Finite volume methods for dissipative problems

Claire Chainais-Hillairet

CEMRACS summer school
Marseille, July 2019, 15-20

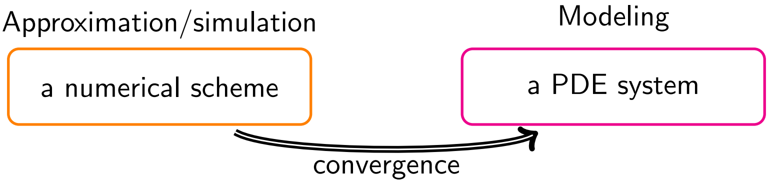
Laboratoire Paul Painlevé

Université de Lille 1 Sciences et Technologies

cnrs

Inria INVENTEURS DU MONDE NUMÉRIQUE

General approach



General approach

Approximation/simulation

a numerical scheme

Modeling

a PDE system

convergence

Analysis of PDEs

existence of a solution ? uniqueness ?
structural properties ?

positivity, dissipation of entropy, bounds, asymptotic behaviors,...

General approach

Approximation/simulation

a numerical scheme

Modeling

a PDE system

convergence

Numerical analysis

Analysis of PDEs

existence of a solution ? uniqueness ?
structural properties ?

positivity, dissipation of entropy, bounds, asymptotic behaviors,...

Outline of the course

1. Introduction to the finite volume method
2. Dissipative problems and long time behavior
3. Finite volume schemes and long time behavior

Lecture 1 :

Introduction to

the finite volume method

Outline of the chapter

1. Presentation of the finite volume method
   - Basic principles
   - Some examples

2. Analysis of the finite volume scheme for the Poisson equation
   - Some reminders
   - Presentation of the scheme and first properties
   - Convergence of the scheme

Outline of the chapter

For which type of equations ?

❏ Eymard, Gallouët, Herbin, 2000

➟ Discretization in space of conservation laws

$$\mathrm{div}\mathbf{J} = g.$$

➟ For evolutive equations

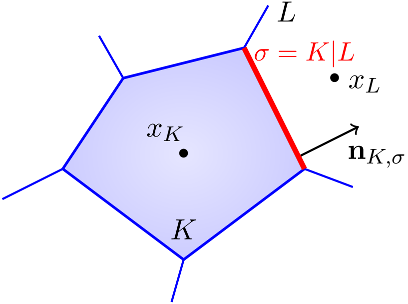
$$\partial_t u + \mathrm{div}\mathbf{J} = g,$$

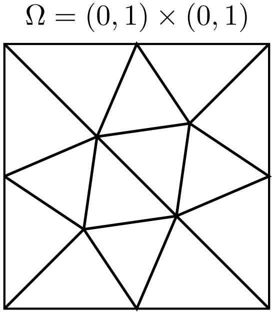
finite difference method for the time discretization.

Fluxes

- Diffusion fluxes : $\mathbf{J} = \nabla u$, $\mathbf{J} = \nabla r(u)$, $\mathbf{J} = \mathbb{K}\nabla u$,...
- Convection fluxes : $\mathbf{J} = \mathbf{v}u$, $\mathbf{J} = \mathbf{v}f(u)$, $\mathbf{J} = \mathbf{F}(u)$,...
- Every combination...

Mesh : definitions and notations





- $\mathcal{T}$: set of control volumes K (open, convex, polygonal)

$$\bar{\Omega} = \bigcup_{K \in \mathcal{T}} \bar{K}, \quad h = \max_{K \in \mathcal{T}} \operatorname{diam}(K)$$

- $\mathcal{E}$: set of edges σ
- $\mathcal{P}$: set of points $x_K \in K$

Space of approximate solution

➟ Reconstruction of piecewise constant approximate solutions

Discrete unknowns

$$(u_K)_{K\in\mathcal{T}}$$

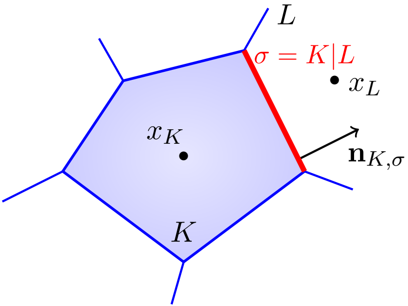
Approximate solution

$$u_{\mathcal{T}} = \sum_{K\in\mathcal{T}} u_K \mathbf{1}_K$$

Space of approximate solutions

$$X(\mathcal{T}) = \left\{ u_{\mathcal{T}} = \sum_{K\in\mathcal{T}} u_K \mathbf{1}_K \right\}$$

How to get a finite volume scheme ?



Integration over each cell K of

$$\mathrm{div}\mathbf{J} = g$$

$$\Downarrow$$

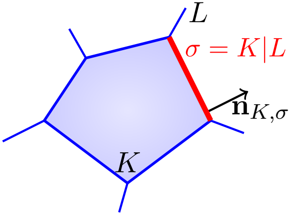
$$\sum_{\sigma \in \mathcal{E}_K} \int_\sigma \mathbf{J} \cdot \mathbf{n}_{K,\sigma} = \int_K g$$

Finite volume scheme

$$\sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{K,\sigma} = \mathrm{m}(K) g_K \text{ for all } K \in \mathcal{T},$$

$$\text{with } \begin{cases} \mathcal{F}_{K,\sigma} \text{ “good” approximation of } \displaystyle\int_\sigma \mathbf{J} \cdot \mathbf{n}_{K,\sigma}, \\ g_K = \dfrac{1}{\mathrm{m}(K)} \displaystyle\int_K g \qquad \text{(or an approximation).} \end{cases}$$

Crucial properties of the numerical fluxes



Conservativity of the numerical fluxes

$$\mathcal{F}_{K,\sigma} + \mathcal{F}_{L,\sigma} = 0 \quad \forall \sigma = K|L.$$

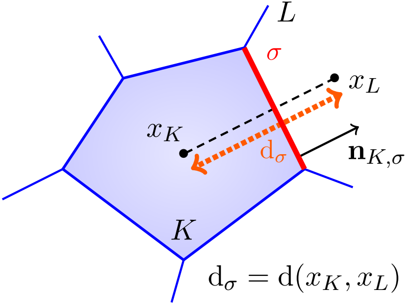
Consistency of the numerical fluxes

- $\widetilde{\mathcal{F}}_{K,\sigma}$: evaluation of the numerical flux for an exact and smooth solution of the problem
- $\widetilde{\mathbf{J}}$: exact flux

$$\frac{1}{\mathrm{m}(\sigma)} \left(\widetilde{\mathcal{F}}_{K,\sigma} - \int_{\sigma} \widetilde{\mathbf{J}} \cdot \mathbf{n}_{K,\sigma} \right) = O(h).$$

Outline of the chapter

Diffusion flux : $\mathbf{J} = -\nabla u$



$\mathrm{d}_\sigma = \mathrm{d}(x_K, x_L)$

$$\mathbf{J} = -\nabla u$$

$$\Downarrow$$

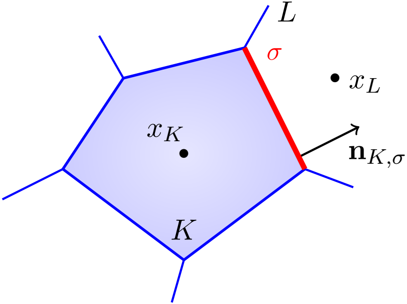
$$\int_\sigma \mathbf{J} \cdot \mathbf{n}_{K,\sigma} = -\int_\sigma \nabla u \cdot \mathbf{n}_{K,\sigma}$$

$$\mathcal{F}_{K,\sigma} = -\mathrm{m}(\sigma)\frac{u_L - u_K}{\mathrm{d}(x_K, x_L)} = -\frac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}(u_L - u_K).$$

- Conservativity : OK
- Consistency : OK... if $(x_K x_L) \perp \sigma$

➟ Admissible mesh : $\forall \sigma = K|L,\ (x_K x_L) \perp \sigma$

Convection flux : $\mathbf{J} = \mathbf{v}u$



$$\mathbf{J} = \mathbf{v}u$$

$$\Downarrow$$

$$\int_\sigma \mathbf{J} \cdot \mathbf{n}_{K,\sigma} = \int_\sigma \mathbf{v} \cdot \mathbf{n}_{K,\sigma} u$$

$$v_{K,\sigma} = \frac{1}{\mathrm{m}(\sigma)} \int_\sigma \mathbf{v} \cdot \mathbf{n}_{K,\sigma}$$

Centered fluxes

$$\mathcal{F}_{K,\sigma} = \mathrm{m}(\sigma) v_{K,\sigma} \frac{u_K + u_L}{2}$$

Upwind fluxes

$$\mathcal{F}_{K,\sigma} = \mathrm{m}(\sigma) v_{K,\sigma} \begin{cases} u_K \text{ if } v_{K,\sigma} \geq 0 \\ u_L \text{ if } v_{K,\sigma} \leq 0 \end{cases}$$

Convection flux : $\mathbf{J} = \mathbf{v}u$

Centered fluxes

$$\mathcal{F}_{K,\sigma} = \mathrm{m}(\sigma) v_{K,\sigma} \frac{u_K + u_L}{2}$$

Upwind fluxes

$$\mathcal{F}_{K,\sigma} = \mathrm{m}(\sigma) v_{K,\sigma} \begin{cases} u_K \text{ if } v_{K,\sigma} \geq 0 \\ u_L \text{ if } v_{K,\sigma} \leq 0 \end{cases}$$

- Conservativity OK $(v_{K,\sigma} + v_{L,\sigma} = 0)$
- Consistency OK

  ➠ Without any assumption on the mesh.

  ➠ But : well-known instability problems with the centered scheme for pure convection equations.

  ➠ A monotony hypothesis is needed in this case.

Outline of the chapter

Theoretical reminder

Problem under study

Ω regular bounded open subset of $\mathbb{R}^d$, $f \in L^2(\Omega)$

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega. \end{cases}$$

Theorem

There exists a unique $u \in H_0^1(\Omega)$ such that

$$\forall v \in H_0^1(\Omega) \quad \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v.$$

Proof : application of Lax-Milgram lemma.

Theoretical reminder

Key points

- Poincaré inequality : $\exists C_P(\Omega)$ such that $\forall u \in H_0^1(\Omega)$

$$\int_\Omega |u|^2 \leq C_P(\Omega) \int_\Omega |\nabla u|^2.$$

- $|u|_{H_0^1(\Omega)} = \left(\int_\Omega |\nabla u|^2 \right)^{1/2}$ is a norm on $H_0^1(\Omega)$.

Qualitative properties

- Monotony : $v = u^-$ in the weak formulation

$$f \geq 0 \text{ a.e.} \Longrightarrow u \geq 0 \text{ a.e.}$$

- Energy estimate : $v = u$ in the weak formulation

$$\int_\Omega |\nabla u|^2 \leq C_P(\Omega) \int_\Omega f^2$$

Finite element scheme / finite volume scheme

Conformal finite element scheme

- Approximation of $H_0^1(\Omega)$ by finite-dimension subspaces :

$$(V_n)_{n\geq 0} \text{ such that } V_n \subset H_0^1(\Omega)\ \forall n$$

  The scheme :

$$\forall v_n \in V_n \quad \int_\Omega \nabla u_n \cdot \nabla v_n = \int_\Omega f v_n.$$

- Poincaré inequality can be applied.
- Compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$.

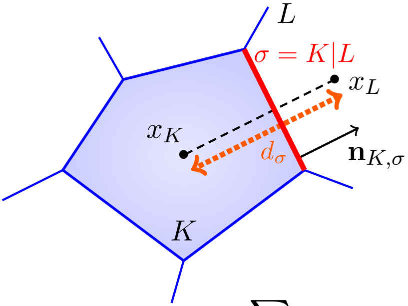
Finite volume scheme

$$X(\mathcal{T}) \not\subset H^1(\Omega)$$

- Discrete counterpart of Poincaré inequality ?
- Compactness arguments must be adapted.

Outline of the chapter

The scheme



$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma = \partial\Omega. \end{cases}$$

$$\mathcal{F}_{K,\sigma} \approx -\int_\sigma \nabla u \cdot \mathbf{n}_{K,\sigma}$$

$$\sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{K,\sigma} = \mathrm{m}(K) f_K \quad \forall K \in \mathcal{T},$$

$$\mathcal{F}_{K,\sigma} = \begin{cases} -\dfrac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}(u_L - u_K) & \text{if } \sigma \in \mathcal{E}_{int},\ \sigma = K|L, \\ -\dfrac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}(0 - u_K) & \text{if } \sigma \in \mathcal{E}_{ext}. \end{cases}$$

$$(\mathrm{d}_\sigma = \mathrm{d}(x_K, \sigma) \text{ if } \sigma \in \mathcal{E}_{ext})$$

Matricial form of the scheme

$$\begin{cases} \sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{K,\sigma} = \mathrm{m}(K) f_K \quad \forall K \in \mathcal{T} \\ \mathcal{F}_{K,\sigma} = -\tau_\sigma (u_{K,\sigma} - u_K) \quad \forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_K. \end{cases}$$

with $\tau_\sigma = \dfrac{\mathrm{m}(\sigma)}{\mathrm{d}_\sigma}$ and $u_{K,\sigma} = \begin{cases} u_L & \text{if } \sigma \in \mathcal{E}_{K,int}, \sigma = K|L, \\ 0 & \text{if } \sigma \in \mathcal{E}_{K,ext}. \end{cases}$

Linear system of equations $\mathbb{A}U = B$

- Unknown : $U = (U_K)_{K \in \mathcal{T}}$,
- Right hand side : $B = (B_K)_{K \in \mathcal{T}}$ with $B_K = \mathrm{m}(K) f_K$,
- Matrix : $\mathbb{A} = (\mathbb{A}_{K,L})_{K \in \mathcal{T}, L \in \mathcal{T}}$

$$\mathbb{A}_{K,K} = \sum_{\sigma \in \mathcal{E}_K} \tau_\sigma$$

$$\mathbb{A}_{K,L} = -\tau_\sigma \text{ if } \exists \sigma = K|L, \ 0 \text{ else}$$

Properties of the matrix $\mathbb{A}$

Proposition 1 : $\mathbb{A}$ is a positive definite symmetric matrix

➟ Calculation of $U^T A U$ for $U \in \mathbb{R}^{|\mathcal{T}|}$

$$
\begin{aligned}
U^T \mathbb{A} U &= \sum_{K\in\mathcal{T}} u_K (\mathbb{A}U)_K \\
&= \sum_{K\in\mathcal{T}} u_K \sum_{\sigma\in\mathcal{E}_K} \mathcal{F}_{K,\sigma} \\
&= \sum_{K\in\mathcal{T}} u_K \sum_{\sigma\in\mathcal{E}_K} (-\tau_\sigma(u_{K,\sigma} - u_K)) \\
&= \sum_{\substack{\sigma\in\mathcal{E}_{int},\\ \sigma=K|L}} \tau_\sigma (u_L - u_K)^2 + \sum_{K\in\mathcal{T}} \sum_{\sigma\in\mathcal{E}_{K,ext}} \tau_\sigma u_K^2
\end{aligned}
$$

➟ $U^T \mathbb{A} U \geq 0 \ \forall U \in \mathbb{R}^{|\mathcal{T}|}$ and $U^T \mathbb{A} U = 0 \Longrightarrow U = 0$.

→ Existence and uniqueness of a solution.

Properties of the matrix $\mathbb{A}$

Proposition 2 : $\mathbb{A}$ is monotone : $B \geq 0 \Longrightarrow U \geq 0$

➠ Calculation of $(U^-)^T \mathbb{A} U$ for $U \in \mathbb{R}^{|\mathcal{T}|}$ $(s^- = \min(s,0))$

$$\begin{aligned}
(U^-)^T A U &= \sum_{K \in \mathcal{T}} u_K^- (\mathbb{A}U)_K \\
&= \sum_{K \in \mathcal{T}} u_K^- \sum_{\sigma \in \mathcal{E}_K} \mathcal{F}_{K,\sigma} \\
&= \sum_{K \in \mathcal{T}} u_K^- \sum_{\sigma \in \mathcal{E}_K} (-\tau_\sigma (u_{K,\sigma} - u_K)) \\
&= \sum_{\substack{\sigma \in \mathcal{E}_{int}, \\ \sigma = K|L}} \tau_\sigma (u_L^- - u_K^-)(u_L - u_K) \\
&\qquad + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{K,ext}} \tau_\sigma (u_K^-)^2
\end{aligned}$$

➠ $B \geq 0 \Longrightarrow (U^-)^T B = (U^-)^T \mathbb{A} U \leq 0 \Longrightarrow U^- = 0$ and $U \geq 0$.

Discrete energy estimate ?

$$\mathbb{A}U = B \Longrightarrow U^T \mathbb{A} U = U^T B$$

$$\sum_{\substack{\sigma \in \mathcal{E}_{int}, \\ \sigma = K|L}} \tau_\sigma (u_L - u_K)^2 + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{K,ext}} \tau_\sigma u_K^2 = \sum_{K \in \mathcal{T}} \mathrm{m}(K) f_K u_K$$

$$\leq \left(\sum_{K \in \mathcal{T}} \mathrm{m}(K) f_K^2 \right)^{1/2} \left(\sum_{K \in \mathcal{T}} \mathrm{m}(K) u_K^2 \right)^{1/2}$$

Discrete Poincaré inequality ? (admitted yet)

$$\sum_{K \in \mathcal{T}} \mathrm{m}(K) u_K^2 \leq C_P^d \left(\sum_{\substack{\sigma \in \mathcal{E}_{int}, \\ \sigma = K|L}} \tau_\sigma (u_L - u_K)^2 + \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{K,ext}} \tau_\sigma u_K^2 \right)$$

Discrete energy estimate ?

$(u_K)_{K\in\mathcal{T}}$ solution to the scheme satisfies :

$$\left(\sum_{\substack{\sigma\in\mathcal{E}_{int},\\ \sigma=K|L}} \tau_\sigma (u_L-u_K)^2 + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,ext}} \tau_\sigma u_K^2\right)^{1/2} \le C_P^d \|f\|_{L^2(\Omega)}.$$

Norms on $X(\mathcal{T})$ $\quad u_{\mathcal{T}} = \sum_{K\in\mathcal{T}} u_K \mathbf{1}_K$

$$\|u_{\mathcal{T}}\|^2_{L^2(\Omega)} = \sum_{K\in\mathcal{T}} \mathrm{m}(K) u_K^2$$

$$|u_{\mathcal{T}}|^2_{1,2,\Gamma,\mathcal{T}} = \sum_{\substack{\sigma\in\mathcal{E}_{int},\\ \sigma=K|L}} \tau_\sigma (u_L-u_K)^2 + \sum_{K\in\mathcal{T}}\sum_{\sigma\in\mathcal{E}_{K,ext}} \tau_\sigma u_K^2$$

→ Discrete H^1 estimate for the approximate solution.

Outline of the chapter

Construction of a sequence of approximate solutions

Sequence of meshes $(\mathcal{T}_n, \mathcal{E}_n, \mathcal{P}_n)_{n\geq 0}$

- Admissible meshes,
- $\lim_{n\to\infty} h_n = 0$.

Sequence of approximate solutions $(u_n)_{n\geq 0}$

$$u_n = u_{\mathcal{T}_n} \in X(\mathcal{T}_n).$$

Discrete H^1 estimate

There exists C, not depending on n, such that

$$|u_n|_{1,2,\Gamma,\mathcal{T}_n} \leq C \quad \forall n \geq 0.$$

Main result

Theorem

Let

- $(\mathcal{T}_n, \mathcal{E}_n, \mathcal{P}_n)_{n \geq 0}$ a sequence of admissible meshes, with $\lim h_n = 0$,
- $(u_n)_{n \geq 0}$ a sequence of approximate solutions given by the finite volume scheme.

Then,

$$\lim_{n \to \infty} u_n = u \text{ in } L^2(\Omega)$$

with

- $u \in H_0^1(\Omega)$,
- and u is the unique solution to the Poisson equation :

$$\forall v \in H_0^1(\Omega) \quad \int_\Omega \nabla u \cdot \nabla v = \int_\Omega f v.$$

Proof of the theorem : the main steps

- Compactness :
  - Consequence of the discrete H^1 estimate,
  - Application of Riesz-Fréchet-Kolmogorov theorem.
- Strong convergence in $L^2(\Omega)$ of a subsequence.
- Let u be the limit : $u \in H_0^1(\Omega)$.
- Passage to the limit in the numerical scheme to show that u is a weak solution.
- Uniqueness of the solution $\Longrightarrow$ convergence of the whole sequence.

About the compactness step

Compactness theorem by Kolmogorov

Let Ω a bounded domain of $\mathbb{R}^N$, $N \geq 1$.

Let $A \subset L^q(\Omega)$, $1 \leq q < +\infty$.

> A is relatively compact in $L^q(\Omega)$ if and only if there exists $\{p(u), u \in A\} \subset L^q(\mathbb{R}^N)$ such that :

1. $p(u) = u$ a.e. in Ω, for all $u \in A$,
2. $\{p(u), u \in A\}$ is bounded in $L^q(\mathbb{R}^N)$,
3. $\|p(u)(\cdot + \eta) - p(u)(\cdot)\|_{L^q(\mathbb{R}^N)} \longrightarrow 0$ when η tends to 0, uniformly with respect to $u \in A$.

Application to the scheme

$$A = \{u_{\mathcal{T}};\ \text{solution to the scheme on } (\mathcal{T}, \mathcal{E}, \mathcal{P})\} \subset L^2(\Omega)$$

$$p(u_{\mathcal{T}}) = \begin{cases} u_{\mathcal{T}} & \text{on } \Omega \\ 0 & \text{on } \mathbb{R}^N \setminus \Omega \end{cases}$$

1. satisfied by definition
2. $\{p(u_{\mathcal{T}}), u_{\mathcal{T}} \in A\}$ is bounded in $L^2(\mathbb{R}^N)$:

$$\|p(u_{\mathcal{T}})\|_{L^2(\mathbb{R}^N)} = \|u_{\mathcal{T}}\|_{L^2(\Omega)}$$

3. It can be proved that

$$\|p(u_{\mathcal{T}})(\cdot + \eta) - p(u_{\mathcal{T}})(\cdot)\|^2_{L^2(\mathbb{R}^N)} \leq |\eta|(|\eta| + 2h)|u_{\mathcal{T}}|^2_{1,2,\Gamma,\mathcal{T}}.$$

- Eymard, Gallouët, Herbin, 2000
- Gallouët, Latché, 2013