

Long-time Behaviour of a Model of Rigid Structure Floating in a Viscous Fluid

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Outline

1. A Model of rigid structure in a viscous fluid
2. Mittag-Leffler functions
3. Solutions unbounded case
4. Long-time behavior viscous case

The equilibrium problem

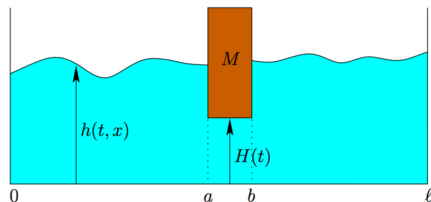


Figure 1: Configuration, Image from Tucsnak et al.

Notations

- $\mu \geq 0$ is the coefficient of viscosity of the fluid.
- $\mathcal{E} = (0, a) \cup (b, a + b)$.
- $\mathcal{I} = [a, b]$.

Equations Return to the Equilibrium

Proposition [Tucsna et al.,2018]

If $l = a + b$, then, the return to the equilibrium problem, the position of the solid is completely determined by the integro-differential equation

$$\begin{cases} \left(1 + \frac{(b-a)^3}{12}\right) \ddot{H} = -H_0 \frac{|\mathcal{I}|^2}{|\mathcal{E}|^2} - \frac{(b-a)^2}{2} F * \dot{H} - \mu(b-a)\dot{H} - (b-a)H, \\ H(0) = H_0, \quad \dot{H}(0) = 0, \end{cases} \quad (1)$$

and with F such that $\hat{F}(s) = \sqrt{1+s\mu} \tanh\left(\frac{sa}{\sqrt{1+s\mu}}\right)$.

Unbounded case

Proposition [Tucsnak et al.,2018]

If $\mathcal{E} = (-\infty, a) \cup (b, \infty)$, then the position of the solid is completely determined by the integro-differential equation

$$\begin{cases} A\ddot{H} + BF * \dot{H} + C\dot{H} + DH = 0, \\ H(0) = H_0, \quad \dot{H}(0) = \dot{H}_0, \end{cases} \quad (2)$$

where $A = 1 + \frac{(b-a)^3}{12}$, $B = (b-a)^2$, $C = (b-a)\mu$, $D = (b-a)$, and

$$F(t) = \sqrt{\mu} \left(\frac{1 - e^{-\frac{t}{\mu}}}{2\sqrt{\pi t^3}} \right) + D^{1/2} \delta_0.$$

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Goal: Understand the long-time behavior of solutions $H(t)$ for the unbounded case

Mittag-Leffler functions

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Definition

The two-parametric Mittag-Leffler function, is the complex-valued function defined by

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \text{with } \alpha > 0, \beta \in \mathbb{C}. \quad (3)$$

In the case when $\beta = 1$ the function is known has the classical Mittag-Leffler function and denoted by $E_{\alpha}(z)$.

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Examples

- $E_{1,1}(z) = e^z$,
- $E_{2,1}(z) = \cosh(\sqrt{z})$,
- $E_{\frac{1}{2},1}(z) = e^{z^2} \operatorname{erfc}(-z)$.

Mittag-Leffler functions

Theorem [Popov-Sedletskii, 2013]

For any $0 < \alpha < 1$, $\beta \in \mathbb{C}$, $p \in \mathbb{N}$, and $z \in \mathcal{L}_{1/\alpha} = \{z \in \mathbb{C} : |\arg z| \leq \frac{\pi}{\alpha}\}$, the following asymptotics hold

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} e^{z^{1/\alpha}} - \sum_{k=1}^p \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + R_m^{[1]}(z; \alpha, \beta), \quad (4)$$

where the remainder $R_m^{[1]}$ admits the estimate

$$\left| R_m^{[1]}(z; \alpha, \beta) \right| \leq \begin{cases} \frac{2^{\frac{b+2}{2}} \Gamma(b+1) e^{\left(\frac{5\pi}{4} |\Im\beta|\right)}}{\alpha \pi |z|^{p+1}}, & \text{if } b = \alpha(p+1) - \Re\beta \geq 0 \\ \frac{\left(6 + \frac{2}{\alpha\pi}\right) e^{\left(\frac{5\pi}{4} |\Im\beta|\right)}}{|z|^{p+1}}, & \text{if } b < 0. \end{cases} \quad (5)$$

The first of estimates in (10) is valid for all $z \in \mathcal{L}_{1/\alpha}$, and the second under the additional condition $|z| \geq 2$.

Mittag-Leffler functions

Theorem [Popov-Sedletskii, 2013]

For any $0 < \alpha < 1$, $\beta \in \mathbb{C}$, $p \in \mathbb{N}$ and $z \in \mathbb{C} \setminus \mathcal{L}_{1/\alpha}$, $z \neq 0$, the following asymptotics formulas hold

$$E_{\alpha,\beta}(z) = - \sum_{k=1}^p \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + R_m^{[2]}(z; \alpha, \beta), \quad (6)$$

where the remainder $R_m^{[2]}$ admits the estimate

$$\left| R_m^{[2]}(z; \alpha, \beta) \right| \leq \begin{cases} \frac{2^{\frac{b+2}{2}} \Gamma(b+1) e^{\left(\frac{3\pi}{4} |\Im\beta|\right)}}{\alpha\pi |z|^{p+1}}, & \text{if } b \geq 0 \\ \frac{\left(6 + \frac{2}{\alpha\pi}\right) e^{\left(\frac{3\pi}{4} |\Im\beta|\right)}}{|z|^{p+1}}, & \text{if } b < 0. \end{cases} \quad (7)$$

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Laplace transform of Mittag Leffler functions

In this work we are interested in the case when $\alpha = \beta > 0$ and $z = \lambda t^\alpha$, with $\lambda \in \mathbb{C}$ and $t \in \mathbb{R}$. For simplicity we introduce the notation

$$\mathcal{E}_\alpha(\lambda, t) := E_{\alpha, \alpha}(\lambda t^\alpha).$$

Lemma

The Laplace transform of $\mathcal{E}_\alpha(\lambda, t)$ is given by

$$\mathcal{L}[\mathcal{E}_\alpha(\lambda, t)](s) = (s^\alpha - \lambda)^{-1},$$

where $\Re s > 0$ and $|\lambda s^{-\alpha}| < 1$.

Solutions unbounded case

Inviscid case

If we consider $\mu = 0$ in equation (2), the model is reduced to

$$\begin{cases} A\ddot{H} + B\dot{H} + DH = 0, \\ H(0) = H_0, \quad \dot{H}(0) = \dot{H}_0. \end{cases}$$

Applying Laplace transform to the equation above, and after simplifications, we obtain

$$\hat{H}(s) [As^2 + Bs + D] = H_0 [As + B] + A\dot{H}_0.$$

Solutions of the model, inviscid case

Proposition [V-H, Sueur, Tucsna, 2019]

The solution $H(t)$ of equation (2) with $\mu = 0$ is given by

$$e^{-\frac{Bt}{2A}} \left\{ H_0 \left(\cosh \left(\frac{t \sqrt{\frac{B^2}{4} - AD}}{A} \right) + \frac{B \sinh \left(\frac{t \sqrt{\frac{B^2}{4} - AD}}{A} \right)}{2 \sqrt{\frac{B^2}{4} - AD}} \right) + \dot{H}_0 \frac{2A \sin \left(\frac{t \sqrt{4AD - B^2}}{2A} \right)}{\sqrt{4AD - B^2}} \right\}.$$

Proposition [V-H, Sueur, Tucsna, 2019]

The solution of the equation (2) with $\mu > 0$ is given by

$$H(t) = e^{-\frac{t}{\mu}} \sum_{i=1}^4 \left[H_0 r_i + \dot{H}_0 \dot{r}_i \right] \mathcal{E}_{\frac{1}{2}}(\lambda_i, t),$$

where each λ_i is a root of the polynomial

$$p(\lambda) = A\lambda^4 + \bar{B}\lambda^3 + \left(C - \frac{2A}{\mu} \right) \lambda^2 - \frac{\bar{B}}{\mu} \lambda + \frac{A}{\mu^2}, \quad (8)$$

and the parameters r_i and \dot{r}_i depends of the roots.

Long-time Behavior Viscous Case

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In this section we denote by $\Lambda = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ the set of roots of the polynomial $P(\lambda)$ in eq. (8).

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Remark

Since the coefficients of the polynomial $p(\lambda)$ in eq. (8) admit different signs, we conclude that

- $\Lambda \cap \mathcal{L}_{1/2} \neq \emptyset$, and
- $\Lambda \cap \mathbb{C} \setminus \mathcal{L}_{1/2} \neq \emptyset$.

Long-time Behavior Viscous Case

Proposition [V-H, Sueur, Tucsnak, 2019]

Let $\Theta_i = (H_0 r_i + \dot{H}_0 \dot{r}_i)$ for $i = 1, 2, 3, 4$, and let $\Lambda_1 = \mathcal{L}_{1/2} \cap \Lambda$. For all $p \in \mathbb{N}$, the following asymptotic formula hold:

$$H(t) = \sum_{\lambda_i \in \Lambda_1} \frac{\Theta_i}{2} \lambda_i \sqrt{t} e^{(\lambda_i \sqrt{t})^2 - \frac{t}{\mu}} - \sum_{i=1}^4 \Theta_i e^{-\frac{t}{\mu}} \left[\sum_{k=1}^p \frac{(\lambda_i \sqrt{t})^{-k}}{\Gamma\left(\frac{1}{2} - \frac{k}{2}\right)} \right] + \mathcal{R}_p, \quad (9)$$

where the remainder \mathcal{R}_p admits the estimate

$$|\mathcal{R}_p| \leq e^{-\frac{t}{\mu}} \cdot \frac{2^{\frac{p+8}{4}} \Gamma(1 + p/2)}{\pi |\sqrt{t}|^{p+1}} \cdot \sum_{\lambda_i \in \Lambda} \frac{1}{|\lambda_i|^{p+1}}. \quad (10)$$

Perspectives

- Implement numerical simulations of the results.
- Study the unbounded case with a control as a source term.
- Extend the results to the bounded case.

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