A class of finite volume schemes for the 2D shallow water equations with Coriolis force

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SW equations with Coriolis source term

Ω being an open bounded domain of $\mathbb{R}^2$, **flat bottom** and $T > 0$.

$$\begin{cases} \partial_t h + \nabla \cdot (h\boldsymbol{u}) &= 0 & \text{in } \Omega \times (0,T), \\ \partial_t (h\boldsymbol{u}) + \nabla \cdot (h\boldsymbol{u} \otimes \boldsymbol{u}) + g\,h \nabla h &= -\omega\, h\, \boldsymbol{u}^{\perp}, & \text{in } \Omega \times (0,T), \\ \boldsymbol{u} \cdot \boldsymbol{n} &= 0 & \text{on } \partial\Omega \times (0,T), \\ h(\boldsymbol{x},0) &= h_0 & \text{in } \Omega, \\ \boldsymbol{u}(\boldsymbol{x},0) &= \boldsymbol{u}_0 & \text{in } \Omega. \end{cases}$$

Energy balance equation:

$$\partial_t \mathcal{E} + \nabla \cdot ((\frac{1}{2}|\boldsymbol{u}|^2 + gh)\, h\boldsymbol{u}) = 0, \text{ with } \mathcal{E} = \frac{1}{2} g\, h^2 + \frac{1}{2} h |\boldsymbol{u}|^2.$$

Linearised SW equations with Coriolis source term

Linear equations around $u_0 = 0$ and $h_0 > 0$:

$$\begin{cases} \partial_t h + h_0 \nabla \cdot \boldsymbol{u} = 0 & \text{in } \Omega \times (0, T), \\ \partial_t \boldsymbol{u} + g \nabla h = -\omega \boldsymbol{u}^{\perp}, & \text{in } \Omega \times (0, T). \end{cases}$$

Energy balance equation:

$$\partial_t \mathcal{E} + \nabla \cdot (\mathcal{E} \boldsymbol{u}) = 0, \text{ with } \mathcal{E} = \frac{1}{2} g h^2 + \frac{1}{2} |\boldsymbol{u}|^2$$

Geostrophic equilibrium: $g \nabla h + \omega \boldsymbol{u}^{\perp} = 0.$

Linearised SW equations with Coriolis source term

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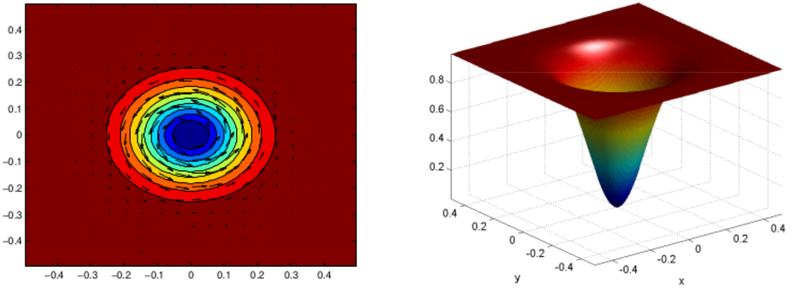


(a) Contour of pressure and vector field

(b) Pressure r

Source : M. H. Do, *Mathematical analysis of finite volume schemes for the simulation of quasi-geostrophic flows at low Froude number, 2017.*

Outline

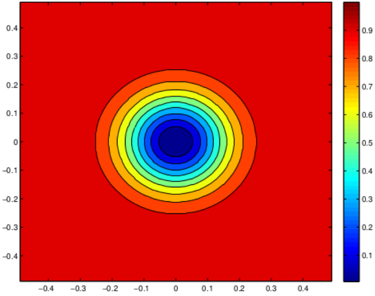
Aim: 2D entropic scheme, consistent kernel with geostrophic equilibrium.

1. State of art
   - Inaccuracy of the classic Godunov scheme
   - Linearised SW with Coriolis source term
   - Energy dissipative scheme for SW
2. Collocated semi-discrete scheme
   - Modified equations
   - Non-linear equations
   - Linear equations
3. Mixed semi-discrete scheme
   - Non-linear scheme
   - Linear scheme

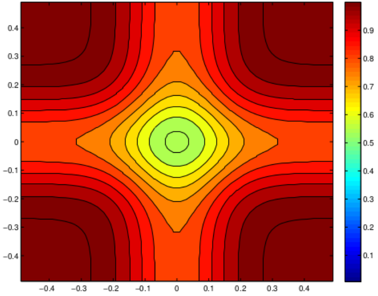
Inaccuracy of the classical Godunov scheme

Modified equations:

$$\begin{cases} \partial_t r + a_\star \nabla \cdot u - \kappa_r a_\star (\Delta x \partial_x^2 r + \Delta y \partial_y^2 r) & = & 0 \\ \partial_t u + a_\star \nabla_x r - \kappa_u a_\star \Delta x \partial_x^2 u & = & \omega v \\ \partial_t v + a_\star \nabla_y r - \kappa_u a_\star \Delta y \partial_y^2 v & = & -\omega u \end{cases}$$



(a) Initial condition



(b) Classical Godunov scheme

Source : E. Audusse, M. H. Do, P. Omnes, and Y. Penel [1]

Linearised SW with Coriolis source term

[1] E. Audusse, M. H. Do, P. Omnes, and Y. Penel, *Analysis of modified godunov type schemes for the two-dimensional linear wave equation with coriolis source term on cartesian meshes*, JCP, 2018.

Cell-centered semi-discrete scheme

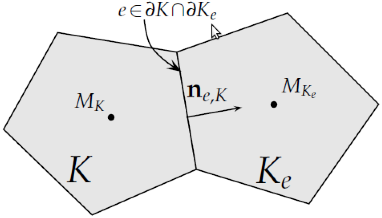
$$\frac{\mathrm{d}r_{i,j}(t)}{\mathrm{dt}} + a_*[\nabla_h \cdot \boldsymbol{u}_h]_{i,j} - \nu_r \left[\nabla_h \cdot \left(\nabla_h r_h + \frac{\omega}{a_*} u_h^\perp\right)\right]_{i,j} = 0,$$

$$\frac{\mathrm{d}\boldsymbol{u}_{i,j}(t)}{\mathrm{dt}} + a_*[\nabla_h \boldsymbol{r}_h]_{i,j} - \nu_u [\nabla_h(\nabla_h \cdot \boldsymbol{u}_h)]_{i,j} = -\omega \boldsymbol{u}_{i,j}^\perp.$$

- preserves geostrophic equilibrium,
- $(\nabla_h r_h + \frac{\omega}{a_*} u_h^\perp = 0) \implies (\nabla_h \cdot \boldsymbol{u}_h = 0)$,
- full discrete energy dissipation ($\nu_r = 0$),
- vertex-based version.

Energy dissipative scheme for SW

[2] F. Couderc, A. Duran and J.-P. Vila, *An explicit asymptotic preserving low Froude scheme for the multilayer shallow water model with density stratification*, JCP, 2017.



Colocated explicit version

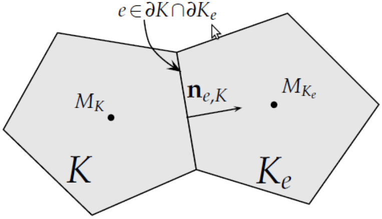
$$h_K^{n+1} = h_K^n - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \mathcal{F}_e^n \cdot \vec{n}_{e,K} m_e \,,$$

$$h_K^{n+1}\mathbf{u}_K^{n+1} = h_K^n\mathbf{u}_K^n - \frac{\Delta t}{m_K} \sum_{e \in \partial K} \Big(\mathbf{u}_K^n \, (\mathcal{F}_e^n \cdot \vec{n}_{e,K})^+ - \mathbf{u}_{K_e}^n \, (\mathcal{F}_e^n \cdot \vec{n}_{e,K})^- \Big) m_e$$

$$- \frac{\Delta t}{m_K} g h_K^n \sum_{e \in \partial K} h_e^{*,n} \, \vec{n}_{e,K} m_e \,.$$

$$h_e = \frac{1}{2}(h_K + h_{K_e})$$

Energy dissipative scheme for SW



Colocated explicit version

$$h_K^{n+1} = h_K^n - \frac{\Delta t}{m_K}\sum_{e\in\partial K}\mathcal{F}_e^n\cdot\vec{n}_{e,K}\,m_e\,,$$

$$h_K^{n+1}\mathbf{u}_K^{n+1} = h_K^n\mathbf{u}_K^n - \frac{\Delta t}{m_K}\sum_{e\in\partial K}\Big(\mathbf{u}_K^n\,(\mathcal{F}_e^n\cdot\vec{n}_{e,K})^+ - \mathbf{u}_{K_e}^n\,(\mathcal{F}_e^n\cdot\vec{n}_{e,K})^-\Big)m_e$$

$$-\frac{\Delta t}{m_K}gh_K^n\sum_{e\in\partial K}h_e^{*,n}\,\vec{n}_{e,K}\,m_e\,.$$

$$h_e = \frac{1}{2}(h_K + h_{K_e})$$

$$\mathcal{F}_e^n = (h\mathbf{u})_e^n - \gamma\, g\,\Pi_e^n \quad,\quad \Pi_e^n \longrightarrow \nabla_e[h] = \Big(h_{K_e}^n - h_K^n\Big)\,\vec{n}_{e,K}\,.$$

$$h_e^* = h_e^n - \alpha\Lambda_e^n \quad,\quad \Lambda_e^n \longrightarrow \nabla_e\cdot[h\mathbf{u}] = \Big(h\mathbf{u}_{K_e}^n - h\mathbf{u}_K^n\Big)\cdot\vec{n}_{e,K}\,.$$

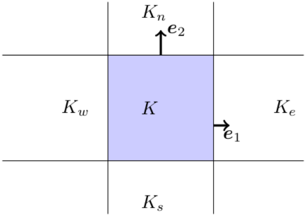
Proposed scheme

Semi-discrete scheme

$$\begin{cases} \dfrac{\mathrm{d}h_K}{\mathrm{dt}} &= -\nabla_K \cdot \mathcal{F} \\ \dfrac{\mathrm{d}h_K \boldsymbol{u}_K}{\mathrm{dt}} &= -\nabla_K^{up} \cdot (\boldsymbol{u}, \mathcal{F}) - g h_K \nabla_K h - \omega h_K \boldsymbol{u}_K^{\perp} + \omega \Pi_K^{\perp} \end{cases}$$

where :

- $\nabla_K^{up} \cdot (\boldsymbol{u}, \mathcal{F}) = \sum_{i \in \{e,w,n,s\}} \dfrac{|i|}{|K|} \left(\boldsymbol{u}_K (\mathcal{F}_{K,i} \cdot \boldsymbol{e}_{K,K_i})^+ + \boldsymbol{u}_i (\mathcal{F}_{K,i} \cdot \boldsymbol{e}_{K,K_i})^- \right)$
- $\nabla_K \cdot \mathcal{F} = \dfrac{\mathcal{F}_{K,e} - \mathcal{F}_{K,w}}{2\Delta x} \cdot \boldsymbol{e}_1 + \dfrac{\mathcal{F}_{K,n} - \mathcal{F}_{K,s}}{2\Delta y} \cdot \boldsymbol{e}_2$
- $\nabla_K h = \dfrac{h_e - h_w}{2\Delta x} \boldsymbol{e}_1 + \dfrac{h_n - h_s}{2\Delta y} \boldsymbol{e}_2$
- $\mathcal{F}_{K,i} = \frac{1}{2}(h_K \mathbf{u}_K + h_i \mathbf{u}_i - \Pi_K - \Pi_i)$
- $\Pi_K = \gamma \Delta t \, h_k (g \nabla_K h + \omega \boldsymbol{u}_K^{\perp})$



Modified equations associated scheme

Semi-discrete scheme

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{dt}} h_k & = & -\nabla_K \cdot h\boldsymbol{u}_K + \nabla_K \cdot \boldsymbol{\Pi}_K \\ \frac{\mathrm{d}}{\mathrm{dt}} h_K \boldsymbol{u}_K & = & -\nabla_K^{up} \cdot (\boldsymbol{u}, \mathcal{F}) - g h_K \nabla_K h - \omega\, h\boldsymbol{u}_K^{\perp} + \omega \boldsymbol{\Pi}_K^{\perp} \end{cases}$$

with $\boldsymbol{\Pi}_K = \gamma \Delta t\ h_k (g \nabla_K h + \omega \boldsymbol{u}_K^{\perp})$

Modified non linear equations

$$\begin{cases} \partial_t h & = & -\nabla \cdot h\boldsymbol{u} + \nabla \cdot \boldsymbol{\Pi} \\ \partial_t (h\boldsymbol{u}) & = & -\nabla \cdot (\mathcal{F} \otimes \boldsymbol{u}) - g\, h\, \nabla h - \omega\, h \boldsymbol{u}^{\perp} + \omega \boldsymbol{\Pi}^{\perp} \end{cases}$$

where : $\boldsymbol{\Pi} = \gamma \Delta t (g\, h \nabla h + \omega\, h\, \boldsymbol{u}^{\perp})$.

Mechanic energy balance of the modified equations :

$$\partial_t \left(\frac{1}{2}(gh^2 + h\boldsymbol{u}^2) \right) + \nabla \cdot \left((gh + \frac{1}{2}|\boldsymbol{u}|^2) h\boldsymbol{u} \right) = -\gamma \Delta t \left| g\, h \nabla h + \omega\, h\, \boldsymbol{u}^{\perp} \right|^2.$$

Energy of the scheme

$$\frac{d}{dt}\left(\sum_{K\in\mathbb{T}}\frac{1}{2}(gh_K^2+h_K|\boldsymbol{u}|_K^2)\right)=\sum_{K\in\mathbb{T}}|K|\left(gh_K\nabla_K\cdot\boldsymbol{\Pi}+\omega\boldsymbol{\Pi}_K^{\perp}\cdot\boldsymbol{u}_K+R_K\right)$$

where

- $R_K=\frac{1}{2}\frac{1}{|K|}\sum_{e\in\partial K}|e|\|\boldsymbol{u}_{Ke}-\boldsymbol{u}_K\|^2(\mathcal{F}_e\cdot\boldsymbol{n}_{e,K})^-\leq 0$
- $\sum_{K\in\mathbb{T}}(gh_K\nabla_K\cdot\boldsymbol{\Pi}+\omega\boldsymbol{\Pi}_K^{\perp}\boldsymbol{u}_K)=-\sum_{K\in\mathbb{T}}((g\nabla_K h+\omega\boldsymbol{u}_K^{\perp})\cdot\boldsymbol{\Pi}_K)\leq 0$
  $\boldsymbol{\Pi}_K=\gamma\Delta t(g\,h_K\nabla_K h+\omega\,h\,\boldsymbol{u}^{\perp})$.thanks to the grad-div duality, preserved by our discretisations

Energy decreasing property

$$\frac{d}{dt}\left(\sum_{K\in\mathbb{T}}\frac{1}{2}(gh_K^2+h_K|\boldsymbol{u}|_K^2)\right)\leq 0$$

Linearisation of the scheme

Semi-discrete linearised scheme around $h_0 > 0$ and $u_0 = 0$

$$\begin{aligned} \frac{d}{dt} h_K + \nabla_K \cdot (h_0\, \boldsymbol{u}_K - \boldsymbol{\Pi}_K) &= 0, \\ h_0\, \frac{d}{dt} \boldsymbol{u}_K + g\, h_0\, \nabla_K h &= -\omega (h_0\, \boldsymbol{u}_K - \boldsymbol{\Pi}_K)^{\perp}. \end{aligned}$$

where : $\boldsymbol{\Pi}_K = h_0\, \gamma \Delta t \,(g \nabla_K h + \omega \boldsymbol{u}_K^{\perp})$.

Remark : Very similar to Audusse and al [1].

Modified equations associated: $\begin{cases} \partial_t h + \nabla \cdot \mathcal{F} &= 0, \\ \partial_t h_0 \boldsymbol{u} + g\, h_0\, \nabla h &= -\omega \mathcal{F}^{\perp}, \end{cases}$

where: $\mathcal{F} = h_0 \boldsymbol{u} - \gamma\, \Delta t\, h_0\, (g\, \nabla h + \omega\, \boldsymbol{u}^{\perp})$.

Geostrophic equlibrium: $g \nabla h + \omega\, \boldsymbol{u}^{\perp} = 0$

The kernel

The discrete kernel associated to the geostrophic equilibrium is :

$$\mathcal{K}_k = \{(h_k, \boldsymbol{u}_K) \mid g\nabla_K h + \omega \boldsymbol{u}_K^\perp = 0\}$$

and

Properties of the kernel

- is consistent with the continuous one,
- is preserved by the linearised version of our scheme. That is : if $(h_k(0), \boldsymbol{u}_k(0)) \in \mathcal{K}_k$ then $(h_k(t), \boldsymbol{u}_k(t)) \in \mathcal{K}_k$, $\forall t \geq 0$.

Remark : The geostrophic equilibrium is, by definition, not an equilibrium for the non linear shallow water model.

Mixed semi-discrete scheme

Non-linear scheme

$$\left\{ \begin{array}{rcl} \frac{d}{dt} h_K + \nabla_K \cdot (\mathcal{F}) & = & 0, \\ \frac{d}{dt}(h\boldsymbol{u})_K + \nabla_K^{up} \cdot (\boldsymbol{u}\mathcal{F}) + g h_K \nabla_K h & = & -\omega\, h_K\, \boldsymbol{u}_K^{\perp} + \omega \sum_{e \in \partial K} \frac{1}{2} \boldsymbol{\Pi}_e^{\perp} \end{array} \right.$$

Mixed semi-discrete scheme

Non-linear scheme

$$\left\{\begin{array}{rcl} \frac{d}{dt}h_K + \nabla_K \cdot (\mathcal{F}) & = & 0, \\ \frac{d}{dt}(h\boldsymbol{u})_K + \nabla_K^{up} \cdot (\boldsymbol{u}\mathcal{F}) + gh_K \nabla_K h & = & -\omega\, h_K\, \boldsymbol{u}_K^\perp + \omega \sum_{e\in\partial K} \frac{1}{2} \boldsymbol{\Pi}_e^\perp \end{array}\right.$$

- $\mathcal{F} = h\boldsymbol{u} - \gamma \Delta t\, h\, (g \nabla h + \omega\, \boldsymbol{u}^\perp)$
- $\boldsymbol{\Pi}_e = \gamma \Delta t\, h_e\, (g \nabla_e h + \omega\, \boldsymbol{u}_e^\perp)$
- $\nabla_e h = \dfrac{|e|}{|K|}(h_{Ke} - h_K)\boldsymbol{e}_1$, with $e = K|K_e$
- $h_e = \frac{1}{2}(h_{Ke} + h_K)$ and $\boldsymbol{u}_e = \frac{1}{2}(\boldsymbol{u}_{Ke} + \boldsymbol{u}_K)$

Mixed semi-discrete scheme

Non-linear scheme

Discrete mechanic energy:

$$\mathcal{E}(t) = \sum_{K \in \mathbb{T}} |K| \frac{1}{2}(gh_K^2 + h_K |\boldsymbol{u}|_K^2).$$

Decreasing of the mechanic energy

$$\mathcal{E}'(t) \leq 0.$$

Mixed semi-discrete scheme

Linear scheme

$$\begin{cases} \frac{d}{dt} h_K + \nabla_K \cdot (\mathcal{F}) & = & 0, \\ h_0 \frac{d}{dt} \boldsymbol{u}_K + g h_0 \nabla_K h & = & -\omega h_0 \boldsymbol{u}_K^{\perp} + \omega \sum_{e \in \partial K} \frac{1}{2} \boldsymbol{\Pi}_e^{\perp} \end{cases}$$

- $\mathcal{F} = h_0 \boldsymbol{u} - \gamma \Delta t \, h_0 \, (g \nabla h + \omega \boldsymbol{u}^{\perp})$
- $\boldsymbol{\Pi}_e = \gamma \Delta t \, h_0 \, (g \nabla_e h + \omega \boldsymbol{u}_e^{\perp})$

Mixed semi-discrete scheme

Linear scheme

Decreasing of the mechanic energy

$$\mathcal{E}'(t) \ \leq 0.$$

Mixed semi-discrete scheme

Linear scheme

Decreasing of the mechanic energy

$$\mathcal{E}'(t) \le 0.$$

Discrete kernel: $\mathcal{K}_K = \{(h_K, \boldsymbol{u}_K) \mid \boldsymbol{\Pi}_K = 0\}$.

Well balanced property ?

$$(h_K, \boldsymbol{u}_K)(t=0) \in \mathcal{K}_K \not\Rightarrow (h_K, \boldsymbol{u}_K)(t>0) \in \mathcal{K}_K,$$

Since $\Pi_K = 0 \not\Rightarrow \Pi_e = 0$.

We have to choose between dissipative energy and preservation of the geostrophic equilibrium, cannot have both.

Conclusion

Results

- Several schemes for non-linear and linear shallow water with Coriolis force
- Semi-discrete energy stability
- Preservation of the geostrophic equilibrium for the collocated semi-discrete scheme
- Staggered semi-discrete scheme

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Perspectives

- Time discretisation of Coriolis source term (work in progress)
- Fully discrete energy inequality
- Addition of a pressure correction
- Numerical simulation (work in progress)