Multidimensional waves in dispersive media

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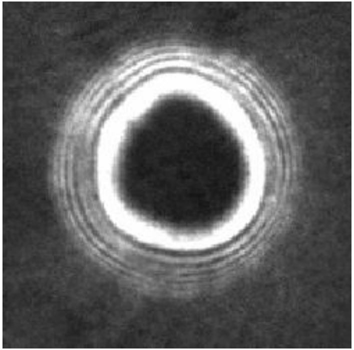
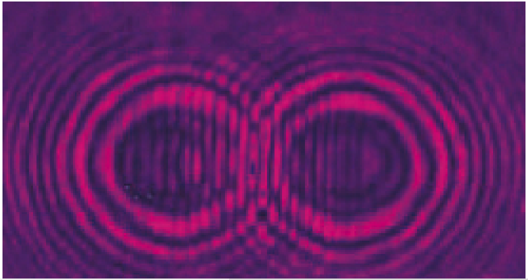
Dispersion





Classic dispersive models:

- Serre-Green-Naghdi equations
- Schrödinger equation
- Bubbly fluids model of Iordanskii-Wijngaarden-Kogarko type

Advantages:

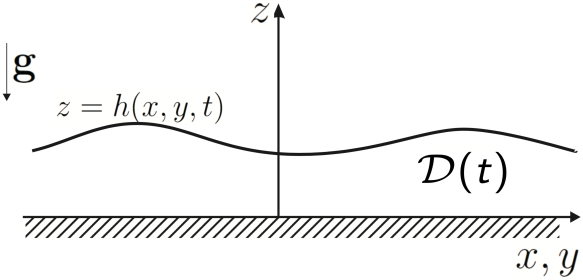
- Accurate description of dispersive phenomenas
- Good mathematical properties (Galilean invariance, entropy condition satisfied, admit a variational formulation, ...)

Disadvantages:

- Huge calculation time
- Boundary conditions are very sophisticated
- Impossible to impose discontinuous initial conditions

Outline

- Serre-Green-Naghdi (SGN) model as Euler-Lagrange equations, variational formulation
- Extended Lagrangian concept
- Hyperbolic version of SGN model
- Numerical methods
- Numerical results



Serre-Green-Naghdi model:

$$\frac{\partial h}{\partial t} + \mathrm{div}(h\mathbf{u}) = 0,$$

$$\frac{\partial h\mathbf{u}}{\partial t} + \mathrm{div}(h\mathbf{u} \otimes \mathbf{u} + pI) = 0, \qquad p = \frac{gh^2}{2} + \frac{1}{3}h\ddot{h}$$

Euler-Lagrange equations

SGN equations admit a variational formulation with the Lagrangian:

$$\mathcal{L} = \int\limits_{\mathcal{D}(t)} \left(\frac{h|\mathbf{u}|^2}{2} + \frac{h\dot{h}^2}{6} - \frac{gh^2}{2} \right) d\mathcal{D}.$$

Functional of action:

$$a = \int\limits_{t_0}^{t_1} \mathcal{L} dt.$$

Hamilton's principle:

$$\delta a = 0, \qquad a = \int\limits_{t_0}^{t_1} \mathcal{L} dt.$$

Complemented by a geometric constraint $h_t + \operatorname{div}(h\mathbf{u}) = 0$, the Euler-Lagrange equations for $\mathcal{L}$ read:

$$\begin{aligned} &\frac{\partial h}{\partial t} + \operatorname{div}(h\mathbf{u}) = 0, \\ &\frac{\partial h\mathbf{u}}{\partial t} + \operatorname{div}(h\mathbf{u} \otimes \mathbf{u} + pI) = 0, \qquad p = \frac{gh^2}{2} + \frac{1}{3}h\ddot{h}. \end{aligned}$$

Extended Lagrangian concept

Lagrangian of the SGN system:

$$\mathcal{L} = \int_{\mathcal{D}(t)} \left(\frac{h|\mathbf{u}|^2}{2} + \frac{h\dot{h}^2}{6} - \frac{gh^2}{2} \right) d\mathcal{D}.$$

The extended Lagrangian:

$$\hat{\mathcal{L}} = \int_{\mathcal{D}(t)} \left(\frac{h|\mathbf{u}|^2}{2} + \frac{h\dot{\eta}^2}{6} - \frac{gh^2}{2} - \frac{\lambda h}{6} \left(1 - \frac{\eta}{h}\right)^2 \right) d\mathcal{D}.$$

The penalty term:

$$-\frac{\lambda h}{6} \left(1 - \frac{\eta}{h}\right)^2.$$

Extended Serre-Green-Naghdi model:

$$
\begin{aligned}
&\frac{\partial h}{\partial t} + \mathrm{div}(h\mathbf{u}) = 0,\\
&\frac{\partial h\mathbf{u}}{\partial t} + \mathrm{div}(h\mathbf{u}\otimes\mathbf{u} + pI) = 0, \qquad p = \frac{gh^2}{2} - \frac{\lambda\eta}{3}\left(\frac{\eta}{h} - 1\right)\\
&\frac{\partial h\eta}{\partial t} + \mathrm{div}\left(h\eta\mathbf{u}\right) = hw,\\
&\frac{\partial hw}{\partial t} + \mathrm{div}\left(hw\mathbf{u}\right) = -\lambda\left(\frac{\eta}{h} - 1\right).
\end{aligned}
$$

The model admits the energy conservation law:

$$
\frac{\partial E}{\partial t} + \mathrm{div}\left(E\mathbf{u} + p\mathbf{u}\right) = 0,
$$

$$
E = \frac{h|\mathbf{u}|^2}{2} + \frac{h\dot{\eta}^2}{6} + \frac{gh^2}{2} + \frac{\lambda h}{6}\left(1 - \frac{\eta}{h}\right)^2.
$$

Hyperbolicity of the extended model

SGN system in the primitive form:

$$\mathbf{W}_t + A(\mathbf{W})\mathbf{W}_x = 0,$$

where

$$\mathbf{W} = \begin{pmatrix} h \\ u \\ v \\ \eta \\ w \end{pmatrix}, \qquad A = \begin{pmatrix} u & h & 0 & 0 & 0 \\ \frac{p_h}{h} & u & 0 & \frac{p_\eta}{h} & 0 \\ 0 & 0 & u & 0 & 0 \\ 0 & 0 & 0 & u & 0 \\ 0 & 0 & 0 & 0 & u \end{pmatrix}.$$

The model is unconditionally hyperbolic. The eigenvalues are:

$$\xi_{1,2,3} = u, \qquad \xi_{3,4} = u \pm \sqrt{p_h} = u \pm \sqrt{gh + \frac{\lambda}{3}\frac{\eta^2}{h^2}}$$

Genuine non-linearity of characteristic fields

Eigenvalues and corresponding right eigenvectors of A:

$$
\begin{aligned}
\xi_1 &= u, & \mathbf{v}_1 &= (0,0,1,0,0)^T, \\
\xi_2 &= u, & \mathbf{v}_3 &= \left(0, -\frac{p_h}{h}, 0, 1, 0\right)^T, \\
\xi_3 &= u, & \mathbf{v}_2 &= (0,0,0,0,1)^T, \\
\xi_4 &= u - \sqrt{p_h}, & \mathbf{v}_4 &= \left(-\frac{h}{\sqrt{p_h}}, 1, 0, 0, 0\right)^T, \\
\xi_5 &= u + \sqrt{p_h}, & \mathbf{v}_5 &= \left(\frac{h}{\sqrt{p_h}}, 1, 0, 0, 0\right)^T,
\end{aligned}
$$

Contact characteristics ($\xi_{1,2,3} = u$) are linearly degenerate:

$$\nabla_{\mathbf{W}} \xi_{1,2,3} \cdot \mathbf{v}_{1,2,3} \equiv 0, \qquad \nabla_{\mathbf{W}} = (\partial_h, \partial_u, \partial_v, \partial_\eta, \partial_w)^T .$$

Sound characteristics ($\xi_{4,5} = u \pm \sqrt{p_h}$) are genuinely non-linear:

$$\nabla_{\mathbf{W}} \xi_{4,5} \cdot \mathbf{v}_{4,5} = 1 + \frac{h p_{hh}}{2 p_h} \neq 0.$$

Numerical resolution

Conservative formulation of the system:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = \mathbf{S}(\mathbf{U}).$$

Conservative variables and source terms:

$$\mathbf{U} = (h, hu, hv, h\eta, hw)^T, \qquad \mathbf{S}(\mathbf{U}) = \left(0, 0, 0, hw, \lambda\left(1 - \frac{\eta}{h}\right)\right)^T,$$

Flux vectors:

$$\mathbf{F}(\mathbf{U}) = (hu, hu^2 + p, huv, hu\eta, huw)^T, \qquad \mathbf{G}(\mathbf{U}) = (hv, hvu, hv^2 + p, hv\eta, hvw)^T.$$

Numerical resolution of the system is divided in two steps via the classic splitting method:
1) Hyperbolic step:

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{U})}{\partial y} = 0.$$

2) ODE step:

$$\frac{\partial \mathbf{U}}{\partial t} = \mathbf{S}(\mathbf{U}).$$

Numerical resolution: hyperbolic step

Godunov method:

$$\tilde{\mathbf{U}}_{i,j}^{n+1} = \mathbf{U}_{i,j}^{n} - \frac{\Delta t}{\Delta V}\left(\Delta y\left(\mathbf{F}_{i+\frac{1}{2},j}^{n} + \mathbf{F}_{i-\frac{1}{2},j}^{n}\right) + \Delta x\left(\mathbf{G}_{i,j+\frac{1}{2}}^{n} + \mathbf{G}_{i,j-\frac{1}{2}}^{n}\right)\right).$$

The Riemann fluxes are calculated with the HLLC solver:

$$\mathbf{F}_{i+\frac{1}{2},j} = HLLC\big(\mathbf{U}(i,j), \mathbf{U}(i+1,j)\big),$$
$$\mathbf{G}_{i,j+\frac{1}{2}} = HLLC\big(\mathbf{U}(i,j), \mathbf{U}(i,j+1)\big).$$

The method has a stability constraint:

$$\Delta t < \frac{\Delta x}{c_{max}^{n}}, \qquad c_{max}^{n} = \max_{i}\left\{|u_i^n| + a_i^n\right\}.$$

Numerical resolution: ODE step

The ODE consists in resolution of the following system:

$$\frac{\partial h}{\partial t} = 0, \qquad \frac{\partial u}{\partial t} = 0, \qquad \frac{\partial v}{\partial t} = 0, \qquad \frac{\partial \eta}{\partial t} = w, \qquad \frac{\partial w}{\partial t} = -\frac{\lambda}{h}\left(\frac{\eta}{h} - 1\right). \tag{1}$$

The system (1) admits the exact solution:

$$h^{n+1} = h^n, \qquad u^{n+1} = u^n, \qquad v^{n+1} = v^n.$$

$$\eta^{n+1} = h^n + (\eta^n - h^n)\cos\left(\sqrt{\lambda}\frac{\Delta t}{h^n}\right) + \frac{h^n w^n}{\sqrt{\lambda}}\sin\left(\sqrt{\lambda}\frac{\Delta t}{h^n}\right),$$

$$w^{n+1} = -\sqrt{\lambda}\left(\frac{\eta^n}{h^n} - 1\right)\sin\left(\sqrt{\lambda}\frac{\Delta t}{h^n}\right) + w^n\cos\left(\sqrt{\lambda}\frac{\Delta t}{h^n}\right).$$

Numerical results 1D, 1st order in time and space

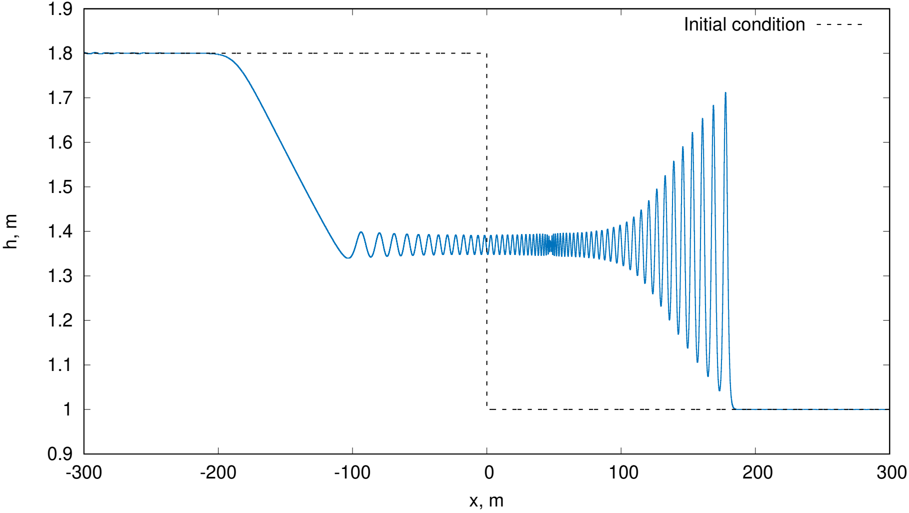


Figure: Numerical solution to the Riemann problem: Dispersive shock wave. $Ncells = 768000$, $\lambda = 300 m^2/s^2$, $t = 44.434 s$.

IMEX: Second order method

Second order implicit-explicit scheme for conservative equations:

$$\mathbf{U}_i^* = \mathbf{U}_i^n - \gamma \frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}}^n - \mathbf{F}_{i-\frac{1}{2}}^n\right) + \gamma \Delta t \mathbf{S}(\mathbf{U}_i^*),$$

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n - (\gamma - 1)\frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}}^n - \mathbf{F}_{i-\frac{1}{2}}^n\right) - (2-\gamma)\frac{\Delta t}{\Delta x}\left(\mathbf{F}_{i+\frac{1}{2}}^* - \mathbf{F}_{i-\frac{1}{2}}^*\right) + (1-\gamma)\Delta t \mathbf{S}(\mathbf{U}_i^*) + \gamma \Delta t \mathbf{S}(\mathbf{U}_i^{n+1}).$$

First step:

- Calculate Riemann fluxes $\mathbf{F}_{i\pm\frac{1}{2}}^n$ from n-th layer
- Solve the implicit equation for $\mathbf{U}_i^*$

Second step:

- Calculate Riemann fluxes $\mathbf{F}_{i\pm\frac{1}{2}}^*$ from $\mathbf{U}_i^*$, calculated on the previous step
- Solve the implicit equation for $\mathbf{U}_i^{n+1}$

Numerical results 1D, 2nd order in time and space

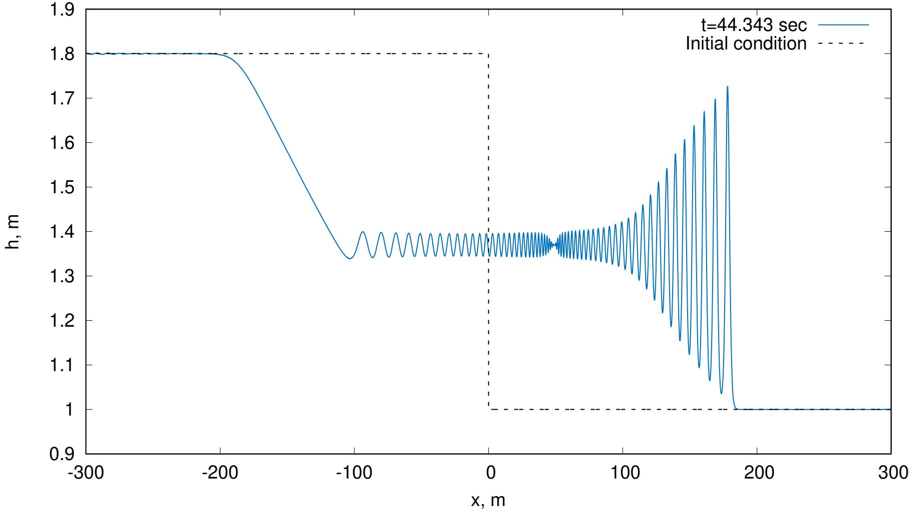


Figure: Numerical solution to 1D Riemann problem for the extended SGN model: a dispersive shock wave. $Ncells = 8000$, $\lambda = 300m^2/s^2$, $t = 44.434s$.

Numerical results 2D

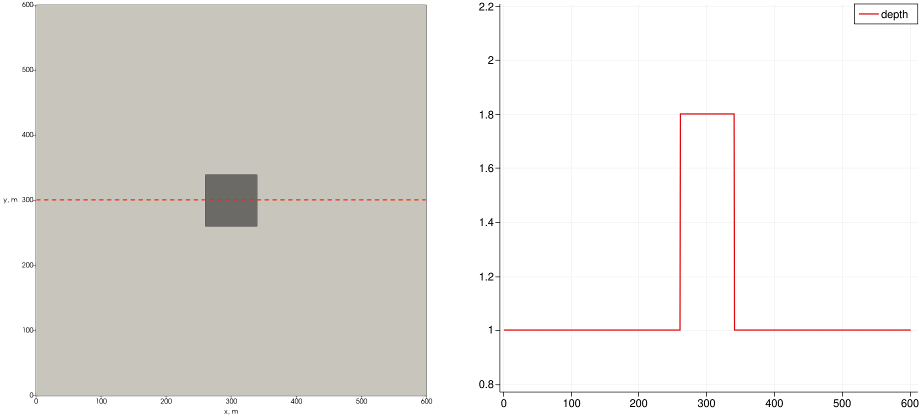


Figure: Initial condition of a 2D Riemann problem for the extended SGN model. Initial water depth is $h_l = 1.8\ m$ inside the dark gray region $\{(x, y) : 260\ m < x, y < 340\ m\}$ and $h_r = 1.0\ m$ outside. Initial velocity is $u_0 = v_0 = 0\ m/s$.

Numerical results 2D

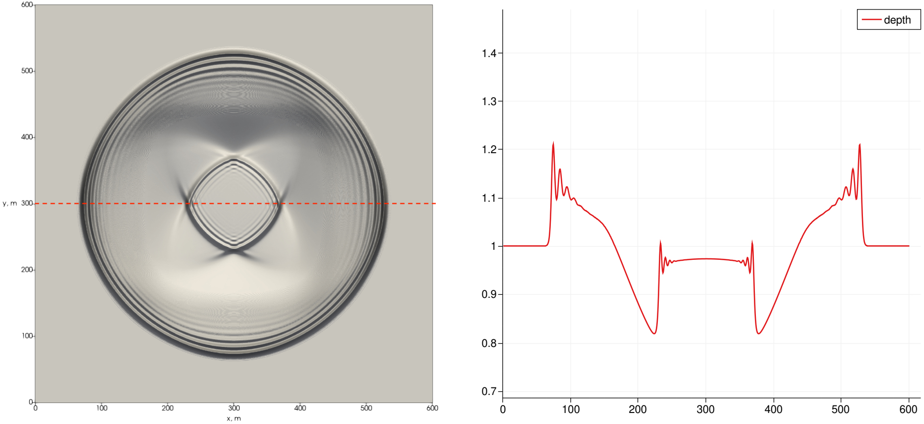


Figure: Numerical solution to 2D Riemann problem for the extended SGN model, $t = 20s$.

Conclusions and perspectives

Conclusions:

- Extended hyperbolic SGN model is derived
- 1D Riemann problem is numerically solved (1st and the 2nd order)
- 2D Riemann problem is numerically solved (1st order)

Perspectives:

- 2D code for bubbly fluids
- Higher order methods for 2D extended models