} D_\alpha(f) &= \frac{4\alpha\beta}{(\alpha+\beta)^2}\int_\Omega |\nabla f^{(\alpha+\beta)/2}|^2 \\ &\geq \frac{4\alpha\beta}{(\alpha+\beta)^2}\left(\frac{\alpha+1}{C_B(p,q)}\right)^{\frac{\alpha+\beta}{\alpha+1}}\left(E_\alpha(f)\right)^{\frac{\alpha+\beta}{\alpha+1}}. \end{aligned}$$

Polynomial decay of the entropies

Theorem [CCH-AJ-SS]

- $\alpha > 0$, $\beta > 1$
- $f_0 \in L^\infty(\Omega)$, $\inf_\Omega f_0 \geq 0$
- f positive regular solution

Then

$$E_\alpha(f(t)) \leq \frac{1}{\left(C_1 t + C_2\right)^{\frac{\alpha+1}{\beta-1}}}.$$

avec

$$C_1 = \frac{4\alpha\beta(\beta-1)}{(\alpha+1)(\alpha+\beta)^2}\left(\frac{\alpha+1}{C_B(p,q)}\right)^{(\alpha+\beta)/(\alpha+1)},$$

$$C_2 = E_\alpha(f_0)^{-(\beta-1)/(\alpha+1)}$$

Towards the exponential decay

$$\|u\|^{2-q}_{L^q(\Omega)}\left(\int_\Omega |u|^q - \left(\int_\Omega |u|^{1/p}\right)^{pq}\right) \le C'_B(p,q)\|\nabla u\|^2_{L^2(\Omega)},$$

$$0 \le q < 2, pq \ge 1$$

Application : $u = f^{\frac{\alpha+\beta}{2}}$, $p = \dfrac{\alpha+\beta}{2}$, $q = \dfrac{2(\alpha+1)}{\alpha+\beta}$

$$\|f\|^{\beta-1}_{L^{\alpha+1}}\left(\int_\Omega f^{\alpha+1} - \left(\int_\Omega f\right)^{\alpha+1}\right) \le C'_B(p,q)\left\|\nabla f^{(\alpha+\beta)/2}\right\|^2_{L^2(\Omega)}$$

Consequence $(\|f\|_{L^{\alpha+1}(\Omega)} \ge \|f\|_{L^1(\Omega)} = \|f_0\|_{L^1(\Omega)})$

$$\begin{aligned} D_\alpha(f) &= \frac{4\alpha\beta}{(\alpha+\beta)^2}\int_\Omega |\nabla f^{(\alpha+\beta)/2}|^2 \\ &\ge \frac{4\alpha\beta(\alpha+1)}{(\alpha+\beta)^2}\frac{\|f_0\|^{\beta-1}_{L^1(\Omega)}}{C'_B(p,q)}E_\alpha(f) \end{aligned}$$

Exponential decay of the entropies

Theorem [CCH-AJ-SS]

- $0 < \alpha \leq 1$, $\beta > 1$
- $f_0 \in L^\infty(\Omega)$, $\inf_\Omega f_0 \geq 0$
- f regular positive solution

Then

$$E_\alpha(f(t)) \leq E_\alpha(f_0) e^{-\lambda t}.$$

with

$$\lambda = \frac{4\alpha\beta(\alpha+1)}{C'_B(p,q)(\alpha+\beta)^2} \|f_0\|_{L^1(\Omega)}^{\beta-1}$$