A macroscopic model to reproduce self-organization near exits

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Outline

Conservation laws with flux constraint

Study of the problem

$$\begin{cases} \partial_t \rho + \partial_x (f(\rho)) = 0 \\ \rho(x,0) = \rho_0(x) \\ f(\rho(0\pm, t)) \leq q(t) \\ q(t) = p\left(\int_{\mathbb{R}} \rho(x,t)\mu(x)\mathrm{d}x\right) \end{cases} \tag{1}$$

- Car/Pedestrian mean density $\rho(x,t) \in [0,1]$ at $(x,t) \in \mathbb{R} \times [0,T]$.
- Flux function $f : [0,1] \to \mathbb{R}^+$ of the kind $f(\rho) = V\rho(1-\rho)$.
- $\rho_0 \in \mathrm{L}^1(\mathbb{R};[0,1]) \cap \mathrm{BV}(\mathbb{R})$.
- Lipschitz weight function μ to average the density upstream the exit.
- $p : [0,1] \to \mathbb{R}^+$, Lipschitz, non-increasing function (exit efficiency).

Notions of solution

Definition

A function $\rho \in L^\infty(\mathbb{R} \times [0, T])$ is a solution to (1) if
(i) for all $\kappa \in [0, 1]$, we have in $\mathcal{D}^+(\mathbb{R} \times [0, T])$

$$\partial_t|\rho - \kappa| + \partial_x\Phi(\rho, \kappa) \leq 2(f(\kappa) - \min\{f(\kappa), q(t)\})\delta_{x=0}$$

(ii) for a.e. $t \in (0, T)$,

$$f(\rho(0\pm, t)) \leq q(t)$$

(iii) for a.e. $t \in (0, T)$,

$$q(t) = p\left(\int_{\mathbb{R}} \rho(x, t)\mu(x)\mathrm{d}x\right)$$

- Upside: uniqueness and stability easy to obtain.
- Downside: the presence of the traces..

Notions of solution

Definition

A function $\rho \in L^\infty(\mathbb{R} \times [0, T])$ is a solution to (1) if
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$$\partial_t|\rho - \kappa| + \partial_x \Phi(\rho, \kappa) \leq 2(f(\kappa) - \min\{f(\kappa), q(t)\})\delta_{x=0} \qquad \text{(E.I.)}$$

(ii) for all $\psi \in C_c^\infty((0, T))$, $\psi \geq 0$ and for all $\varphi \in C_c^\infty(\mathbb{R})$ which verifies $\varphi(0) = 1$,

$$\mp \int_0^T \int_{\mathbb{R}^\pm} \rho \partial_t(\varphi\psi) + f(\rho)\partial_x(\varphi\psi)\, \mathrm{d}x\mathrm{d}t \leq \int_0^T q(t)\psi(t)\mathrm{d}t \qquad \text{(C.I.)}$$

(iii) for a.e. $t \in (0, T)$,

$$q(t) = p\left(\int_{\mathbb{R}} \rho(x, t)\mu(x)\mathrm{d}x\right)$$

Formulation well-suited for passage to the limit in a.e. convergent sequences of approximate solutions.

Stability and uniqueness

Theorem (ADR2014, Theorem 1)

Fix ρ^1, ρ^2 two solutions to (1) corresponding to initial data ρ_0^1, ρ_0^2. Then, for a.e. $t \in (0, T)$,

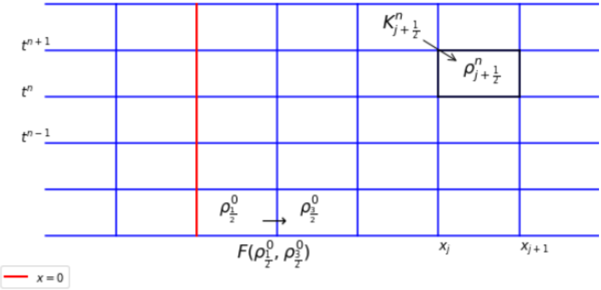
$$||\rho^1(.,t) - \rho^2(.,t)||_{L^1} \leq e^{\alpha t} ||\rho_0^1 - \rho_0^2||_{L^1}, \ \alpha = 2||p'||_{L^\infty} ||\mu||_{L^\infty}. \qquad (2)$$

In particular, problem (1) admits at most one solution.

Proof.

A stability estimate characteristic of (1) yields Lipschitz dependence $q \mapsto \rho$ for $\rho \in \mathrm{C}([0,T]; \mathrm{L}^1)$ and $q \in \mathrm{L}^1(0,T)$. Gronwall's lemma concludes. □

Finite volume scheme



- Time mesh size Δt and spatial mesh size Δx,

$$\forall n \in \mathbb{N},\ \ t^n = n\Delta t,\ \ \forall j \in \mathbb{Z},\ \ x_j = j\Delta x$$

- ρ_0 and μ discretized with $\left(\rho^0_{j+\frac{1}{2}}\right)_{j\in\mathbb{Z}}$ and $\left(\mu_{j+\frac{1}{2}}\right)_{j\in\mathbb{Z}}$.

Marching formula

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ f(\rho(0\pm, t)) \leq q(t) \\ q(t) = p\left(\int_{\mathbb{R}} \rho(x,t)\mu(x)\mathrm{d}x\right) \end{cases} \overrightarrow{F.V.} \begin{cases} \left(\rho^{n+1}_{j+\frac{1}{2}} - \rho^n_{j+\frac{1}{2}}\right)\Delta x + \left(f^n_{j+1} - f^n_j\right)\Delta t = 0 \\ f^n_j = \begin{cases} F(\rho^n_{j-\frac{1}{2}}, \rho^n_{j+\frac{1}{2}}) & \text{if } j \neq 0 \\ \min\left\{F(\rho^n_{-\frac{1}{2}}, \rho^n_{\frac{1}{2}}), q^n\right\} & \text{if } j = 0 \end{cases} \\ q^n = p\left(\sum_{j\in\mathbb{Z}} \rho^n_{j+\frac{1}{2}}\mu_{j+\frac{1}{2}}\Delta x\right) \end{cases}$$

- F is a monotonic numerical flux associated to f
- "Cellwise" constant approximate solution:

$$\rho^\Delta(x,t) = \rho^n_{j+\frac{1}{2}} \text{ if } (x,t) \in K^n_{j+\frac{1}{2}}.$$

- CFL condition

$$2\frac{\Delta t}{\Delta x}||f'||_{\mathrm{L}^\infty} \leq 1.$$

Stability, discrete entropy inequalities

- L^∞ stability.

$$\forall n \in \mathbb{N},\ \forall j \in \mathbb{Z},\ \rho^n_{j+\frac{1}{2}} \in [0,1].$$

- Discrete entropy inequalities with ad hoc numerical fluxes: for all $\kappa \in [0,1]$,

$$\begin{aligned}&\left(|\rho^{n+1}_{j+\frac{1}{2}} - \kappa| - |\rho^n_{j+\frac{1}{2}} - \kappa|\right)\Delta x + \left(\Phi^n_{j+1} - \Phi^n_j\right)\Delta t \\ &\leq \left(f(\kappa) - \min\{f(\kappa), q^n\}\right)\left(\delta_{j=-1} + \delta_{j=0}\right)\Delta t + \ldots\end{aligned} \qquad \text{(D.E.I.)}$$

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$$\partial_t|\rho - \kappa| + \partial_x\Phi(\rho,\kappa) \leq 2(f(\kappa) - \min\{f(\kappa), q(t)\})\delta_{x=0} \qquad \text{(E.I.)}$$

Approximate constraint inequalities

- Approximate coupling equality:

$$q^{\Delta}(t) = p\left(\int_{\mathbb{R}} \rho^{\Delta}(x,t)\mu(x)\mathrm{d}x\right)$$

- Approximate constraint inequalities with ad hoc approximate flux: for all $\psi \in \mathrm{C}^{\infty}_{\mathrm{c}}((0,T))$, $\psi \geq 0$ and for all $\varphi \in \mathrm{C}^{\infty}_{\mathrm{c}}(\mathbb{R})$ such that $\varphi(0) = 1$,

$$-\int_{\Delta t}^{T}\int_{\mathbb{R}+} \rho^{\Delta}\partial_t(\varphi\psi)\,\mathrm{d}x\mathrm{d}t - \int_{0}^{T}\int_{\mathbb{R}+} f^{\Delta}(x,t)\partial_x(\varphi\psi)\,\mathrm{d}x\mathrm{d}t \leq \int_0^T q^{\Delta}(t)\psi(t)\mathrm{d}t + C(\Delta x + \Delta t)$$

and

$$\int_{\Delta t}^{T}\int_{\mathbb{R}-} \rho^{\Delta}\partial_t(\varphi\psi)\,\mathrm{d}x\mathrm{d}t + \int_{0}^{T}\int_{-\infty}^{-\Delta x} f^{\Delta}(x,t)\partial_x(\varphi\psi)\,\mathrm{d}x\mathrm{d}t \leq \int_0^T q^{\Delta}(t)\psi(t)\mathrm{d}t + C(\Delta x + \Delta t).$$

Compactness: local BV estimates and convergence

Theorem (BGKT2008, Lemma 4.1, Lemma 4.2)

Fix $0 < \epsilon < X$ and let $\Omega(X,\epsilon)$ be the open subset

$$\Omega(X,\epsilon) = (-X,-\epsilon) \cup (\epsilon, X).$$

Then, there exists $C > 0$ such that for all $t \geq 0$,

$$TV(\rho^{\Delta}(.,t)_{|\Omega(X,\epsilon)}) \leq TV(\rho_0) + \frac{C}{\epsilon}$$

and

$$\int_{\Omega(X,\epsilon)} |\rho^{\Delta}(x,t+\Delta t) - \rho^{\Delta}(x,t)|dx \leq C\Delta t.$$

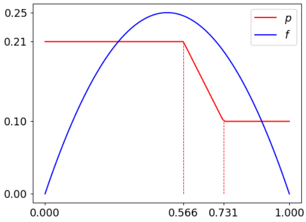
In particular, up to an extraction, there exists $\rho \in L^{\infty}(\mathbb{R} \times [0,T])$ such that $(\rho^{\Delta})_{\Delta}$ converges a.e. to ρ.

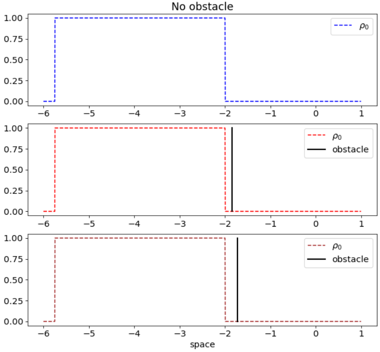
The function ρ constructed above is the solution to Problem (1).

Braess' paradox

Example. Crowd evacuation.

- $\rho_0(x) = \mathbb{1}_{[-5.75\ ;-2]}(x)$
- $\mu(x) = 2(1+x)\mathbb{1}_{[-1;0]}(x).$





Some references

Given constraint: $f(\rho(0\pm, t)) \leq q(t)$

Colombo-Goatin (2007), Andreianov-Goatin-Seguin (2010)

- Well-posedness in BV (front-tracking) and L^∞ (finite-volume scheme).

Non-local constraint: $f(\rho(0\pm, t)) \leq p\left(\int_{\mathbb{R}} \rho(x, t)\mu(x)\mathrm{d}x\right)$

Andreianov-Donadello-Rosini (2014) and Andreianov-Donadello-Rosini-Razafison (2016)

- Well-posedness in BV setting.
- Braess' paradox, "Faster Is Slower" effect.
- Attempt to reproduce self-organization.

Compactness

- Coclite-Risebro (2005) Singular mapping technique.
- Burger-Garcia-Karlsen-Towers (2008) Local BV estimates.
- Towers (2018) One-sided Lipschitz condition.

To go further: two levels of constraint

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \xi(t) = \int_{\mathbb{R}} \rho(x,t)\mu(x)\mathrm{d}x \end{cases}$$

disorganized state
$f(\rho(0\pm,t)) \leq p_{\min}(\xi(t))$

$p_{\min} \leq p_{\max}$

organized state
$f(\rho(0\pm,t)) \leq p_{\max}(\xi(t))$

Two levels of constraint and an organization parameter

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \xi(t) = \int_{\mathbb{R}} \rho(x,t)\mu(x)\mathrm{d}x \end{cases}$$

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$f(\rho(0\pm,t)) \leq p_{\max}(\xi(t))$

$\omega(t) \in (0,1)$
organization parameter

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$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \xi(t) = \int_{\mathbb{R}} \rho(x,t)\mu(x)\mathrm{d}x \end{cases}$$

disorganized state $p_{\min}$

$p_{\min} \leq p_{\max}$

organized state $p_{\max}$

$\omega(t) \in (0,1)$
organization parameter

set
$$q(t) = (1 - \omega(t))p_{\min}(\xi(t)) + \omega(t)p_{\max}(\xi(t))$$

Two levels of constraint and an organization parameter

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ f(\rho(0\pm, t)) \leq q(t) \\ \xi(t) = \int_{\mathbb{R}} \rho(x,t)\mu(x) \mathrm{d}x \end{cases}$$

disorganized state $p_{\min}$

$p_{\min} \leq p_{\max}$

organized state $p_{\max}$

$$q(t) = (1 - \omega(t))p_{\min}(\xi(t)) + \omega(t)p_{\max}(\xi(t))$$

$$\omega(t) \in (0,1)$$

organization parameter, following the dynamic

$$\dot{\omega}(t) = K\left(\xi(t), \dot{\xi}(t)\right)\omega(t)(1 - \omega(t))$$

A coupled PDE-ODE system

To summarize

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ \rho(x,0) = \rho_0(x) \\ f(\rho(0\pm,t)) \leq q(t) \\ \dot{\omega}(t) = K\left(\xi(t), \dot{\xi}(t)\right) \omega(t)(1-\omega(t)) \\ \omega(0) = \omega_0 \end{cases} \tag{3}$$

with

- $\xi(t) = \int_{\mathbb{R}} \rho(x,t)\mu(x)\mathrm{d}x$, the mean subjective density.
- $q(t) = (1-\omega(t))p_{\min}(\xi(t)) + \omega(t)p_{\max}(\xi(t))$.
- $K : \mathbb{R}^2 \to \mathbb{R}$, locally Lipschitz, for example:

$$K(\xi, \dot{\xi}) = \mathrm{C}\left(1 - \alpha\ \xi\ \max\{\dot{\xi}, 0\}\right)$$

Notion of solution

Definition

A couple $(\rho, \omega) \in \mathrm{L}^\infty(\mathbb{R} \times [0, T]) \times \mathrm{W}^{1,\infty}(0, T))$ is a solution to (3) if
(i) for all $\kappa \in [0, 1]$, we have in $\mathcal{D}^+(\mathbb{R} \times [0, T])$

$$\partial_t|\rho - \kappa| + \partial_x \Phi(\rho, \kappa) \leq 2(f(\kappa) - \min\{f(\kappa), q(t)\})\delta_{x=0}$$

(ii) for all $\psi \in \mathrm{C}_c^\infty((0, T))$, $\psi \geq 0$ and for all $\varphi \in \mathrm{C}_c^\infty(\mathbb{R})$ which verifies $\varphi(0) = 1$,

$$\mp \int_0^T \int_{\mathbb{R}^\pm} \rho \partial_t(\varphi\psi) + f(\rho)\partial_x(\varphi\psi) \, \mathrm{d}x\mathrm{d}t \leq \int_0^T q(t)\psi(t)\mathrm{d}t,$$
$$q(t) = (1 - \omega(t))p_{\min}(\xi(t)) + \omega(t)p_{\max}(\xi(t))$$

(iii) Weak ODE formulation: for a.e. $t \in (0, T)$,

$$\omega(t) = \omega_0 + \int_0^t K\left(\xi(s), \dot{\xi}(s)\right)\omega(s)(1 - \omega(s))\mathrm{d}s. \qquad \text{(ODE)}$$

Stability and uniqueness

Theorem (Andreianov-Sylla (2018))

Fix (ρ^1, ω^1) and (ρ^2, ω^2) two solutions to problem (3) corresponding to initial data (ρ_0^1, ω_0^1) and (ρ_0^2, ω_0^2). Then, there exist $\alpha, \beta \geq 0$ and $g \in C([0, T])$ such that for a.e. $t \in (0, T)$,

$$||\rho^1(.,t) - \rho^2(.,t)||_{L^1} \leq ||\rho_0^1 - \rho_0^2||_{L^1} g(t) + \alpha |W_0^1 - W_0^2| \int_0^t g(s)ds$$

and

$$|\omega^1(t) - \omega^2(t)| \leq \frac{|W_0^1 - W_0^2|}{4} + \beta \int_0^t \left(\alpha |W_0^1 - W_0^2|(t-s) + ||\rho_0^1 - \rho_0^2||_{L^1}\right) g(s)ds.$$

In particular, problem (3) admits at most one solution.

Proof.

The map $\omega \mapsto q$ is obviously Lipschitz. Then, the ODE yields Lipschitz dependence $\rho \mapsto \omega$. Gronwall's lemma concludes. □

Finite volume scheme

- Initialization with

$$\omega^0 = \omega_0 \in (0,1) \text{ and } \xi^0 = \int_{\mathbb{R}} \rho_0(x)\mu(x)\mathrm{d}x$$

$$\begin{cases} \partial_t \rho + \partial_x f(\rho) = 0 \\ f(\rho(0\pm, t)) \leq q(t) \\ q(t) = q(\omega(t), \xi(t)) \\ \text{ODE of } \omega \end{cases} \quad \underset{F.V.}{\Longrightarrow} \quad \begin{cases} \left(\rho^{n+1}_{j+\frac{1}{2}} - \rho^n_{j+\frac{1}{2}}\right)\Delta x + \left(f^n_{j+1} - f^n_j\right)\Delta t = 0 \\ f^n_j = \begin{cases} F(\rho^n_{j-\frac{1}{2}}, \rho^n_{j+\frac{1}{2}}) & \text{if } j \neq 0 \\ \min\left\{F(\rho^n_{-\frac{1}{2}}, \rho^n_{\frac{1}{2}}), q^n\right\} & \text{if } j = 0 \end{cases} \\ q^n = (1-\omega^n)p_{\min}(\xi^n) + \omega^n p_{\max}(\xi^n) \\ \xi^{n+1} = \sum_{j\in\mathbb{Z}} \rho^{n+1}_{j+\frac{1}{2}} \mu_{j+\frac{1}{2}} \Delta x \\ \omega^{n+1} = \omega^n + K\left(\xi^{n+1}, \left(\frac{\xi^{n+1} - \xi^n}{\Delta t}\right)\right)\omega^n(1-\omega^n)\Delta t \end{cases}$$

Existence result

Theorem (Andreianov-Sylla 2018)

The problem (3) admits an unique solution (ρ, ω). It can be obtained as the limit of the previous finite volume scheme.

Proof.

- Discrete/Approximate entropy inequalities
- Approximate constraint inequalities
- Compactness for $(\rho^\Delta)_\Delta$
- $(\omega^\Delta)_\Delta$ verifies for a.e. $t \in (0, T)$,

$$\omega^\Delta(t) = \omega_0 + \int_0^t K\left(\xi^\Delta(s+\Delta t), \dot{\xi}^\Delta(s)\right)\omega^\Delta(s)(1-\omega^\Delta(s))\mathrm{d}s + \ldots$$

- Compactness for $(\omega^\Delta)_\Delta$: Arzela-Ascoli

□

Figure 1.

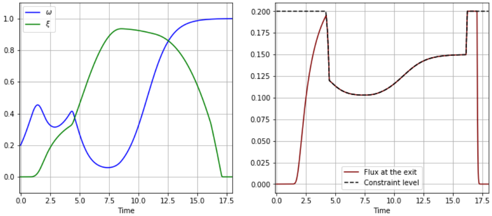
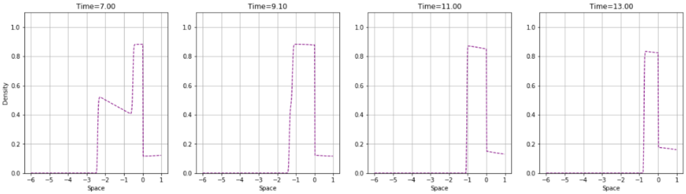


Figure 2.



- Partial reproduction of the self-organization (Figure 1)
- Organization not helpful once a jam arises near the exit (Figure 2)

1. Introduction to balance laws with non-local flux constraint
2. A model to reproduce self-organization
3. Conclusions and perspectives

The model (3) provides what it was designed for:

- easy proof of existence, uniqueness and stability
- promising signs showing its capacity to reproduce some of the self-organization features
- an appreciable flexibility due to the rather non-restrictive assumption on K.

Possible perspectives:

- Study of the model

$$\begin{cases} \partial_t \rho + \partial_x \mathcal{F}_{\omega(t)}(\rho) = 0 \\ \mathcal{F}_{\omega(t)}(\rho) = (1-\omega(t)) f_{\min}(\rho) + \omega(t) f_{\max}(\rho) \\ \mathcal{F}_{\omega(t)}(\rho(0\pm, t)) \leq q(t) \\ \dot{\omega}(t) = K\left(\xi(t), \dot{\xi}(t)\right) \omega(t)(1-\omega(t)) \\ \omega(0) = \omega_0 \end{cases} \tag{4}$$

- A major issue is the dependence of these models regarding the choice of K. Which choices of K can be taken, on phenomenological basis ?
- Put a Brownian motion in the ODE of ω to take into account small random fluctuations of the state of self-organization:

$$\mathrm{d}\omega(t) = \omega(t)(1-\omega(t))\left(K(\xi(t), \dot{\xi}(t))\mathrm{d}t + \epsilon\, \mathrm{d}W(t)\right).$$